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TAME ALGEBRAS AND DERIVED CATEGORIES

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0. Introduction

Let A be a finite dimensional algebra over an algebraically closed field k. One of the main tasks of the representation theory of algebras is the study of the category A of modules. In the best understood cases, this task is achieved by classifying all the indecomposable A-modules (up to isomorphism). From this point of view, the simplest situation occurs when there are only finitely many isomorphism classes of indecomposable A-modules, that is, A is *representation finite*. The problem of determining if a given algebra A is representation-finite and, in case it is, classifying all the indecomposable modules, is well understood, see for examples [7].

Tame algebras are precisely those algebras for which one may expect to reach the classification of the indecomposable modules. Indeed, the *tame-wild dichotomy* tell us that any algebra A is either tame and hence the indecomposable modules of each fixed dimension are classified in a finite number of one-parametric or discrete families of modules, or A is wild and then there is a $A - k\langle x,y\rangle$-bimodule M which is free as right $k\langle x,y\rangle$-module and such that the functor $M \otimes -\colon$ k$\langle x,y\rangle \to$ A preserves indecomposability and reflects isomorphism classes.

For certain families of algebras there are good criteria to determine the representation type (tame or wild). Disgracefully there is no general solution for this problem or for the problem of classifying the indecomposable modules over a tame algebra. For the best known examples of tame algebras A, also the *derived category* $D^b(A)$ of the category A is well understood. Namely, for

$A = kQ$ an *hereditary algebra* of finite or tame type (that is, A is the path algebra of a quiver Q of Dynkin or extended Dynkin type), the description of the indecomposable objects over $D^b(A)$ is given in [8]; in the case, A is a *tubular algebra*, the description was given in [9]. It is worth recalling that the category of coherent sheaves on the projective space $\mathbb{P}^n$ is derived equivalent to the module category modA_n of a certain finite dimensional algebra A_n.

Two derived-equivalent algebras A and B (that is, $D^b(A)$ and $D^b(B)$ are triangle equivalent) share many other invariants, among which the Grothendieck group, equipped with the Euler bilinear form, is most important. Work has been done to characterize algebras A which are derived-equivalent to a tame hereditary or to a tubular algebra.

The purpose of these lectures is to present the main concepts of tame and wild algebras and to consider the associated derived categories. For the understanding of tame algebras and their modules categories, the introduction of some geometric concepts and techniques is of fundamental importance. In section 2 we introduce the notion of module varieties and show some geometric characterizations of tame algebras. In section 3 we describe some examples of derived categories of tame algebras and in section 4 we present some recent characterizations of algebras which are derived-equivalent to tame hereditary or tubular algebras.

For some basic notions, the reader may refer to [7] and [12]. For further information on tame algebras to [13]. Most of the recent results reported on these notes come from the papers [3], [5].

1. Tame and wild algebras

1.1. A problem from analysis. Consider the system of m differential equations of degree s in n indeterminates

$$M_s \frac{d^s x}{dt^s} + M_{s-1} \frac{d^{s-1} x}{dt^{s-1}} + \cdots + M_1 \frac{dx}{dt} + M_0 x = f(t),$$

where $M_0, M_1, \dots, M_s$ are complex $m \times n$-matrices, x is a column formed by the indeterminates and $f(t)$ is a vector function. The problem is to solve the system.

The case of degree $s = 1$ was consider originally by K. Weiestrass and only partially solved. The complete solution for that case was given years alter by Kronecker. We shall start examining the general problem.

Given two invertible matrices P of size $m \times m$ and Q of size $n \times n$, we may transform the system to

$$M'_s \frac{d^s z}{dt^s} + M'_{s-1} \frac{d^{s-1} z}{dt^{s-1}} + \cdots + M'_1 \frac{dz}{dt} + M'_0 z = f'(t),$$

where $M'_i = PM_iQ$, $z = Q^{-1}x$ and $f'(t) = Pf(t)$. In case we find a solution for the transformed system, we immediately find a solution for the original system.

The aim is therefore, to find adequate matrices P and Q which transform the system in an 'easy to solve' problem. For instance, in case of an homogeneous system of degree $s = 1$ and

$$M_0 = \begin{pmatrix} 3 & -1 & 0 \\ -2 & 2 & 0 \\ 1 & -4 & 1 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & -2 & 1 \end{pmatrix}$$

we may choose $P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ which yields a new system $M'_1 \frac{dz}{dt} + M'_0 z = 0$ where

$$M'_0 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M'_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The new system is readily solved by

$$\begin{aligned} (z_1(t), z_2(t)) &= e^{At}, \qquad \text{where } A = -\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \\ \text{and } z_3(t) &= e^t. \end{aligned}$$

1.2. The Weierstrass-Kronecker solution. Consider the problem (1.1) when $s = 1$. The aim is to find 'normal forms' for pairs of matrices (M_0, M_1), given that two pairs (M_0, M_1) and (M_0', M_1') are equivalent if $M_i' = PM_iQ$ $(i = 0, 1)$ for some invertible matrices P and Q. The solution is as follows.

Given two matrices A and B we define the *direct sum* $A \oplus B$ as $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Given two pairs (A_0, A_1) and (B_0, B_1), the sum $(A_0, A_1) \oplus (B_0, B_1)$ is defined coordinatewise. A pair (A_0, A_1) is *indecomposable* if it is not equivalent to the direct sum of two non-trivial pairs. Clearly, every pair (A_0, A_1) is equivalent to a direct sum of finitely many indecomposable pairs.

Representatives of the equivalence classes of indecomposable pairs are the following:

(a) $(J_n(\lambda), \mathbf{1}_n)$ of size $n \times n$, where $J_n(\lambda)$ is the Jordan block with eigenvalue $\lambda \in \mathbb{C}$.

(b) $(\mathbf{1}_n, J_n(0))$ of size $n \times n$.

(c) For each n, a pair of size $n \times (n+1)$ of the form $(P_0^{(n)}, P_1^{(n)})$ where

$$P_0^{(n)} = \left(\mathbf{1}_n \,\middle|\, 0\right) \qquad \text{and} \qquad P_1^{(n)} = \left(0 \,\middle|\, \mathbf{1}_n\right)$$

(d) For each n, a pair of size $(n+1) \times n$ of the form $(I_0^{(n)}, I_1^{(n)})$ where

$$I_0^{(n)} = \begin{pmatrix} \mathbf{1}_n \\ -- \\ 0 \end{pmatrix} \qquad \text{and} \qquad I_1^{(n)} = \begin{pmatrix} 0 \\ -- \\ \mathbf{1}_n \end{pmatrix}$$

1.3. Trying to solve the system for degree $s > 1$. To fix ideas, consider the case $s = 2$. We shall restrict to the following type of system

$$\frac{d^2x}{dt^2} + M_1 \frac{dx}{dt} + M_0 x = f(t)$$

where M_0 and M_1 are $n \times n$-matrices. Two such systems $(M_0, M_1, \mathbf{1}_n)$ and $(M_0', M_1', \mathbf{1}_n)$ are equivalent if and only if there are invertible $n \times n$-matrices P and Q such that

$$M_0' = PM_0Q, \quad M_1' = PM_1Q \quad \text{and} \quad \mathbf{1}_n = PQ.$$

Therefore, the pairs (M_0, M_1) and (M_0', M_1') are equivalent by simultaneous conjugation.

The problem of finding 'normal forms' for pairs of square matrices where equivalence is defined by simultaneous conjugation is a well-known *open problem* that we will consider in more detail. In this moment we only conclude that there is no hope to solve the systems of differential equations of degree $s > 1$ by giving a set of normal forms.

1.4. Quivers, algebras and representations. Consider the following oriented graph

consisting of two vertices a and b and s arrows $\alpha_1, \dots, \alpha_s$. A *representation* X of Δ_s over the complex numbers $\mathbb{C}$ is given as a couple of finite dimensional vector spaces X_a and X_b (say a dimension m and n respectively) and for each arrow α_i, a linear map $X(\alpha_i)\colon X_a \to X_b$. Fixing basis for X_a and X_b, we may identify X with a tuple $(X(\alpha_1), \dots, X(\alpha_s))$ of s matrices of size $m \times n$.

Two representation X and Y of Δ_s are equivalent if there are linear isomorphisms $P_a\colon X_a \to Y_a$ and $P_b\colon X_b \to Y_b$ such that $Y(\alpha_i) = P_b X(\alpha_i) P_a^{-1}$, for $i = 1, \dots, s$.

In case $s = 2$, the pairs described in (1.2a,b,c,d) form a complete set of representatives of the isomorphism classes of indecomposable representations of Δ_2.

Associated with Δ_s we define an algebra $A_s = \mathbb{C}\Delta_s$ with a basis $\alpha_1, \dots, \alpha_s$, e_a, e_b (hence of dimension $s+2$) and multiplication given by $e_a^2 = e_a$, $e_b^2 = e_b$, $\alpha_i e_a = \alpha_i$, $e_b \alpha_i = \alpha_i$, $i = 1, \dots, s$ and all other products equal to zero. Then $\{e_a, e_b\}$ is a complete set of pairwise orthogonal idempotents. Moreover, observe that the category of representations of Δ_s is equivalent to the category A_s of finitely generated left A_s-modules (indeed, if X is an A_s-module, then $X_a = e_a X$, $X_b = e_b X$ and $X(\alpha_i)\colon X_a \to X_b$, $m \mapsto \alpha_i m$ yields a representation of Δ_s; conversely, if Y is a Δ_s-representation, then $Y_a \oplus Y_b$ gets a structure of A_s-module).

The above notions may be easily generalized. Let Δ be any finite *quiver* (= oriented graph). By $k\Delta$ we denote the *path algebra* whose underlying vector space has as basis all the oriented paths in Δ (including a 'lazy' path e_x for each vertex x); the multiplication of a path γ from a to b and γ' from b' to c is the path $\gamma'\gamma$ if $b = b'$ and 0 otherwise. The importance of path algebras arises from the following observation of Gabriel:

Let A be a finite dimensional algebra over an algebraically closed field k. Then there exists a unique quiver Δ and an ideal I of $k\Delta$ with $I \subset J^2$ where J is the ideal of $k\Delta$ generated by the arrows of Δ, such that the category A is equivalent to $\mathrm{k}\Delta/I$.

A *representation* X of a quiver Δ is a collection $X = ((X_a)_{a\in\Delta_0}, (X(\alpha)\colon X_a \to X_b)_{a\to b})$, where X_a is a finite dimensional k-vector space for each a in the set Δ_0 of vertices of Δ and a linear map $X(\alpha)\colon X_a \to X_b$ for each arrow $a \xrightarrow{\alpha} b$ in Δ. A map $f\colon X \to Y$ between representations is a family of linear maps $f = ((f_a\colon X_a \to Y_a)_{a\in\Delta_0})$ such that $f_b X(\alpha) = Y(\alpha) f_a$ for every arrow $a \xrightarrow{\alpha} b$ in Δ. The *dimension vector* of X is $\underline{\dim}\, X = (\dim_k X_a)_{a\in\Delta_0}$.

A representation X of Δ is said to *satisfy* an ideal I of $k\Delta$ if for every element $\rho = \sum_{i=1}^{t} \lambda_i \alpha_i, \dots, \alpha_{is_i} \in I$, where $\lambda_i \in k$ and α_{ij} is an arrow, the matrix

$$X(\rho) = \sum_{i=1}^{t} \lambda_i X(\alpha_{i1}) \dots X(\alpha_{is_i})$$

is zero.

Finally, as in the case of Δ_s above, it is easy to see that the category of representations of Δ satisfying an ideal I of $k\Delta$ is equivalent to $\mathrm{k}\Delta/I$. In what follows we usually consider algebras A of the form $A = k\Delta/I$ and identify left A-modules with representations satisfying I.

1.5. Tame algebras. An algebra A is said to be *tame* if for each dimension d there is a finite family of $A - k[T]$-bimodules $M_1, \dots, M_s$ which are free as right $k[T]$-modules and such that almost every indecomposable A-module X of dimension d is isomorphic to some module of the form $M_i \otimes_{k[T]} S_\lambda$ for some $1 \leq i \leq s$ and some $\lambda \in k$, where $S_\lambda = k[T]/(T-\lambda)$ is a simple $k[T]$-module.

Examples: (a) Every representation-finite algebra is tame.

(b) The algebra $A_2 = k\Delta_2$ where $\Delta_2 \colon \cdot$ is tame. Indeed, for every $s \in \mathbb{N}$, consider the $A - k[T]$-bimodule M defined by

According to (1.2), every indecomposable A-module X with $\dim_k X_a = s = \dim_k X_b$ is isomorphic to $M \otimes_{k[T]} S_\lambda$ for some $\lambda \in k$.

(c) Let Δ be a connected quiver without oriented cycles. Then $A = k\Delta$ is a finite dimensional hereditary algebra. Gabriel showed that algebras of this form are representation-finite if and only if Δ is of Dynkin type. Dlab and Ringel showed that $k\Delta$ is tame if and only if Δ is of extended Dynkin type, see [7].

1.6. Wild algebras. We denote by $k\langle T_1, T_2\rangle$ the algebra of polynomials in two non-commutative variables T_1 and T_2.

We say that an algebra A is *wild* if there is an $A - k\langle T_1, T_2\rangle$-bimodule M which is free as right $k\langle T_1, T_2\rangle$-module and such that the functor $M \otimes_{k\langle T_1, T_2\rangle} - \colon$ k$\langle T_1, T_2\rangle \to$ A *insets* indecomposables, that is, for every indecomposable $k\langle T_1, T_2\rangle$-module X, the A-module $M \otimes X$ is indecomposable and in case $M \otimes X$ and $M \otimes Y$ are isomorphic, then X and Y are isomorphic.

Examples: (a) The algebra $A_3 = k\Delta_3$ with $\Delta_3 \colon \cdot \quad \cdot$ is wild. Indeed, consider the $A_3 - k\langle T_1, T_2\rangle$-bimodule M defined as follows

It is an easy exercise to check that $M \otimes_{k\langle T_1,T_2\rangle} -$ insets indecomposables.

(b) The five dimensional algebra $k\langle x,y\rangle/(x^2, xy, y^2x, y^3)$ is wild. The proof is left to the reader.

To get a clear idea of the behavior of wild algebras we show the following remark.

Proposition. *Let A be a wild algebra and let B be any finitely generated k-algebra. Then there exists an $A-B$-bimodule N such that the functor $N \otimes_B -: B \to A$ insets indecomposables.*

Proof: Let $b_1, \dots, b_s$ be a set of generators of B. Consider the $k\langle T_1, T_2\rangle - B$-bimodule L given as

then $L \otimes_B -:$ B $\to$ k$\langle T_1, T_2\rangle$ insets indecomposables. Since A is wild, there is an $A - k\langle T_1, T_2\rangle$-bimodule M such that $M \otimes_{k\langle T_1,T_2\rangle} -:$ k$\langle T_1, T_2\rangle \to$ A insets indecomposable. The $A - B$-bimodule $L \otimes_B M$ yields the result.

1.7. Tame and wild dichotomy. In 1980, Drozd proved the following fundamental result.

Theorem. [6] *Every finite dimensional algebra over an algebraically closed field is either tame or wild and not both.*

The known proofs of the Theorem use some rather sophisticated techiniques (bocses, subspace categories). Some insight may be gained by considering some geometric aspects of the modules varieties associated with an algebra. This is the purpose of the following section.

2. Some geometric aspects.

2.1. Module varieties. Let $A = kQ/I$ be a finite dimensional k-algebra and L be a finite set of generators of I with $L \subset \bigcup\limits_{i,j \in Q_0} e_j I e_i$. Let $v \in \mathbb{N}^{Q_0}$ be a dimension vector. The *module variety* $\mathrm{mod}_A(v)$ is the closed subset, with respect to the Zariski topology, of the affine space $k^v = \prod\limits_{i \to j} k^{v(j)v(i)}$ defined by the polynomial equations given by the entries of the matrices

$$m_\rho = \sum_{i=1}^{t} \lambda_i m_{\alpha_{i1}} \dots m_{\alpha_{is_i}}, \text{ where } \rho = \sum_{i=1}^{t} \lambda_i \alpha_{i1} \dots \alpha_{is_i} \in L$$

and for each arrow $x \xrightarrow{\alpha} y$, m_α is the matrix of size $v(y) \times v(x)$

$$m_\alpha = (X_{\alpha_{ij}})_{ij}$$

where $X_{\alpha_{ij}}$ are pairwise different indeterminates. We shall identify points in the variety $\mathrm{mod}_A(v)$ with representations X of A with vector dimension $\underline{\dim} X = v$.

The group $G(v) = \prod\limits_{i \in Q_0} GL_{v(i)}(k)$ acts on k^v by conjugation, that is, for $X \in k^v$, $g \in G(v)$ and $x \xrightarrow{\alpha} y$, then $X^g(\alpha) = g_y X(\alpha) g_x^{-1}$. By restriction of this action, $G(v)$ also acts on $\mathrm{mod}_A(v)$. Moreover, there is a bijection between the isoclasses of A-modules X with $\underline{\dim} X = v$ and the $G(v)$-orbits in $\mathrm{mod}_A(v)$.

Given $X \in \mathrm{mod}_A(v)$, we denote by $G(v)X$ the $G(v)$-orbit of X. Then

$$\dim G(v)X = \dim G(v) - \dim \mathrm{Stab}_{G(v)} X$$

where the stabilizer $\mathrm{Stab}_{G(v)}(X) = \{g \in G(v) \colon X^g = X\} = \mathrm{Aut}_A(X)$ is the group of automorphisms of X. Since $\mathrm{Aut}_A(X)$ is open on the affine variety $\mathrm{End}_A(X)$, we get

$$\dim \mathrm{Stab}_{G(v)} X = \dim \mathrm{Aut}_A(X) = \dim \mathrm{End}_A(X).$$

Observe that an orbit is always irreducible and locally closed.

2.2. Examples. (a) The orbit $G(v)X$ is closed if and only if X is semisimple. For the proof one shows that for an exact sequence $0 \to X' \to X \to X'' \to 0$, the direct sum $X' \oplus X''$ *is a degeneration* of X, that is, $X' \oplus X''$ belongs to the closure $\overline{G(v)X}$ of the orbit of X.

(b) The subset $\mathrm{ind}_A(v)$ of $\mathrm{mod}_A(v)$ formed by the points corresponding to indecomposable A-modules is a constructible subset of $\mathrm{mod}_A(v)$. Indeed, the set of pairs

$$\{(X,f) \colon X \in \mathrm{mod}_A(v),\ f \in \mathrm{End}_A(X) \text{ with } 0 \neq f \neq 1_x \text{ and } f^2 = 1_x\}$$

is a locally closed subset of $\mathrm{mod}_A(v) \times k^{d^2}$, where $d = \sum_{i \in Q_0} v(i)$. The projection $\pi_1 \colon \mathrm{mod}_A(v) \times k^{d^2} \to \mathrm{mod}_A(v)$ is a regular map with image $\mathrm{mod}_A(v) \setminus \mathrm{ind}_A(v)$. Hence $\mathrm{mod}_A(v) \setminus \mathrm{ind}_A(v)$ (and therefore $\mathrm{ind}_A(v)$) is constructible by Chevalley's Theorem.

(c) Let $F = k\langle T_1, \dots, T_m\rangle$ be a free algebra in m indeterminates. Let M be a $A - F$-bimodule which is free as right F-module. Then the functor $M \otimes_F - \colon \mathrm{F} \to \mathrm{A}$ induces a family of regular maps $f_M^n \colon \mathrm{mod}_F(n) \to \mathrm{mod}_A(nv)$ for some vector $v \in \mathbb{N}^{Q_0}$ and every $n \in \mathbb{N}$. Indeed, $v(i) = rk_F M_i$ and for an arrow $i \xrightarrow{\alpha} j$ in Q, $M(\alpha) \colon M_i \to M_j$ is a $v(j) \times v(i)$-matrix with entries in F. Therefore, for an element $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathrm{mod}_F(n)$, each λ_i is a $n \times n$ matrix over k and

$$f_M^n(\lambda) = (M(\alpha)_{st}(\lambda_1, \dots, \lambda_m))_{s,t} \in \mathrm{mod}_A(nv).$$

(d) Suppose $V \subset k^n$ is defined by certain polynomials $f(T_1, \dots, T_n)$ for $x \in V$, define $d_x f = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i}(x)(T_i - x_i)$ the *derivative* of f at x. Then the *tangent space* of V at x is the linear variety $T_x(V)$ in k^n defined by the vanishing of all $d_x f$ as $f(T)$ ranges over the polynomials in the ideal defining V and $x \in V$.

There is an interesting result going back to Voigt, see [13].

Theorem. *Let $X \in \text{mod}_A(v)$. Consider $T_x(G(v)X)$ as a linear subspace of $T_X(\text{mod}_A(v))$. Then there exists a natural linear monomorphism*

$$T_X(\text{mod}_A(v))/T_X(G(v)X) \hookrightarrow \text{Ext}^1_A(X,X).$$

(e) As a consequence of the above we get that a module X with $\text{Ext}^1_A(X,X) = 0$ has an open orbit $G(v)X$.

(f) If $A = kQ$ is a hereditary algebra, then for any $X \in \text{mod}_A(X)$ we get an isomorphism

$$T_X(\text{mod}_A(v))/T_X(G(v)X) \xrightarrow{\sim} \text{Ext}^1_A(X,X).$$

On the other hand, consider the algebra $B = k[T]/(T^2)$ and the simple B-module S. Then $\text{mod}_B(1) = G(1)S$ but $\text{Ext}^1_B(S,S) \neq 0$.

2.3. Geometric characterizations of tameness. The powerful tame and wild dichotomy accepts the following geometric interpretations.

Theorem. [12] *Let $A = kQ/I$ be a finite dimensional k-algebra. The following are equivalent:*

(a) *A is tame*

(b) *For each $v \in \mathbb{N}^{Q_0}$, there is a constructible subset C of $\text{mod}_A(v)$ satisfying $\dim C \leq 1$ and $\text{ind}_A(v) \subset G(v)C$.*

(c) *For each $v \in \mathbb{N}^{Q_0}$, if C is a constructible subset C of $\text{mod}_A(v)$ intersecting each orbit of $G(v)$ in at most one point, then $\dim C \leq 1$.*

(d) *For each $v \in \mathbb{N}^{Q_0}$ and $t \in \mathbb{N}$, the closed subset $\text{mod}_A(v,t) = \{X \in \text{mod}_A(v)$:*
$\dim_k \text{End}_A(X) \geq t\}$ of $\text{mod}_A(v)$ satisfies

$$\dim \text{mod}_A(v,t) \leq \frac{1}{2}|v| + v^2 - t,$$

where $|v| = \sum_{i\in Q_0} v(i)$ *and* $v^2 = \sum_{i \in Q_0} v(i)^2$.

2.4. An algebra is not tame and wild simultaniously. Indeed, if A is a wild algebra, there is a regular injective map $f\colon \operatorname{mod}_{k\langle T_1,T_2\rangle}(1) \to \operatorname{mod}_A(v)$ which preserves indecomposable modules (2.2c). Then $\operatorname{Im} f$ is a constructible set in $\operatorname{ind}_A(v)$ intersecting only once each $G(v)$-orbit and with $\dim \operatorname{Im} f = 2$. By (2.3), A cannot be tame.

2.5. The Tits form. The following criterion is useful.

Proposition. *Let* $A = kQ/I$ *be a tame algebra and* $v \in \mathbb{N}^{Q_0}$. *Then* $\dim \operatorname{mod}_A(v) \leq \dim G(v) = v^2$.

The converse of the above result is not true as may be shown by considering $B = k[T_1, T_2, T_3]/(T_iT_j\colon 1 \leq i \leq j \leq 3)$ which is a wild algebra with

$$\dim \operatorname{mod}_B(n) = \left\{ \begin{array}{ll} n^2 & \text{if } n \text{ even} \\ n^2 - 1 & \text{if } n \text{ odd.} \end{array} \right\} \leq n^2$$

On the other hand, $\dim \operatorname{mod}_B(2n, n^2+1) = 4n^2$ and the characterization (2.3) yields $\dim \operatorname{mod}_A(2n, n^2+1) \leq 3n^2 + 2n - 1$ for any tame local algebra A.

An interesting application of the above Proposition is the following.

Let $A = kQ/I$ and L be a minimal set of generators of I with $L \subset \bigcup_{i,j\in Q_0} e_jIe_i$. Consider the quadratic form $q_A\colon \mathbb{Z}^{Q_0} \to \mathbb{Z}$ given by

$$q_A(v) = \sum_{i \in Q_0} v(i)^2 - \sum_{i \to j} v(i)v(j) + \sum_{i,j\in Q_0} r(i,j)v(i)v(j),$$

where $r(i,j)$ is the cardinality of $L \cap e_jIe_i$. For any algebra and any set L we have for $v \in \mathbb{N}^{Q_0}$,

$$q_A(v) \geq \dim G(v) - \dim \operatorname{mod}_A(v).$$

The quadratic form q_A is called the *Tits form* of A.

Corollary. *Let* $A = kQ/I$ *be a tame algebra, then* q_A *is weakly non-negative, that is,* $q_A(v) \geq 0$ *for any* $v \in \mathbb{N}^{Q_0}$.

The usefullness of the Tits form may be appreciated in the following *examples*

(a) Let $A = k\Delta$ be a hereditary algebra where

Then $q_A(v) = \sum_{i=1}^{6} v(i)^2 - \sum_{i=2}^{6} v(1)v(i)$ and $q_A(w) = -1$. Hence A is not tame (and therefore wild).

(b) Consider an hereditary algebra $A = k\Delta$, then (1.5c) may be reformulated in the following way. The algebra A is representation-finite if and only if q_A is positive. The algebra A is tame if and only if q_A is non-negative. If A is tame not of finite type, then $\operatorname{corank} q_A = 1$. In fact, there is a vector $v_0 \in \mathbb{N}^{\Delta_0}$ such that

$$\{v \in \mathbb{Z}^{\Delta_0} \colon q_A(v) = 0\} = \mathbb{Z}v_0.$$

(c) As a non-hereditary example consider the algebra $A = kQ/I$, where

and ideal I generated by $\beta\alpha$. Then $q_A(v) = \sum_{i=1}^{6} v(i)^2 - \sum_{i=1}^{5} v(i)v(6) + v(3)v(5)$ with $q_A(w) = -1$.

3. Derived categories.

3.1. Definitions. Let A be a finite dimensional k-algebra. A *differential complex* C in A is a family $(C^i, d_C^i)_{i\in\mathbb{Z}}$ formed by A-modules C^i and morphisms $d_C^i\colon C^i \to C^{i+1}$ such that $d_C^{i+1}d_C^i = 0$ for every $i \in \mathbb{Z}$. A *morphism of differential complexes* $f\colon C \to D$ is a family $(f^i)_{i\in\mathbb{Z}}$ of morphisms $f^i\colon C^i \to D^i$ such that $f^{i+1}d_C^i = d_D^i f^i$ for each $i \in \mathbb{Z}$. The morphism $f\colon C \to D$ is null homotopic if there exists a family $(h^i)_{i\in\mathbb{Z}}$ of morphisms $h^i\colon C^i \to D^{i-1}$ such that $f^i = d_D^{i-1}h^i + h^{i+1}d_C^i$ for each $i \in \mathbb{Z}$. More generally, two morphisms $f, g\colon C \to D$ are *homotopic* if $f - g$ is null homotopic. A complex C is bounded if $C^i \neq 0$ only for finitely many $i \in \mathbb{Z}$.

Denote by $K(A)$ (resp. $K^b(A)$) the category whose objects are the differential complexes (resp. bounded differential complexes) and whose space of morphisms are the homotopy classes of morphisms. Given $C \in K^b(A)$, we consider $H^i(C) = \ker d_C^{i+1}/\operatorname{Im} d_C^i$ the *i-th cohomology* of C. A morphism $f\colon C \to D$ induces a family of maps $H^i(f)\colon H^i(C) \to H^i(D)$, for $i \in \mathbb{Z}$. In case $H^i(f)$ is an isomorphism for every $i \in \mathbb{Z}$ we say that f is a *quasi-isomorphism*.

Given $C \in K^b(A)$, consider the category $\mathcal{J}_C$ with objects (X, s) where $X \in K^b(A)$ and $s\colon X \to C$ a quasi-isomorphism. A map $f\colon (X, s) \to (X', s')$ is a morphism $f\colon X \to X'$ such that $s'f = s$. Then the *derived category* $D^b(A)$ of A has as objects the bounded differential complexes and as morphisms from C to D;

$$\operatorname{Hom}_{D^b(A)}(C, D) = \varinjlim_{(X,s)\in\mathcal{J}_C} \operatorname{Hom}_{K^b(A)}(X, D).$$

3.2. Triangulated structure of $D^b(A)$. We observe that we have an automorphism $T\colon K^b(A) \to K^b(A)$ given by $TX = (X^{i+1}, d_X^{i+1})_{i\in\mathbb{Z}}$ for every complex $X = (X^i, d_X^i)_{i\in\mathbb{Z}}$. For each morphism $f\colon D \to E$ in $K^b(A)$ we have the *cone* C_f which is a complex associated to f with $C_f^i = E^{i-1} \oplus D^i$ and

$$d_{C_f}^i = \begin{bmatrix} d_E^{i-1} & (-1)^i f^i \\ 0 & d_D^i \end{bmatrix} : E^{i-1} \oplus D^i \to E^i \oplus D^{i+1}.$$

With the canonical maps α_f and w_f we get an exact sequence

$$C_f \xrightarrow{\alpha_f} D \xrightarrow{f} E \xrightarrow{w_f} TC_f$$

The set $\mathcal{T}$ of all sextuples (C, D, E, α, f, w) with $C, D, E \in K^b(A)$, morphisms $\alpha\colon C \to D$, $f\colon D \to E$ and $w\colon E \to TC$ such that there is an isomorphism $\varphi\colon C \to C_f$ with $\alpha_f \varphi = \alpha$ and $(T\varphi)w = w_f$, is called the set of *triangles*.

The triple $(K^b(A), T, \mathcal{T})$ satisfies the axioms defining a *triangulated category* in the sense of Verdier (see [8]).

Consider the quotient functor $\pi\colon K^b(A) \to D^b(A)$. Then there is an automorphism $\bar{T}\colon D^b(A) \to D^b(A)$ with $\bar{T}\pi = \pi T$. Moreover, let $\bar{\mathcal{T}}$ be the set of sextuples $s = (X, Y, Z, x, y, z)$ in $D^b(A)$ such that there is a triangle t in $\mathcal{T}$ with s isomorphic to $\pi(t)$. Then $(D^b(A), \bar{T}, \bar{\mathcal{T}})$ is a triangulated category. We shall write $X[1]$ for $\bar{T}(X)$.

3.3. The hereditary case. Let A be a finite dimensional k-algebra. For each A-module M we denote by $M[i]$ the complex in $D^b(A)$ with 0 in all degrees except in degree $-i$ where it has M. Hence $\bar{T}M[i] = M[i+1]$.

Lemma. *Let $A = kQ$ be a hereditary algebra. Each indecomposable object $C \in D^b(A)$ is isomorphic to $M[i]$ for some $i \in \mathbb{Z}$ and some indecomposable A-module M. Moreover*

$$\operatorname{Hom}_{D^b(A)}(M[i], N[j]) = \operatorname{Ext}_A^{j-i}(M, N).$$

Proof: Let $C \in K^b(A)$. We show that there is a complex $P(C) \in K^b(A)$ whose modules in each degree are projective and a quasi-isomorphism $\varepsilon\colon P(C) \to C$.

Indeed, assume $C^n \neq 0$ and $C^m = 0$ for all $m > n$. Let $\varepsilon^n\colon P^n \to C^n$ be a projective cover of C^n, $d_p^n = 0$. Consider the following construction:

where the D^i are defined as fibered products, the $\pi_i \colon P^i \to D^i$ are projective covers and all parts commute. Since $\operatorname{gl.dim} A$ is finite, $P(C)$ is a bounded complex and $\varepsilon = (\varepsilon^i)_{i\in\mathbb{Z}} \colon P(C) \to C$ is easily verified to be a quasi-isomorphism.

Let now $P \in D^b(A)$ be an indecomposable complex whose modules in each degree are projective. Assume $P^n \neq 0$ and $P^m = 0$ for $m > n$. Since A is hereditary, $Q = \operatorname{Im} d_p^{n-1}$ is a projective A-module. In case $Q = 0$, then P is isomorphic to $P^n[-n]$. Assume that $Q \neq 0$, then the complex

$$\cdots \longrightarrow 0 \longrightarrow Q \overset{i}{\hookrightarrow} P^n \longrightarrow 0 \longrightarrow \cdots$$

is a direct summand of P, hence isomorphic to P. This last complex is isomorphic to $(\operatorname{coker} i)[-n]$.

The last claim is easy to verify.

3.4 The Grothendieck group of the derived category. Let $A = kQ/I$ be a finite dimensional k-algebra with $\operatorname{gl.dim} A < \infty$. The Grothendieck group $K_0(A)$ is the free abelian group $\mathbb{Z}^{Q_0}$. We consider the *homological bilinear form* given by

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}_A^i(X,Y)$$

where X and Y are A-modules.

The canonical embedding A $\to D^b(A)$ induces an isomorphism $K_0(A) \to K_0(D^b(A))$, whose inverse is given by $[X^\cdot] \mapsto \sum_{n\in\mathbb{Z}} (-1)^n [X^n]$. There is also a bilinear form

$$K_0(D^b(A)) \times K_0(D^b(A)) \to \mathbb{Z}, \quad ([X],[Y]) \mapsto \sum_{n\in\mathbb{Z}} (-1)^n \dim_k \operatorname{Hom}_{D^b(A)}(X, T^n Y).$$

Clearly, the isomorphism $K_0(A) \to K_0(D^b(A))$ is an isometry.

Associated with the homological bilinear form we consider the quadratic form $\chi_A(v) = \langle v, v\rangle_A$ which is called the *Euler form* of A.

Two algebras A and B are said to be *derived equivalent* if the derived categories $D^b(A)$ and $D^b(B)$ are triangle equivalent. If $F \colon D^b(A) \to D^b(B)$ is a triangle equivalence, then there is an induced isometry $f^b\colon K_0(D^b(A)) \to K_0(D^b(B))$. Therefore, we get an isometry $f\colon K_0(A) \to K_0(B)$ with $\chi_{_A} = \chi_{_B} f$. In particular, $\chi_{_A}$ is positive (resp. non-negative) if and only if so is $\chi_{_B}$. In that case $\operatorname{corank} \chi_{_A} = \operatorname{corank} \chi_{_B}$.

Examples: (a) If A is derived equivalent to a tame hereditary algebra kQ, then $\chi_{_A}$ is non-negative with $\operatorname{corank} \chi_{_A} \leq 1$.

Proof: By the above we may suppose that $A = kQ$. Since $\operatorname{gl.dim} A = 1$, then for a module X with $v = \underline{\dim}\, X$,

$$\begin{aligned} \chi_{_A}(v) &= \dim_k \mathrm{Hom}_A(X,X) - \dim_k \mathrm{Ext}^1_A(X,X) \\ &= \sum_{i,j\in Q_0} v(i)v(j)[\dim_k \mathrm{Hom}_A(S_i,S_j) - \dim_k \mathrm{Ext}^1_A(S_i,S_j)] \\ &= \sum_{i\in Q_0} v(i)^2 - \sum_{i\to j} v(i)v(j) = q_A(v), \end{aligned}$$

where $(S_i)_{i\in Q_0}$ is a set of representatives of the isomorphism classes of simple A-modules. As we have seen in (2.5), q_A is non-negative of $\operatorname{corank} q_A \leq 1$. $\square$

(b) The algebra $A = kQ/I$ given by the quiver

and ideal I generated by $\gamma\beta\alpha$ and $\delta\beta\alpha$ (this is denoted by the dotted edges), is not derived equivalente to a representation-finite hereditary algebra.

Indeed, the Euler form of A is

$$\chi_{_A}(v) = \sum_{i=1}^{5} v(i)^2 - v(1)v(2) - \sum_{i=2,4,5} v(3)v(i) + v(1)v(4) + v(1)v(5),$$

as easily follows from the definition. Then $\chi_{_A}(-1,0,1,1,1) = 0$ and therefore $\chi_{_A}$ is not positive.

3.5. Reflections and tilting. Let $A = kQ/I$ be a finite dimensional algebra and x be a source in the quiver Q. Then the indecomposable projective A-module $P_x = Ae_x$ has radical $R = \operatorname{rad} P_x$ with $R_x = 0$. Hence R is a module over the quotient algebra $B = A/(e_x)$. We say that $A = B[R]$ is a *one-point extension* of B by the module R. Dually we define a *one-point coextension* $[M]B$ of an algebra B by a module M.

Consider again $A = B[R]$ with $R = \operatorname{rad} P_x$. The one-point coextension $[R]B$ is called the *reflection* $S_x^- A$ of A at x. As an example, in (3.4b), the vertex 1 is a source in A and $S_1^- A$ is given as kQ' where

Coreflections $S_y^+ A$ of A are defined dually.

A *tilting complex* in $D^b(A)$ is a differential complex T satisfying that $\operatorname{Hom}_{D^b(A)}(T, T[n]) = 0$ for every $n \neq 0$ and that T generates $D^b(A)$, that is, the smallest triangulated full subcategory of $D^b(A)$ containing T is $D^b(A)$ itself.

As an example we remark that for a reflection $S_x^- A$, there is a tilting complex T in $D^b(A)$ such that $\operatorname{End}_{D^b(A)}(T) \cong S_x^- A$. More generally we have

Theorem. [16] *Let A and B be two algebras. There is a triangle equivalence φ: $D^b(A) \to D^b(B)$ if and only if there is a tilting complex T in $D^b(A)$ such that $\operatorname{End}_{D^b(A)}(T) \cong B$.*

3.6. Extending triangle equivalences. The following result is a useful tool in the proof of several theorems considered in the next section.

Theorem. [2] *Let A and B be two algebras and φ: $D^b(A) \to D^b(B)$ be a triangle equivalence. Let M_1 be an A-module such that $\varphi(M_1[0]) = M_2[0]$ for* *some B-module M_2. Then φ may be extended to a triangle equivalence*

$$\bar{\varphi}\colon D^b(A[M_1]) \to D^b(B[M_2]).$$

Sketch of proof: Let $\bar{M}_2$ be the $B[M_2]$-module whose radical is M_2. Then the complex

$$T = \varphi(A[0]) \oplus \bar{M}_2[0]$$

in $D^b(B[M_2])$ is tilting and $\mathrm{End}_{D^b(B[M_2])}(T) \cong A[M_1]$. Then the result follows from (3.5). □

3.7. The repetitive category. Let $A = kQ/I$ be a finite dimensional algebra. The repetitive category $\hat{A}$ of A has object set $\mathrm{Ob}\,\hat{A} = Q_0 \times \mathbb{Z}$ (we shall write $s[i]$ instead of (s,i)); morphism spaces are given by $\hat{A}(r[i], s[i]) = A(r,s) \times \{i\}$,

$$\hat{A}(r[i], s[i-1]) = \mathrm{Hom}_k(A(s,r),k) \times \{i\} \quad \text{and} \quad \hat{A}(r[i], s[j]) = 0$$

if $i \neq j, j+1$. Composition is given in the natural way: for $e^* \in \hat{A}(r[i+1], s[i])$, $f \in \hat{A}(s[i], t[i])$ and $g^* \in \hat{A}(t[i], u[i-1])$ then $fe^* = e^*(f\cdot ?)$ and $g^*f = g^*(? \cdot f)$.

As an example consider $A = kQ/I$ where

and I is generated by $\beta\alpha$. Then $\hat{A}$ is given as kQ'/I' where and I' is generated by the differences of the paths whose extremal vertices are joined by dotted edges.

The importance of the construction of the repetitive category is due to the following result.

Theorem. [8] *Let A be a finite dimensional algebra with* $\operatorname{gl.dim} A < \infty$. *Then $D^b(A)$ is triangle equivalent to $\underline{\operatorname{mod}}\,\hat{A}$ which is the quotient category of A obtained by 'killing' the morphisms factorizing through projective objects.*

4. Algebras whose Euler form is non-negative.

4.1. Hereditary and canonical tame algebras. There are two classes of algebras for which the derived category has been completely described. On one hand, the tame hereditary algebras $A = k\Delta$ for which Δ is of Dynkin or extended Dynkin type. As we have seen, in that case χ_A is either positive or non-negative with $\operatorname{corank} \chi_A = 1$. The other described case is that of tame *canonical algebras*.

Let $n \geq 2$ and $p_1, \dots, p_n$ be natural numbers ≥ 2 and $\lambda_3, \dots, \lambda_n$ be pairwise different elements in $k \setminus \{0, 1\}$. The *canonical algebra* $C = C(p_1, \dots, p_n, \lambda_3, \dots, \lambda_n)$ is given by the quiver

and bounded by relations $\alpha_{1p_1} \dots \alpha_{11} + \lambda_i \alpha_{2p_2} \dots \alpha_{21} = \alpha_{ip_i} \dots \alpha_{i1}$ for $i = 3, \dots, n$.

Theorem. [12] *A canonical algebra $C = C(p_1, \dots, p_n, \lambda_3, \dots, \lambda_n)$ is tame if and only if either C is tilting equivalent to a tame hereditary algebra or*

$(p_1, \ldots, p_n)$ is of the form $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. In the latter case, χ_C is non-negative with $\operatorname{corank} \chi_C = 2$.

4.2. Strongly simply connected algebras. We say that an algebra A is said to be *strongly simply connected* if for every algebra B convex in A, the first Hochschild cohomology $H^1(B)$ vanishes. Equivalently, A is strongly simply connected if and only if every algebra B convex in A is *separated*, that is, $B = kQ_B/I'$ and for every vertex x in Q_B the following condition is satisfied: let $\operatorname{rad} P_x = \bigoplus_{i=1}^{t} M_i$ be a decomposition into indecomposable modules of the B-module $\operatorname{rad} P_x$, then for any $i \neq j$, the support of M_i and M_j are contained in different connected components of $Q_B \setminus \{y\colon$ there is a path from y to $x\}$.

Examples: (a) If $A = kQ_A/I$ is a *tree algebra* (that is, the underlying graph of Q_A is a tree), then A is strongly simply connected.

(b) Let $A = k[S]$ be a *poset algebra* (that is, S is a poset and $\dim_k A(x,y) = 1$ if $x \leq y$ and $\dim_k A(x,y) = 0$ otherwise). Then A is strongly simply connected if and only if A has no crowns. We recall that a *crown* in A is an algebra C, fully contained in A, of the form

and such that the convex closure $\overline{\{a_i, b_i\}}$ of $\{a_i, b_i\}$ intersects $\overline{\{a_{i+1}, b_i\}}$ (resp. $\{a_i, b_{i-1}\}$) in b_i (resp. in a_i), for $i = 1, \ldots, m$ and $a_{m+1} = a_1$, $b_0 = b_m$.

The following results are central in our considerations.

Theorem. *Let A be a strongly simply connected algebra.*

(i) [1,4] *A is derived equivalent to a tame hereditary algebra $k\Delta$ if and only if χ_A is non-negative with* $\operatorname{corank} \chi_A = 1$. *In this case, Δ is of type $\tilde{\mathbb{D}}_n$ $(n \geq 4)$ or $\tilde{\mathbb{E}}_p$ $(p = 6, 7, 8)$.*

(ii) [3] *If Q_A has more than 6 vertices, then A is derived equivalent to a tame canonical algebra if and only if χ_A is non-negative with* $\operatorname{corank} \chi_A = 2$ *and $\chi_A^{-1}(1) \cap \chi_A^{-1}(0)^{\perp} = \emptyset$ (where $V^{\perp} = \{w \in K_0(A) \colon \langle v, w\rangle_A = 0$ for all $v \in V\}$).*

4.3. Derived-tame algebras. Following [11], we say that A is *derived-tame* if A has finite global dimension and the repetitive category $\hat{A}$ is tame. Examples of derived-tame algebras are the following:

(a) By [8], hereditary tame algebras are also derived-tame. By [9], tame canonical algebras are also derived-tame.

(b) If A is derived tame and $D^b(A) \simeq D^b(B)$ is a triangular equivalence, then B is also derived-tame, see [11].

(c) Other examples of derived tame algebras are provided by the poset algebras $P(n, m)$ associated to posets of the form

Observe that $\chi_{P(n,m)}$ is non-negative with $\operatorname{corank} \chi_{P(n,m)} = m$.

Remark: (1) All algebras in the above examples have a non-negative Euler form.

(2) We *conjecture* that a strongly simply connected algebra A whose Euler

form χ_A is non-negative is derived-tame. The results of this section point in this direction.

4.4. Non-negative unit forms. Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be an integral quadratic form of the shape $q(v) = \sum_{i=1}^{n} q_i v(i)^2 + \sum_{i<j} q_{ij} v(i)v(j)$. We say that q is a *unit* (resp. *semi-unit*) form if $q_i = 1$ (resp. $q_i \in \{0,1\}$).

Associated with a semi-unit form we define a *bigraph* G_q with vertices $1,\dots,n$; two vertices $i \neq j$ are joined by $|q_{ij}|$ full edges if $q_{ij} < 0$ and by q_{ij} dotted edges if $q_{ij} \geq 0$; for every vertex i, there are $1 - q_i$ full loops at i. We say that q is *connected* if G_q is connected. The following are elementary facts.

(a) If $A = kQ/I$ is a connected and triangular algebra, then χ_A is a connected unit form.

(b) Given a connected graph Δ formed by full edges and at most one loop at each vertex, there is a semi-unit form q_Δ such that $G_{q_\Delta} = \Delta$. Then q_Δ is positive (resp. non-negative) if and only if Δ is a Dynkin diagram (resp. extended Dynkin diagram).

For Dynkin diagrams we consider the following partial order:

$$\mathbb{A}_m \leq \mathbb{A}_n \leq \mathbb{D}_n \leq \mathbb{D}_p \text{ for } m \leq n \leq p \text{ and}$$

$$\mathbb{D}_p \leq \mathbb{E}_p \leq \mathbb{E}_q \text{ for } 6 \leq p \leq q \leq 8.$$

The following result is relevant in our discussion.

Theorem. [4] *Let $q\colon \mathbb{Z}^n \to \mathbb{Z}$ be a connected, non-negative semi-unit form. Then there exists a $\mathbb{Z}$-invertible linear transformation $T\colon \mathbb{Z}^n \to \mathbb{Z}^n$ such that $qT(x_1,\dots,x_n) = q_\Delta(x_1,\dots,x_{n-c})$, where $c = \operatorname{corank} q$ and $\Delta = \operatorname{Dyn}(q)$ is a Dynkin diagram uniquely determined by q. Moreover, if q' is a connected restriction of q, then $\operatorname{Dyn}(q') \leq \operatorname{Dyn}(q)$.*

As a simple example we consider the algebra A given by

the quiver with indicated relations. The Euler form χ_A is non-negative and corank
$\chi_A = 2$ (in fact $\chi_A^{-1}(0)$ is generated by the indicated vectors). The associated Dynkin graph Dyn (χ_A) is $\mathbb{E}_6$.

4.5. The corank two case. The most general result on derived-tame algebras we know is the following.

Theorem. [5] *Let $A = kQ/I$ be a connected finite dimensional k-algebra such that χ_A is non-negative of* corank 2. *Assume that A is in one of the following classes:*

(1) *Tree algebras;* (2) *strongly simply connected poset algebras.*

Then A is derived equivalent to a tame canonical algebra or to a poset algebra $P(n)$ of the form

Moreover, if A has more than 6 vertices, then A is derived equivalent to a tubular algebra (resp. to $P(n)$) if and only if $Dyn(\chi_A) = \mathbb{E}_p$ ($p = 6, 7$ or 8) (resp. $Dyn(\chi_A) = \mathbb{D}_{n-2}$).

Sketch of proof: (1) There exists an algebra B satisfying the following two conditions:

(i) B is derived equivalent to a tree algebra and χ_B is non-negative with corank $\chi_B = 1$.

(ii) A is derived equivalent to $B[M]$ for some indecomposable B-module M.

The proof of this claim is quite technical but mainly combinatorial.

(2) As we mention in (4.3), B is derived equivalent to a tame algebra $k\Delta$ where Δ is either of type $\tilde{\mathbb{D}}_m$ or $\tilde{E}_p$ ($p = 6, 7$ or 8). Consider a triangle equivalence $\varphi\colon D^b(B) \to D^b(k\Delta)$. Since M is an indecomposable B-module, the object $M[0]$ is indecomposable and $\varphi(M[0])$ is isomorphic to an object $N[i]$ where N is an indecomposable $k\Delta$-module and $i \in \mathbb{Z}$ (3.3). By shifting the functor φ we may assume that $i = 0$. Applying (3.6), we get an equivalence $\bar{\varphi}\colon D^b(B[M]) \to D^b(k\Delta[N])$.

(3) We conclude that A is derived equivalent to $D = k\Delta[N]$ and χ_D is non-negative with corank $\chi_D = 2$. Using the Tits form criterion of section 2, it is not hard to get the result (details of this part may be seen in [10]). □

4.6. More examples. We give some *examples* showing that the results similar to the above can not be expected in the non strongly simply connected case.

(a) Let A be the algebra given by the following quiver with commutativity relations as indicated by dotted lines.

Then χ_A is non-negative with corank $\chi_A = 2$ and $\mathrm{Dyn}(\chi_A) = \mathbb{E}_8$. Moreover, A is wild and hence A cannot be derived tame.

(b) Let A be the algebra given by the following quiver with zero relations as indicated by dotted lines.

Then χ_A is non-negative with corank $\chi_A = 3$ and Dyn $(\chi_A) = \mathbb{E}_6$.

Remarks: (1) We have recently shown that a tree algebra A with χ_A non-negative and containing a convex subcategory B derived equivalent to a Dynkin algebra of type $\mathbb{E}_p$ $(p = 6, 7, 8)$ has corank $\chi_A \leq 2$. Therefore A is derived-tame of one of the cases considered in (4.5).

(2) We conjecture that a tree algebra A with χ_A non-negative and Dyn $(\chi_A) = \mathbb{D}_{n-m}$ is derived equivalent to a poset algebra of the form $P(n, m)$ and therefore also derived-tame. Recent results in this direction have been obtained in joint work with Ch. Geiss [8], where we also obtain a description of the Auslander-Reiten quiver of the derived category $D^b(P(n, m))$.

(3) Finally we remark that all the quadratic form criteria mentioned in this paper has been implemented as computer programs.

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