Matemática Contemporânea, Vol. 16, 227–236
http://doi.org/10.21711/231766361999/rmc1612

SOME RESULTS ON INTEGRAL GROUP RINGS OF FROBENIUS GROUPS

César Polcino Milies *

Abstract

We recall several problems regarding the structure of the group of units of an integral group ring. Then, we give some results regarding these problems in the case when the given group is a Frobenius group.

Resumo

Inicialmente lembramos alguns problemas referentes à estrutura do grupo das unidades de um anel de grupo sobre os inteiros. Depois damos alguns resultados ligados a estas questões, no caso em que o grupo dado é um grupo de Frobenius.

1. Introduction

Let G be a finite group. We denote by $\mathbb{Z}G$ the integral group ring of G; i.e., the set of all formal sums $\alpha = \sum_{g\in G} a(g)g$.

Given two elements $\sum_{g\in G} a(g)g$, $\sum_{g\in G} b(g)g$ in $\mathbb{Z}G$, the sum and product of these elements can be defined, repectively, by:

$$(\sum_{g\in G} a(g)g) + (\sum_{g\in G} b(g)g) = \sum_{g\in G}(a(g)+b(g))g.$$

$$\alpha\beta = \sum_{g,h\in G} a(g)b(h)gh.$$

*The author was partially supported by a research grant from CNPq, Proc. 300243/79-0(RN)
AMS Subject Classifiation: Primary 20C05, 16U60, Secondary 16S34, 20C10, 20C12.
Key words and frases: Group ring, Frobenius group, unit groups.

It is easy to see that, with the operations above, $\mathbb{Z}G$ is a ring. The map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ given by $\varepsilon(\sum_{g\in G} a(g)g) = \sum_{g\in G} a(g)$ is a ring homomorphism called the *augmentation mapping* of $\mathbb{Z}G$ and plays a very important role in the theory. The following is a long-standing question, known as the *isomorphism problem*:

(ISO) Given two groups G and H, is it true that

$$\mathbb{Z}G \cong \mathbb{Z}H \Longrightarrow G \cong H?$$

We recall that an isomorphism $\varphi : \mathbb{Z}G \to \mathbb{Z}H$ is called *normalized* if, for every element $\alpha \in \mathbb{Z}G$ we have that $\varphi(\alpha) = \varepsilon(\varphi(\alpha))$ (or, equivalently, if for every element $g \in G$ we have that $\varepsilon(\varphi(g)) = 1$). It is easy to show that if there exists an isomorphism $\varphi : \mathbb{Z}G \to \mathbb{Z}H$ then, there also exists a normalized isomorphism between these rings.

Let $\varphi : \mathbb{Z}H \to \mathbb{Z}G$ be a normalized isomorphism. Note that if $\varphi(g) \in H$, for every element $g \in G$, then φ itself gives, by restriction, an isomorphism from H onto G. Unfortunately, in geral, we know very little about elements of the form $\varphi(h)$, $h \in H$. Note however that, since $|H| = n$, we have that $h^n = 1$ for every $h \in H$ and, as φ is a morphism, it follows that also $\varphi(h)^n = 1$. This means that $\varphi(h)$, $h \in H$, is always an invertible element, of finite order, in $\mathbb{Z}G$.

Let G be a finite group. We define:

$$\mathcal{U}\mathbb{Z}G = \{\alpha \in \mathbb{Z}G \mid \alpha \text{ is invertible}\},$$

$$\mathcal{U}_1\mathbb{Z}G = \{\alpha \in \mathcal{U}\mathbb{Z}G \mid \varepsilon(\alpha) = 1\}.$$

The first set is called the *group of units* of $\mathbb{Z}G$ and the second, which is a normal subgroup of the first, the *group of normalized units* of $\mathbb{Z}G$.

It is then natural to seek information about $\mathcal{U}_1\mathbb{Z}G$ since an adequate knowledge of this group could help us to find solutions of the isomorphism problem.

At the beginning of the seventies, H.J. Zassenhaus formulated several conjectures about units and also about normalized isomorphisms. We list them below, together with the names by which they are commonly refered in the literature.

- **(Aut)** Let $\theta : \mathbb{Z}G \rightarrow \mathbb{Z}G$ be a normalized automorphism. Then, there exists a unit $\alpha \in \mathbf{Q}G$ and an automorphism $\sigma \in Aut(G)$ such that $\theta(g) = \alpha^{-1}\sigma(g)\alpha$, $\forall g \in G$.

- **(ZC1)** Let $u \in \mathcal{U}\mathbb{Z}G$ be an element of finite order. Then, there exists a unit $\alpha \in \mathbf{Q}G$ such that $\alpha^{-1}u\alpha \in G$ (in this case, we say that u is *rationally conjugate* to an element of G).

- **(ZC2)** Let $\mathcal{H}$ be a finite subgroup of $\mathcal{U}_1\mathbb{Z}G)$ such that $|\mathcal{H}| = |G|$. Then, there exists a unit $\alpha \in \mathbf{Q}G$ such that $\alpha^{-1}\mathcal{H}\alpha = G$.

- **(ZC3)** Let $\mathcal{H}$ be a finite subgroup of $\mathcal{U}_1\mathbb{Z}G)$. Then, there exists a unit $\alpha \in \mathbf{Q}G$ such that $\alpha^{-1}\mathcal{H}\alpha \subset G$.

Note that **(ZC2)** is obviously a particular case of **(ZC3)**. Also **(ZC1)** is just **(ZC3)** in the particular case of cyclic groups. Unfortunatelly, this conjecture is not true, as was shown by K.W. Roggenkamp and L. Scott (see L. Klinger [11]).

Finally, we should mention that recently a weaker version of the conjecture was formulated [4]:

- **(p-ZC)** Let $\mathcal{H}$ be a finite p-subgroup of $\mathcal{U}_1\mathbb{Z}G$. Then, there exists a unit $\alpha \in \mathbf{Q}G$ such that $\alpha^{-1}\mathcal{H}\alpha \subset G$.

Note that this conjecture is somehow similar to Sylow's Theorem.

We give below a list of groups for which **(ZC1)** has been verified.

- S_3 (I. Hughes and K.R. Pearson [9]).

- D_4 (C. Polcino Milies [16]).

- Metacyclic groups of the form $G = < x > \rtimes < y >$ where $o(x) = p$, $o(y) = q$ with p, q different primes (A.K, Bhandari and I.S. Luthar [3]).

- Metacyclic groups of the form $G =< x > \rtimes < y >$ where $(o(x), o(y)) = 1$ (C. Polcino Milies, J. Ritter and S.K. Sehgal [17]).
- S_4 (N.A. Fernandes [6]).
- A_5 (I.S. Luthar and I.B.S. Passi [12]).
- S_5 (I.S. Luthar and P. Trama [13]).
- Groups of the form $G =< x > \rtimes H$ where H is an abelian group such that $(o(x), |H|) = 1$ (I.S. Luthar and P. Trama [14]).

The validity of **(ZC3)** has been verified for the following groups.

- Nilpotent groups (A. Weiss [24]).
- Metacyclic groups of the form $G =< x > \rtimes < y >$ where $(o(x), o(y)) = 1$ (A. Valenti [22]).
- S_5, A_5 and $SL(2,5)$ (M. Dokuchaev, S.O. Juriaans and C. Polcino Milies [5]).

Also **(p-ZC)** has been established for the following families of groups by M. Dokuchaev and S.O. Juriaans:

- Groups that are nilpotent-by-nilpotent.
- Solvable groups G such that every p-Sylow subgroup of G is either abelian or generalized quaternion.
- Solvable groups whose orders do not envolve a forth power of a prime number.

We finish this section quoting a result of M. Dokuchaev and S.O. Juriaans [4] which is a very useful tool when discussing Zassenhaus conjectures.

Lemma 1.1. (Co-prime reduction) *Let N be a normal subgroup of a group G, denote by $w : \mathbb{Z}G \to \mathbb{Z}(G/N)$ the natural homomorphism and let $\mathcal{H}$ be a* *subgroup in* $\mathcal{U}_1\mathbb{Z}G$ *such that* $(|\mathcal{H}|, |N|) = 1$. *Then* $\mathcal{H}$ *is rationally conjugate to a subgroup* H *in* G *if and only if* $w(\mathcal{H})$ *is rationally conjugate to* $w(H)$ *in* $\mathbf{Q}(G/N)$.

2. Frobenius groups

In this section, we shall consider integral group rings of Frobenius groups and give an account of the status of the problems mentioned above, in this context.

A finite group G is called a *Frobenius group* if it contains a proper, nontrivial subgroup H such that $H \cap H^x = 1$ for every element $x \in G \setminus H$.

We shall first recall several important facts about Frobenius groups. For a detailed treatment of the subject, the reader is refered to the book of D.S. Passman [15]. We denote $H^* = H \setminus \{1\}$ and consider the union of all conjugates of H^*. The complement of this set, $N = G \setminus (\cup_{\in G}(H^*)^x)$ is a normal subgroup of G called the *Frobenius kernel* of G. It is easy to see that:

$$G = NH \quad \text{and} \quad N \cap H = 1.$$

Any subgroup X of G such that

$$G = NX \quad \text{and} \quad N \cap X = 1.$$

is called a *Frobenius complement* in G.

G.J. Thompsom proved that the Frobenius kernel N is always a nilpotent group. If a Frobenius group G is not solvable then the Frobenius complements are not solvable, but we have a good deal of information about the structure of G. In fact, if N is its Frobenius kernel and X is a Frobenius complement, we have that:

$$G = N \rtimes X,$$

and

- $|X| \mid (|N| - 1)$.

- The Sylow subgroups of X are either cyclic or generalized quaternion.
- There exists a normal subgroup H of X of the form $H_0 \times SL(2,5)$, where H_0 is a metacyclic normal subgroup of X whose order is relativelly prime to 2, 3 and 5, such that $H = X$ or $[X : H] = 2$.

The case when $[X : H] = 2$ is particularly difficult and it is possible to show that this happens if and only if G has the group S_5 as a homomorphic image. By taking adequate quotients and using co-prime reduction, one can prove the following.

Theorem 2.1. ([5]) *Let G be a Frobenius group. Then*
(i) G satisfies **(p-ZC3)** *if $p > 2$.*
(ii) G satisfies **(2-ZC3)** *if G cannot be mapped homomorphically onto S_5.*

Also a particular case of **(ZC1)** can be proven.

Lemma 2.2. ([10]) *Let G be a Frobenius group with Frobenius kernel N and Frobenius complement X and let $u \in \mathcal{U}_1\mathbb{Z}G$ be a torsion unit of prime order. Then u is rationally conjugate to an element in G.*

Since it has not been possible to establish **(ZC1)** in general, a weaker version of this conjecture is stated as Research Problem 8 in [20] and it is as follows.

Problem 8. *Let G be a group and let $u \in \mathcal{U}_1\mathbb{Z}G$ be a torsion unit. Then there exists an element $g \in G$ such that $o(u) = o(g)$.*

We wish to discuss **Problem 8** in the context of Frobenius groups. The results announced below were obtained in joint work with S.O. Juriaans [10].

Since an element g in a Frobenius group G with Frobenius kernel N and Frobenius complement X is such that either $o(g) \mid |N|$ or $o(g) \mid |X|$, one should expect a similar result to hold in $\mathbb{Z}G$. Actually, we have the following.

Theorem 2.3. *Let G be a finite Frobenius group with Frobenius kernel N and a Frobenius complement X. If $u \in \mathcal{U}_1\mathbb{Z}G$ is a unit of finite order, then $o(u)$ divides either $|N|$ or $|X|$.*

An interesting application of this result gives:

Theorem 2.4. *Let G be a finite Frobenius group and let H be a normalized group basis of $\mathbb{Z}G$. Then H is a Frobenius group.*

One can also prove that **Problem 8** has a positive answer for a large family of Frobenius groups.

Theorem 2.5. *Let G be a group whose Sylow subgroups are all cyclic. Then,* **(ZC3)** *holds in $\mathbb{Z}G$.*

The class above includes all Frobenius groups of odd order.

I.N. Herstein raised, in 1953, the question of classifying finite groups of the multiplicative group of a division ring D and showed that, if $char(D) = p > 0$, then these are cyclic p'-groups. The case when $char(D) = 0$ was solved by S. Amitsur in 1955, so we shall refere to finite groups that can be realized in this way as *Amitsur groups*. Later, M. Shirvani showed that Amitsur groups are Frobenius complements (see [21, 2.1.2]). The classification theorem is as follows.

Theorem 2.6. ([6]) *Let G be a finite group. Then G is a subgroup of a division ring of characteristic 0 if and only if G is isomorphic to one of the following groups.*

(i) A group all of whose Sylow subgroups are cyclic.

(ii) The binary octahedral group of order 48.

(iii) A group of the form $C_m \rtimes \mathcal{Q}_{2^n}$ where C_m is a cyclic group of odd order m,

$\mathcal{Q}_{2^n} = \langle a, b \mid a^{2^{n-2}} = b^2,\ b^4 = 1,\ a^b = a^{-1}\rangle$ *denotes a quaternion group of order* 2^n, *a centralizes* C_m *and b inverts elements of* C_m.

(iv) A group of the form $\mathcal{Q} \times M$, *where* $\mathcal{Q}$ *is the quaternion group of order 8, M is a group of odd order, all of whose Sylow subgroups are cyclic and 2 has odd multiplicative order modulo* $|M|$.

(v) A group of the form $SL(2,3) \times M$, *where M is a group of order coprime to 6, all of whose Sylow subgroups are cyclic and 2 has odd multiplicative order modulo* $|M|$.

(vi) The binary icosahedral group $SL(2,5)$.

In this regard, we have the following.

Theorem 2.7. *Let G be an Amitsur group. If* $\alpha \in U_1(\mathbb{Z}G)$ *is a unit of finite order, then there exists an element* $g \in G$ *such that* $o(\alpha) = o(g)$.

Finally, we return to Frobenius groups.

Theorem 2.8. *Let G be a Frobenius group with Frobenius kernel N and Frobenius complement X. If X is an Amitsur group and* $\alpha \in U_1(\mathbb{Z}G)$ *is a unit of finite order, then there exists an element* $g \in G$ *such that* $o(\alpha) = o(g)$.

Theorem 2.9. *Let G be a Frobenius group of odd order. If* $\alpha \in U_1(\mathbb{Z}G)$ *is a unit of finite order, then there exists an element* $g \in G$ *such that* $o(\alpha) = o(g)$.

References

[1] Amitsur, S.A., *Finite subgroups of division rings*, Trans. Amer. Math. Soc., 80 (1955), 361-386.

[2] Berman, S.D., *On the equation* $X^n = 1$ *in an integral group ring*, Ukrain. Math. Zh., 7 (1955), 253-261.

[3] Bhandari, A.K. and Luthar, I.S., *Torsion units of integral group rings of metacyclic groups*, J. Number Theory, 17 (1983), 170-183.

[4] Dokuchaev, M.A., Juriaans, S.O., *Finite Subgroups in Integral Group Rings*, *Canad. J. Math.*, 48 (6), (1996) 1170-1179.

[5] Dokuchaev, M.A., Juriaans, S.O. and Polcino Milies, C., *Integral Group Rings of Frobenius groups and the Conjectures of H.J. Zassenhaus*, Comm. in Algebra, 25, 7 (1997), 3211-3225.

[6] Fernandes, N., *Torsion Units in the integral group ring of S_4*, Bol. Soc. Bras. Mat., 18, 1(1987), 1-10.

[7] Gonçalves, J. Z., *Free subgroups in subnormal subgroups and the residual nilpotence of the group of units of group rings*, Canad. Math. Bull., 27(1984), 365-370.

[8] Higman, G., *The units of group rings*, Proc. London Math. Soc, 2, 46 (1940), 231-248.

[9] Hughes, I. and Pearson, K.R., *The group of units of the integral group ring $\mathbb{Z}S_3$*, Canad. Math, Bull., 15 (1972), 529-534.

[10] Juriaans, S.O. and Polcino Milies, C., *Units of Integral Group Rings of Frobenius Groups*, preprint.

[11] Klinger, L., *Construction of a counterexample to a conjecture of Zassenhaus*, Commun. Algebra 19 (1993), 2303-2330.

[12] Luthar, I.S., Passi, I.B.S., *Zassenhaus conjecture for A_5*, Proc. Indian Acad. Sci 99 (1) (1989), 1-5.

[13] Luthar, I.S. and Trama, P., *Zassenhaus conjecture for S_5*, (preprint).

[14] Luthar, I.S. and Trama, P., *Zassenhaus conjecture for certain integral group rings*, J. Indian Math. Soc., 55 (1990), 199-212.

[15] Passman, D.S., *Permutation groups*, W.A.Benjamin Inc., New York, 1968.

[16] Polcino Milies, C., *The Group of Units of the Integral Group Ring* ZD_4, Bol. Soc. Brasileira de Mat., 4, 2, (1973) 85 - 92.

[17] C. Polcino Milies, J. Ritter e S.K. Sehgal, *A Conjecture of Zassenhaus on Torsion Units of Integral Group Rings II*, Proc. A.M.S., 97, 3(1986) 201 - 206.

[18] C. Polcino Milies e S.K.Sehgal, *Torsion Units of Integral Group Rings of Metacyclic Groups*, J. of Number Theory, 19, 1(1984) 103 - 114.

[19] Sehgal, S.K. , *Topics in Group Rings*, Marcel Dekker, New York, 19978.

[20] Sehgal, S.K. , *Units in Integral Group Rings*, Longman's, New York, 1996.

[21] Shirvani, M. and Wehfritz, B.A.F. *Skew Linear Groups*, London Math. Soc. Lecture Notes Series 118, Cambridge U. Press, Cambridge, 1986.

[22] Valenti, A., *Torsion Units in Integral Group Rings*, Proc. Amer. Math. Soc. 120 (1) (1994), 1-4.

[23] Whitcomb, A., *The group ring problem*, Ph.D. Thesis, University of Chicago, 1968.

[24] Weiss, A., *Rigidity of p-adic torsion*, Ann. Math. 127 (1988), 317-332.

Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 66281
05315-970 - São Paulo - Brasil
e-mail: polcino@ime.usp.br