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REES ALGEBRAS OF COMPLETE BIPARTITE GRAPHS

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Abstract

We study Rees algebras of edge ideals associated to complete bipartite graphs, the main result is a compact expression for their canonical module. Some formulae for the Cohen-Macaulay type and Hilbert series of those algebras are also presented.

1. Introduction

Let G be a graph on the vertex set $V = \{x_1, \dots, x_n\}$ and $R = K[x_1, \dots, x_n]$ a polynomial ring over a field K. The *monomial subring* of G is the K-subalgebra

$$K[G] = K[\{x_ix_j | \, x_i \text{ is adjacent to } x_j\}] \subset R,$$

and the *edge ideal* $I(G)$ of G is the ideal of R generated by all the squarefree monomials x_ix_j so that x_i is adjacent to x_j. To relate properties of $K[G]$ and $I(G)$ it is useful to introduce the *Rees algebra* of $I(G)$ defined as:

$$\mathcal{R}(I(G)) = R[\{x_ix_jT | \, x_i \text{ is adjacent to } x_j\}] \subset R[T].$$

A remarkable connection between the monomial subring $K[G]$ and the edge ideal $I(G)$ occurs when G is a connected graph, in this case $K[G]$ is a normal domain if and only if all the powers of $I(G)$ are complete [8].

Here we examine the Rees algebra $\mathcal{R}(I)$ of the edge ideal I of the complete bipartite graph $G = \mathcal{K}_{m,n}$ using two of its representations. The contents of this note are as follows. First we compute the Hilbert series of $\mathcal{R}(I)$ by representing

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this algebra as a polynomial ring modulo its ideal of relations and applying an explicit description of Conca and Herzog [2] for the Hilbert series of $K[\mathcal{K}_{m,n}]$. It will turn out that $\mathcal{R}(I)$ is defined by a ladder ideal of the simplest kind.

To introduce a second representation of $\mathcal{R}(I)$ recall that the *cone* $C(G)$, over the graph G, is obtained by adding a new vertex x to G and joining every vertex of G to x. According to [9, Section 7.3], there is an isomorphism:

$$\mathcal{R}(I) \overset{\varphi}{\simeq} K[C(G)],$$

thus $\mathcal{R}(I)$ can be regarded as the monomial subring of a graph, this observation is used to find an explicit formula for the canonical module of $\mathcal{R}(I)$ and to compute its last Betti number. The main tool is a result of Danilov-Stanley that provides a useful expression for the canonical module of certain monomial subrings; for details about this result, as well as for basic facts and terminology on polyhedral theory, the reader is refer to [1, Chapter 6].

2. The Hilbert series and canonical module

The following result do not seem to have been noticed, it shows that Rees algebras of edge ideals of complete bipartite graphs have some structure.

Proposition 2.1. *Let B/Q be the presentation of the Rees algebra of the edge ideal $I = (x_iy_j | 1 \leq i \leq m, 1 \leq j \leq n)$ of a complete bipartite graph. Then the toric ideal Q is equal to the ideal $I_2(Y)$ generated by the 2-minors of the ladder*

$$Y = \begin{array}{ccccc} & y_1 & y_2 & \cdots & y_n \\ x_1 & T_{11} & T_{12} & \cdots & T_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_m & T_{m1} & T_{m2} & \cdots & T_{mn} \end{array}$$

and the 2-minors of Y form a Gröbner basis w.r.t the lex ordering induced by $y_1 > \cdots > y_n > x_1 > T_{11} > \cdots > T_{1n} > x_2 > \cdots > T_{mn}$.

Proof. Let K be a field and consider the presentation of the Rees algebra of I:

$$\psi : B = K[x_i\text{'s}, y_j\text{'s}, T_{ij}\text{'s}] \longrightarrow \mathcal{R}(I), \quad \psi(T_{ij}) = Tx_iy_j, \quad Q = \ker(\psi).$$

Note $I_2(Y) \subset Q$. As they are both prime ideals of height $mn-1$, by [3, Section 4] and [6], one has the equality. The last assertion that the 2-minors are a Gröbner basis of $I_2(Y)$ is a general fact of ladder ideals [6]. □

Lemma 2.2. *Let $\mathcal{K}_{m,n}$ be the complete bipartite graph and A/P the presentation of $K[\mathcal{K}_{m,n}]$. If $n \geq m$, then the Hilbert series of A/P is given by*

$$H(A/P,z) = \frac{\sum_{i=0}^{m-1} \binom{m-1}{i}\binom{n-1}{i} z^i}{(1-z)^{n+m-1}}.$$

In particular $e(K[\mathcal{K}_{m,n}])$, the multiplicity of $K[\mathcal{K}_{m,n}]$ is equal to $\binom{m+n-2}{m-1}$.

Proof. Let $V_1 = \{x_1, \ldots, x_m\}$, $V_2 = \{y_1, \ldots, y_n\}$ be disjoint sets of vertices and $R = K[V_1 \cup V_2]$ a polynomial ring over a field K.

Let $T_{ij}, 1 \leq i \leq m,\ 1 \leq j \leq n$ be distinct indeterminates and map

$$\psi : A = K[T_{ij}\text{'s}] \longrightarrow K[\mathcal{K}_{m,n}] = K[x_i y_j\text{'s}] \subset R, \quad \psi(T_{ij}) = x_i y_j.$$

It is known [4] that the toric ideal $P = \ker(\psi)$ equals the ideal $I_2(X)$ generated by the 2-minors of the $m \times n$ matrix $X = (T_{ij})$ and $\dim(K[\mathcal{K}_{m,n}]) = m+n-1$.

Let $I_{r+1}(X)$ be the ideal generated by the $r+1$-minors of X. According to [2] the Hilbert series of $R_{r+1} = K[T_{ij}]/I_{r+1}(X)$ has a description as:

$$H(R_{r+1},z) = \frac{\det(\sum_k \binom{m-i}{k}\binom{n-j}{k} z^k)_{i,j=1,\ldots,r}}{z^{\binom{r}{2}}(1-z)^d},$$

where $d = \dim R_{r+1}$. Making $r=1$ the result follows. □

Proposition 2.3. *Let B/Q be the presentation of $\mathcal{R}(I)$, the Rees algebra of the edge ideal $I = (x_i y_j | 1 \leq i \leq m, 1 \leq j \leq n)$. If $n \geq m$, then the Hilbert series of B/Q can be written as*

$$H(B/Q,z) = \frac{1 + (mn-1)z + \sum_{i=2}^{m} \binom{n}{i}\binom{m}{i} z^i}{(1-z)^{n+m+1}}.$$

In particular $a(B/Q) = -(n+1)$ *and* $e(B/Q) = \binom{m+n}{m} - 1$.

Proof. Let $B = K[x_i, y_j, T_{ij} | 1 \leq i \leq m,\ 1 \leq j \leq n]$ be a polynomial ring over a field K and consider the ladders

$$M_\ell = \begin{matrix} & y_1 & y_2 & \cdots & y_n \\ & T_{11} & T_{12} & \cdots & T_{1n} \\ & \vdots & \vdots & & \vdots \\ x_\ell & T_{\ell 1} & T_{\ell 2} & \cdots & T_{\ell n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_m & T_{m1} & T_{m2} & \cdots & T_{mn} \end{matrix} \qquad N_\ell = \begin{matrix} y_1 & y_2 & \cdots & y_n \\ T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ T_{\ell 1} & T_{\ell 2} & \cdots & T_{\ell n}. \end{matrix}$$

Let I_ℓ and J_ℓ be the ideals generated by the 2-diagonals of M_ℓ and N_ℓ respectively, where a 2-diagonal is the product of the diagonal entries of a 2×2 submatrix.

By Proposition 2.1 and [1, Theorem 4.2.3] one has $H(B/Q, z) = H(B/I_1, z)$. Hence the proof reduces to compute $H(B/I_1, z)$.

For $1 \leq \ell \leq m-1$ consider the graded exact sequences

$$0 \longrightarrow B/K_\ell[-1] \xrightarrow{x_\ell} B/(x_1, \ldots, x_{\ell-1}, I_\ell) \longrightarrow B/(x_1, \ldots, x_\ell, I_{\ell+1}) \longrightarrow 0,$$

where $K_\ell = (T_{ij} | \ell+1 \leq i) + J_\ell + (x_1, \ldots, x_{\ell-1})$. Therefore

$$H(B/I_1, z) = \left(\sum_{\ell=1}^{m-1} H(B/K_\ell, z) \right) z + H(B/(x_1, \ldots, x_{m-1}, I_m), z).$$

As $x_\ell, \ldots, x_m$ is a regular sequence on B/K_ℓ using Lemma 2.2 one readily obtains

$$H(B/K_\ell, z) = \frac{\sum_{i=0}^{\ell} \binom{\ell}{i} \binom{n-1}{i} z^i}{(1-z)^{m+n+1}},$$

and consequently

$$H(B/I_1, z) = \frac{\left(\sum_{\ell=1}^{m-1} \left(\sum_{i=0}^{\ell} \binom{\ell}{i} \binom{n-1}{i} z^i \right) \right) z + \sum_{i=0}^{m} \binom{m}{i} \binom{n-1}{i} z^i}{(1-z)^{m+n+1}}.$$

Using the formula $\sum_{i=0}^{n}\binom{x+i}{i}=\binom{x+n+1}{n}$ yields the required equality. □

Theorem 2.4. *Let K be a field, $S=K[x_iy_j,x_ix,y_jx|1\le i\le m,1\le j\le n]$ the monomial subring of the cone over a complete bipartite graph $\mathcal{K}_{mn}$ and ω_S the canonical module of S. Assume $n\ge m\ge 2$.*

(a) If $n=m$, then $\omega_S=(x_1\cdots x_ny_1\cdots y_nx^{2i}|\,1\le i\le n-1)$.

(b) If $n\ge m+1$, then ω_S is generated by the set of monomials of the form:

(i) $M=x_1\cdots x_my_1\cdots y_nx^b$, $n-m+1\le b\le n+m-1$ and $n+m+b\in 2\mathbb{N}$, or

(ii) $M=x_1^{a_1}\cdots x_m^{a_m}y_1\cdots y_nx^b$, $\sum_{i=1}^{m}a_i=n-b+2$, $b=2,3,\dots,n-m+1$, and $a_i\ge 1$ for all i.

Moreover in both cases (a) and (b) those generating sets for ω_S are minimal.

Proof. For $M=x_1^{a_1}\cdots x_m^{a_m}y_1^{b_1}\cdots y_n^{b_n}x^b$, set $\log(M)=(a_1,\dots,a_m,b_1,\dots,b_n,b)$. Define

$$\mathcal{A}=\{\log M|\,M\in\{x_iy_j,x_ix,y_jx|1\le i\le m,1\le j\le n\}\}.$$

Since S is a normal Cohen-Macaulay domain of dimension $n+m+1$ by [5] and [8, Theorem 1.1], one can use the Danilov-Stanley formula [1, Theorem 6.3.5] to express its canonical module as:

$$\omega_S=(\{M=x_1^{a_1}\cdots x_m^{a_m}y_1^{b_1}\cdots y_n^{b_n}x^b|\ \log(M)\in\mathbb{N}\mathcal{A}\cap(\mathbb{R}_+\mathcal{A})^{\circ}\}),$$

where $(\mathbb{R}_+\mathcal{A})^{\circ}$ is the interior of the cone generated by $\mathcal{A}$ and S has the normalized grading. Let $M=x_1^{a_1}\cdots x_m^{a_m}y_1^{b_1}\cdots y_n^{b_n}x^b$ be a minimal generator of ω_S. According to [10, Theorem 3.2] a vector $\beta\in\mathbb{N}^{m+n+1}$ is in $(\mathbb{R}_+\mathcal{A})^{\circ}$ if and only if β satisfies the inequalities:

$$\begin{aligned} 1 &\leq \beta_i,\ i=1,\ldots,m+n \\ \sum_{i=1}^{m}\beta_i &\leq -1+\beta_{m+n+1}+\sum_{i=m+1}^{m+n}\beta_i \\ \sum_{i=m+1}^{m+n}\beta_i &\leq -1+\beta_{m+n+1}+\sum_{i=1}^{m}\beta_i \\ \beta_{m+n+1} &\leq -1+\sum_{i=1}^{m+n}\beta_i, \end{aligned} \tag{1}$$

Therefore one has:

$$\begin{aligned} 1 &\leq a_i, \quad 1 \leq b_j, \quad 1\leq i\leq m, \quad 1\leq j\leq n \\ \sum_{i=1}^{m}a_i &\leq -1+b+\sum_{i=1}^{n}b_i \\ \sum_{i=1}^{n}b_i &\leq -1+b+\sum_{i=1}^{m}a_i \\ b &\leq -1+\sum_{i=1}^{m}a_i+\sum_{i=1}^{n}b_i \end{aligned} \tag{2}$$

Using that $\alpha\in\mathbb{N}\mathcal{A}$ one also obtains:

$$b+\sum_{i=1}^{m}a_i+\sum_{i=1}^{n}b_i\in 2\mathbb{N}. \tag{3}$$

Observe that (2) and (3) yield $b\geq 2$. We consider the following cases:

I. Assume $a_i\geq 2$ for some i. We claim that $b_i=1$, for all i. For simplicity assume $a_1\geq 2$ and $b_1\geq 2$. Write:

$$b=-i+\sum_{i=1}^{m}a_i+\sum_{i=1}^{n}b_i,\ i\geq 1.$$

If $1\leq i\leq 2$, write $M=M'y_1x$. Since $\log M=\log M'+\log y_1x$ one has that $\log M'$ is in $\mathbb{Z}\mathcal{A}$. It is not hard to see that $\log M'$ satisfies all the equations in (1), thus $\log M'\in(\mathbb{R}_+\mathcal{A})^{\circ}$. By the normality of S one concludes $\log M'\in\mathbb{N}\mathcal{A}$, hence M' is in ω_S, which contradicts the minimality of M. If $i\geq 3$, consider $M=M'x_1y_1$, as before one readily obtains $M'\in\omega_S$, which again contradicts the choice of M.

II. Assume $n=m$. First let us show $a_i=1$ and $b_j=1$, for all i,j. By the previous case and symmetry if $a_i\geq 2$ for some i (resp. $b_i\geq 2$), then $b_i=1$ for all i (resp. $a_i = 1$ for all i). If $a_i \geq 2$ for some i, say $i = 1$, then $M' \in \omega_S$, where $M = M'x_1x$, and this is impossible by the choice of M. Next, using (2) and (3), it follows that $b = 2i$ for some $1 \leq i \leq n-1$, and we are in case (a).

III. Assume $n \geq m+1 \geq 3$. We claim that $b_i = 1$, for all i. If $b_i \geq 2$, say $i = 1$, then by case I one has $a_i = 1$ for all i. As before one readily derives a contradiction by writing $M = M'y_1x$. Altogether one can write:

$$M = x_1^{a_1} \cdots x_m^{a_m} y_1 \cdots y_n x^b, \quad b \geq 2.$$

If $a_i = 1$ for all i, then using (2) and (3) one has

$$n - m + 1 \leq b \leq m + n - 1 \ \text{ and } \ m + n + b \in 2\mathbb{N},$$

and we are in case (b). Next assume $a_i \geq 2$ for some i, say $i = 1$. Combining (2) and (3) one has $n \leq \sum_{i=1}^{m} a_i + b - 2$. If $n \leq \sum_{i=1}^{m} a_i + b - 3$, one derives a contradiction by considering $M = M'x_1x$. Therefore

$$m + 1 \leq \sum_{i=1}^{m} a_i = n - b + 2, \quad b = 2, \ldots, n - m + 1,$$

and we are in case (b).

Thus any minimal generator of ω_S must be as in (a) or (b). To complete the argument observe that any of the monomials occuring in (a) or (b) are in ω_S and they are not multiple of each other. □

Remark 2.5. If $n > m = 1$, then assertion (b) above still holds. To show it replace the very first set of inequalities in (1) by $\beta_i \geq 1$, $i = 2, \ldots, n+m$. On the other hand if $n = m = 1$, then $\omega_S = (x_1^2 y_1^2 x^2)$.

Let B/Q be the presentation of a Cohen-Macaulay monomial subring $K[G]$, where G is a graph. The *Cohen-Macaulay type* of the ring $K[G]$ is the last Betti number in the minimal free resolution of B/Q as a B-module; it will be denoted by type($K[G]$). Set $S = K[G]$. We recall that the *type* of S is also equal to the minimal number of generators of the canonical module ω_S of S.

Corollary 2.6. *Let $S = \mathcal{R}(I)$ be the Rees algebra of the edge ideal I of the* *complete bipartite graph* $\mathcal{K}_{mn}$*. If* $n \geq m \geq 2$ *or* $n > m = 1$*, then*

$$type(S) = \binom{n}{m} + (m-2).$$

Proof. If $n = m \geq 2$ or $n > m = 1$ the formula follows from Theorem 2.4(a) and Remark 2.5. Assume $n \geq m+1 \geq 3$. Observe that there are $m-1$ generators of ω_S as in Theorem 2.4(i), to count the remaining generators of ω_S as in Theorem 2.4(ii) note that the sum from $b = 2$ to $b = n-m+1$ of the m-partitions of $n-b+2$ is equal to

$$\sum_{b=2}^{n-m+1} \binom{n-b+1}{m-1} = \sum_{i=m+1}^{n} \binom{i-1}{m-1} = -1 + \sum_{i=0}^{n-m} \binom{m-1+i}{m-1} = -1 + \binom{n}{m}.$$

Hence type$(S) = (m-1) + \binom{n}{m} - 1 = \binom{n}{m} + (m-2)$. □

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