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LIE ALGEBRAS AND QUANTUM GROUPS

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Abstract

In classical and in quantum mechanics there are two basic concepts: state and observable. In classical mechanics states are points of a manifold M and observables are functions on M. In quantum mechanics states are 1-dimensional subspaces of a Hilbert space H and observables are operators in H. In both cases observables form an associative algebra that is commutative in the classic case and is noncommutative in the quantum case. So, in some way, we can understand quantization as a replacement of commutative algebras by noncommutative ones. (see [1])

If we consider elements in a group G as states and functions on G as observables, to quantize the notion of group it is needed to translate it first to the language of observables. Let us consider the algebra $A = Fun(G)$ consisting of functions on G (smooth if G is a Lie group, regular if G is an algebraic group, etc.). Then A is a commutative associative unital algebra and $Fun(G \times G) = A \otimes A$ (understanding $\otimes$ in the appropiate sense).

The general principle is: the functor $X \to Fun(X)$ is an antiequivalence from the category of spaces in the category of commutative associative unital algebras (may be with some additional property). So the category of groups is antiequivalent to the category of commutative Hopf algebras.

It is possible to define the category of quantum spaces as dual to the category of (not necessarily commutative) associative unital algebras. Denote by $SpecA$ the quantum space corresponding to an algebra A. **A quantum group is the spectrum of a not necessarily commutative Hopf algebra**. So the notions of quantum group and Hopf algebra are equivalent, but the first one has some geometric flavor.

When we try to find natural examples of noncommutative Hopf algebras universal enveloping algebras of a Lie algebra and relations with Kac-Moody algebras appear.

We will try to give here an introduction to the theory of Lie algebras, Kac-Moody algebras and quantum groups, noticing the relations between those notions.

1. Lie algebras

1.1. Definitions and First Properties

We will consider always algebras over a field F, so an algebra is an F- vector space with a bilinear operation $A \times A \to A$.

A Lie algebra is an algebra $\mathcal{G}$ whose map $[,] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ satisfies:

$(L.1)$ $[x,x] = 0\ \forall x \in \mathcal{G}$ (skewsymmetry)

$(L.2)$ $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ (Jacobi identity).

If char$F \neq 2$, then (L.1) is equivalent to $[x,y] = -[y,x]$ for every pair of elements x, y in $\mathcal{G}$. From now on (unless the contrary is explicitly mentioned) we will assume $F = R$ or $F = C$ (real numbers and complex numbers respectively).

Starting with an associative algebra A we can define a new algebra structure over the same underlying vector space with the new product $[a,b] = ab - ba$, (where juxtaposition denotes the original product in the algebra A). We will denote this new algebra by A^-. It is easy to check that A^- is a Lie algebra.

Examples

1. Let $\dim(V) = l + 1$. We will denote by $sl(V)$ or $sl(l+1, F)$ the set of endomorphisms of V having zero trace. Then $sl(V)$ is a Lie algebra called the **special linear algebra**. Its dimension is $(l+1)^2 - 1$. These algebras are also called algebras of type A and $A_l = sl(l+1, F)$

2. Let $\dim(V) = 2l$ with basis $\{v_1, \ldots, v_{2l}\}$. Define a nondegenerate skew-symmetric form f on V by the matrix $S = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$. Denote the **symplectic algebra** by $C_l = sp(V)$ or $sp(2l, F)$ that, by definition, consists of all endomorphisms φ of V such that $f(\varphi(v), w) = -f(v, \varphi(w))$. In matrix terms, the condition for a $2l \times 2l$-matrix over F, $X = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$, to be symplectic is that $SX = -X^t S$, that is, $n^t = n$, $p^t = p$ and $m^t = -q$. It can easily be checked that $\dim(sp(2l, F) = 2l^2 + l$.

3. Let $\dim(V) = 2l + 1$ be odd and take f the nondegenerate symmetric bilinear form on V whose matrix is $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$. The **orthogonal algebra** $B_l = o(V)$ or $o(2l+1, F)$ consists of all endomorphisms φ of V satisfying $f(\varphi(v), w) = -f(v, \varphi(w))$. Again in matrix terms, if φ is represented by the matrix $X = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$, then the condition $SX = -X^t S$ translates to the following set of conditions: $a = 0,\ c_1 = -b_2^t,\ c_2 = -b_1^t,\ q = -m^t, n^t = -n, p^t = -p$. Again we can easily check that $\dim B_l = 2l^2 + l$.

4. We can obtain another **orthogonal algebra** in an identical way to the one followed for B_l, except that $\dim(V) = 2l$ is even and the matrix of the nondegenerate symmetric form is now $S = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$. This algebra $D_l = o(2l, F)$ has dimension $2l^2 - l$.

The following definitions are the natural ones.

A **subalgebra** $\mathcal{H}$ of $\mathcal{G}$ is a vector subspace $\mathcal{H} \subseteq \mathcal{G}$ that is a Lie algebra with the restriction of the map in $\mathcal{G}$. The subalgebra is called **proper** if $\mathcal{H} \neq \mathcal{G}$.

A subalgebra $\mathcal{H}$ satisfying $[\mathcal{G}, \mathcal{H}] \subseteq \mathcal{H}$ is an **ideal**.

If $\mathcal{H}$ and $\mathcal{K}$ are two ideals of the Lie algebra $\mathcal{G}$, then $\mathcal{H} \cap \mathcal{K}$, $[\mathcal{H}, \mathcal{K}]$ and $\mathcal{H} + \mathcal{K}$ are also ideals of $\mathcal{G}$.

The subalgebra (it is an ideal) $\mathcal{G}' = [\mathcal{G}, \mathcal{G}]$ is called the **derived subalgebra** of $\mathcal{G}$.

The **derived series** of $\mathcal{G}$ is constructed inductively by:

$\mathcal{G}^{(1)} = \mathcal{G}'$, $\mathcal{G}^{(i+1)} = [\mathcal{G}^{(i)}, \mathcal{G}^{(i)}]$ si $i \geq 1$.

The **lower central series** is defined by:

$\mathcal{G}_{(1)} = \mathcal{G}'$, $\mathcal{G}_{(i+1)} = [\mathcal{G}, \mathcal{G}_{(i)}]$ for every $i \geq 1$.

A Lie algebra $\mathcal{G}$ is called **solvable** if the derived series of $\mathcal{G}$ ends up with 0 and is called **nilpotent** if the lower central series of $\mathcal{G}$ ends up with 0.

We will give now, without proofs, some well known properties:

1. If $\mathcal{G}$ is nilpotent, then it is solvable. The converse is not true.

2. The Lie algebra $\mathcal{G}$ is solvable if and only if its derived algebra $\mathcal{G}'$ is nilpotent.

3. $\mathcal{G}$ is nilpotent if and only if there exists a natural number $n = n(\mathcal{G})$ such that $ad(x_1) \cdot ad(x_2) \cdots ad(x_n) = 0$ for every $x_1, \ldots, x_n \in \mathcal{G}$.

4. If $\mathcal{G}$ is solvable (resp. nilpotent) and $\mathcal{H}$ is a subalgebra of G, then $\mathcal{H}$ is solvable (resp. nilpotent).

5. If $\mathcal{H}$ and $\mathcal{K}$ are two solvable (resp. nilpotent) ideals of $\mathcal{G}$, then $\mathcal{H} + \mathcal{K}$ is solvable (resp. nilpotent).

6. If $\mathcal{G}$ is nilpotent, then the **center** of $\mathcal{G}$, $\mathcal{Z}(\mathcal{G}) = \{x \in \mathcal{G} | [x, \mathcal{G}] = 0\}$ (which is always an ideal of $\mathcal{G}$) is non zero.

7. There is a unique maximal solvable ideal of $\mathcal{G}$, namely its radical $\mathcal{G}_{rad}$.

For a given subset $\mathcal{K}$ of $\mathcal{G}$ we define its **centralizer** by $C_{\mathcal{G}}(\mathcal{K}) = \{x \in \mathcal{G} | [x, \mathcal{K}] = 0\}$. $C_{\mathcal{G}}(\mathcal{K})$ is always a subalgebra of $\mathcal{G}$.

A **homomorphism** of Lie algebras is a linear map such that the image of a product of two elements is the product of the respective images. If it is also bijective it is called an **isomorphism**. An isomorphism of a Lie algebra to itself is an **automorphism**.

A **derivation** D of a Lie algebra $\mathcal{G}$ is a linear map that satisfies: $[x, y]D = [xD, y] + [x, yD]$, for every pair of elements in the algebra $\mathcal{G}$. Every element $x \in \mathcal{G}$ defines an (inner) derivation by $ad_x : \mathcal{G} \to \mathcal{G}$, $y \to [y, x]$.

The element x is called **nilpotent** if there exists a natural number n such that $(ad_x)^n = 0$ and it is called **locally nilpotent** if for each element $y \in \mathcal{G}$ there exists $n = n(y)$ such that $(ad_x)^n(y) = 0$.

A Lie algebra $\mathcal{G}$ is called **abelian** if $[\mathcal{G}, \mathcal{G}] = 0$, **simple** if it is non abelian and does not contain proper ideals, **semisimple** if it is a direct sum of simple Lie algebras and **reductive** if it is a direct sum of a semisimple Lie algebra and an abelian Lie algebra.

If $\mathcal{K}$ is an ideal of $\mathcal{G}$, the quotient vector space $\mathcal{G}/\mathcal{K}$ can be given the structure of a Lie algebra by defining the product: $[x + \mathcal{K}, y + \mathcal{K}] = [x, y] + \mathcal{K}$,and it is called the **quotient Lie algebra of $\mathcal{G}$ over $\mathcal{K}$**.

If the Lie algebra $\mathcal{G}$ is not solvable, then there is a semisimple subalgebra $\mathcal{S}$ satisfying $\mathcal{S} \simeq \mathcal{G}/\mathcal{G}_{rad}$ and $\mathcal{G} = \mathcal{S} + \mathcal{G}_{rad}$, (semidirect sum, so that $\mathcal{S} \cap \mathcal{G} = 0$.) (Levi decomposition).

It is known (Harish-Chandra Theorem) that $\mathcal{S}$ is a maximal semisimple subalgebra and if $\mathcal{K} \subseteq \mathcal{G}$ is another semisimple subalgebra of $\mathcal{G}$, there exists an automorphism φ with $\varphi(\mathcal{K}) \subseteq \mathcal{S}$.

By Engel's Theorem it is known that if $\mathcal{G}$ is a finite dimensional Lie algebra whose elements are all ad-nilpotent, i.e., for every $x \in \mathcal{G}$ there is some natural number n such that $ad(x)^n = 0$, then the Lie algebra $\mathcal{G}$ is nilpotent.

Cartan's Theorem ensures that if $\mathcal{G} \leq gl(V)$, V finite dimensional and $Tr(xy) = 0$ for every $x \in [\mathcal{G}, \mathcal{G}]$, $y \in \mathcal{G}$, then $\mathcal{G}$ is solvable. In the general case, if $\mathcal{G}$ is a Lie algebra and $Tr(adxady) = 0$ for every $x \in [\mathcal{G}, \mathcal{G}]$, $y \in \mathcal{G}$, then $\mathcal{G}$ is solvable.

Definition 1.1. *A Cartan subalgebra $\mathcal{G}_0$ of $\mathcal{G}$ is a maximal nilpotent subalgebra.*

A Borel subalgebra is a maximal solvable subalgebra.

If the field F is infinite, Cartan subalgebras always exist. A nilpotent subalgebra $\mathcal{H}$ of $\mathcal{G}$ is a Cartan subalgebra if and only if $\mathcal{H} = N_{\mathcal{G}}(\mathcal{H})$.

If the Lie algebra $\mathcal{G}$ is semisimple, its Cartan subalgebras are precisely the maximal abelian subalgebras.

Definition 1.2. *A gradation of a Lie algebra $\mathcal{G}$ is a decomposition $\mathcal{G} = \oplus_{i \in A} \mathcal{G}_i$ satisfying that $[\mathcal{G}_i, \mathcal{G}_j] \subseteq \mathcal{G}_{i+j}$, where A is an abelian group.*

Example 1.1. 1. Let $\mathcal{G} = F(u, v, w)$ be the 3-dimensional Lie algebra with the product: $[u, v] = w$, $[u, w] = [v, w] = 0$. The algebra $\mathcal{G}$ is solvable and nilpotent.

2. The 2-dimensional algebra $\mathcal{G} = F(u, v)$ with $[u, v] = v$ is solvable but is not nilpotent.

3. The 3-dimensional algebra $\mathcal{G} = F(e, f, h)$ with $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$ is simple.

1.2. Representations and modules

Let V be an F- vector space and let us consider $gl(V) = \{\varphi : V \to V |\ \varphi \textit{ is linear}\}$ Then $gl(V)$ is a Lie algebra with the (associative) commutator product of linear mappings, that is, $[\varphi, \psi] = \varphi \cdot \psi - \psi \cdot \varphi$ and it is called the general linear algebra. There is an isomorphism of Lie algebras between $gl(V)$ and the algebra of $n \times n$ matrices over the field F. The isomorphism maps a linear map to the coordinate matrix with respect to a fixed basis.

Definition 1.3. *A representation R of the Lie algebra $\mathcal{G}$ is a homomorphism from $\mathcal{G}$ to $gl(V)$ for some vector space V, i.e.*

$$R : \mathcal{G} \longrightarrow gl(V),\ [x, y] \to R([x, y]) = R(x)R(y) - R(y)R(x)$$

Let us notice that using the above mentioned isomorphism between $gl(V)$ and the $n \times n$ matrices we can consider in a similar way matrix representations.

Example 1.2. 1.Let us consider $\mathcal{G}$ to be the algebra in example 1.1. Then $\mathcal{G}$ has a representation given by:

$$R(u) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R(v) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R(w) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2. The 2-dimensional solvable non nilpotent algebra in example 1.1 has the representation

$$R(u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$R(v) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

3.The simple 3-dimensional algebra in example 1.1 ($sl(2)$) has the representation

$$R(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$R(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$R(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4. The adjoint representation $R_{ad} : \mathcal{G} \longrightarrow gl(\mathcal{G})$, $x \to ad_x$ satisfies $\mathrm{Im}R_{ad} \leq Der(\mathcal{G})$

Definition 1.4. *A $\mathcal{G}$-module is a vector space V together with an action $\bullet : \mathcal{G} \times V \to V$ that satisfies:*

$(\xi x + \mu y) \bullet w = \xi(x \bullet w) + \mu(y \bullet w)$,

$x \bullet (\xi v + \mu w) = \xi(x \bullet v) + \mu(x \bullet w)$,

$[x, y] \bullet w = x \bullet (y \bullet w) - y \bullet (x \bullet w)$ *for arbitrary* $\xi, \mu \in F$, $v, w \in V$, *and* $x, y \in \mathcal{G}$.

The notion of submodule can be defined in the usual way. We can think equivalently in terms of representations or in terms of modules.

An **irreducible** module of a Lie algebra is a module that does not contain proper submodules, that is, distinct from 0 and the proper module.

A **completely reducible** module of a Lie algebra is a module that is a direct sum of irreducible modules.

The adjoint representation of $\mathcal{G}$ is irreducible (resp. completely reducible) if and only if $\mathcal{G}$ is simple (resp. semisimple).

Weyl's theorem ensures that if $\mathcal{G}$ is semisimple and $\phi : \mathcal{G} \to gl(V)$ is a finite dimensional representation, then ϕ is completely reducible.

Definition 1.5. *The Kronecker product $V_R \times V_S$ of two $\mathcal{G}$-modules V_R and V_S is the vector space $V_R \otimes_F V_S$ with the action of $\mathcal{G}$ defined by:* $(R \times S)(x)(v \otimes w) =$

$(R(x)v) \otimes w + v \otimes (S(x)w)$ *for arbitrary elements* $x \in \mathcal{G}$, $v \in V_R$ *and* $w \in V_S$.

The Kronecker product is associative and if $\mathcal{G}$ is semisimple, the Kronecker product of irreducible finite dimensional modules is completely reducible.

Remark 1.1. For an arbitrary algebra we can define in the same way the representation concept, but not the notion of tensor product. The necessary restrictions an algebra must obey in order for the definition of a tensor product to make sense lead to the concept of Hopf algebra.

Definition 1.6. *The universal enveloping algebra of a Lie algebra* $\mathcal{G}$ *is a pair* $(U(\mathcal{G}), i)$, *where* $U(\mathcal{G})$ *is an associative algebra,* i *is a homomorphism of* $\mathcal{G}$ *in* $U(\mathcal{G})^-$ *such that given an arbitrary associative algebra* A *and a homomorphism* $\theta : \mathcal{G} \to A^-$, *there exists a unique homomorphism of associative algebras* $\theta' : U(\mathcal{G}) \to A$ *such that* $\theta = i\theta'$.

One of the main properties of $U(\mathcal{G})$ is that it lets us reduce the representation theory of $\mathcal{G}$ to the representation theory of (the associative algebra) $U(\mathcal{G})$. Notice that $\mathcal{G} \subseteq U(\mathcal{G})^-$.

Let us see a construction of the universal enveloping algebra. Let us denote by $\mathcal{T}(\mathcal{G})$ the tensor algebra over the vector space $\mathcal{G}$. By definition

$$\mathcal{T} = F1 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_i \oplus \cdots$$

where $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_i = \mathcal{G} \otimes \mathcal{G} \otimes \cdots \mathcal{G}$, i times.

The operations of sum and product in the vector space $\mathcal{T}(\mathcal{G})$ are the usual ones:

$$(x_1 \otimes \cdots \otimes x_i) \otimes (y_1 \otimes \cdots \otimes y_j) = x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j.$$

Let $\mathcal{R}$ be the ideal of $\mathcal{T}$ generated by all elements of the form $[a, b] - a \otimes b + b \otimes a$, $a, b \in \mathcal{G}$. Then $U(\mathcal{G}) = \mathcal{T}/\mathcal{R}$

That is, the universal enveloping algebra of $\mathcal{G}$ is the associative algebra spanned by all monomials in elements of a basis of $\mathcal{G}$, under identification of those products that are equal by using the Lie relations.

The Poincaré-Birkoff-Witt theorem ensures that if $\pi : \mathcal{T}(\mathcal{G}) \to \mathcal{U}(\mathcal{G})$ then its restriction to $\mathcal{G}$ is injective.

1.3. Real and Complex Lie Algebras. Killing form. Cartan-Weyl bases

If V is a vector space an endomorphism $\varphi \in End(V)$ is **semisimple** if the roots of its minimal polynomial over F are all distinct. If the field F is algebraically closed this is equivalent to saying that φ is diagonalizable. Let V be a finite dimensional vector space on F and φ an endomorphism of V. Then there exist unique φ_s and φ_n endomorphisms of V, semisimple and nilpotent respectively, such that $\varphi = \varphi_s + \varphi_n$ (Jordan decomposition).

If $\mathcal{G}$ is a semisimple Lie algebra all its derivations are inner and every representation is completely reducible. If $\mathcal{G} \subseteq gl(V)$ then $\mathcal{G}$ contains the semisimple and nilpotent parts of all of its elements. In the general case, $\mathcal{G}$ a semisimple abstract Lie algebra, we can consider the above decomposition for $adx = (adx)_s + (adx)_n$ in the semisimple and nilpotent part. Then there are elements s and n such that $(adx)_s = ads$ and $(adx)_n = adn$ respectively and $x = s + n$ (abstract Jordan decomposition).

Let $\mathcal{G}$ be a semisimple Lie algebra and $\phi : \mathcal{G} \to gl(V)$ a finite dimensional representation. Then if $x = s + n$ is the abstract Jordan decomposition of an element $x \in \mathcal{G}$, $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$.

Let us have a look to the representation theory of $A_1 = sl(2) = F(e, f, h)$ (see example 1.1(3)) that will help to understand the general situation. In example 1.2 (3) we have seen one representation of $sl(2)$.

If V is an A_1-module, since the element h is semisimple h acts diagonally on V. This yields a decomposition of V as direct sum of eigenspaces $V_\lambda = \{v \in V | hv = \lambda v\}$, $\lambda \in F$. When $V_\lambda \neq 0$, that is, λ is an eigenvalue of the endomorphism of V that represents h, λ is called a **weight** of h in V and V_λ a **weight space**. If $v \in V_\lambda$, then $ev \in V_{\lambda+2}$ and $fv \in V_{\lambda-2}$. Since V is finite dimensional, there must exist $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$. This proves that every nonzero vector in V_λ is annihilated by e and is called a **maximal vector** of weight λ.

Theorem 1.1. *Let V be an irreducible module over $\mathcal{G} = sl(2, F)$. Then V is a direct sum of weight spaces V_μ, $\mu = m, m-2, \ldots, -m$, $dim V_\mu = 1\ \forall \mu$. There exists at most one irreducible $\mathcal{G}$-module (up to isomorphism) of each dimension $m+1$, $m \geq 0$*

Definition 1.7. *The* **Killing form** *of the Lie algebra $\mathcal{G}$ is the map $\kappa : \mathcal{G} \times \mathcal{G} \to F$, such that $\kappa(x, y) = tr(ad_x \cdot ad_y)$.*

The Killing form is bilinear, symmetric and associative, i.e., $\kappa([x, y], z) = \kappa(x, [y, z])$ and it is preserved by automorphisms, i.e., $\kappa(\sigma(x), \sigma(y)) = \kappa(x, y)$ for every automorphism σ of $\mathcal{G}$.

The algebra $\mathcal{G}$ is solvable if and only if $\kappa(x, x) = 0$ for every $x \in \mathcal{G}'$.

The algebra $\mathcal{G}$ is semisimple if and only if the Killing form is non-degenerate, i.e., $\kappa(x, x) = 0$ if and only if $x = 0$.

Properties

1. The radical of $\mathcal{G}$ is the orthogonal complement with respect to κ of the derived algebra $\mathcal{G}'$.

2. If $\mathcal{G} = \mathcal{S} \oplus \mathcal{H}$ is a Levi decomposition of $\mathcal{G}$ into its radical $\mathcal{H}$ and its semisimple part $\mathcal{S}$ and $\mathcal{K}$ is the maximal nilpotent subalgebra of $\mathcal{H}$, then κ is non-degenerate on $\mathcal{S}$ and on $\mathcal{H}$ mod $\mathcal{K}$ and is identically zero on $\mathcal{K}$. In particular, if $\mathcal{G}$ is nilpotent, then $\kappa \equiv 0$. Conversely,if $\kappa \equiv 0$, then $\mathcal{G}$ is solvable.

3. If $\mathcal{H}$ is an ideal of $\mathcal{G}$, then $\kappa|_{\mathcal{H}} = \kappa_{\mathcal{H}}$.

4. The ideals in a direct sum $\mathcal{K} \oplus \mathcal{H}$ are mutually orthogonal. As a consequence, the Killing form of a semisimple Lie algebra is already determined by the Killing forms of its simple ideals.

If a Lie algebra is not nilpotent it has non zero subalgebras that consist of ad-semisimple elements. Such algebras are called toral algebras.

Let $\mathcal{G}$ be a semisimple Lie algebra. We can consider a maximal toral algebra, $H = \mathcal{G}_0 = F(h_i)$, $[h_i, h_j] = 0$. Since $\mathcal{G}$ is semisimple, the algebra $H = \mathcal{G}_0$ is abelian and a Cartan subalgebra. Hence $ad_{\mathcal{G}}H$ is simultaneously diagonalizable and thus we get a decomposition of $\mathcal{G}$, $\mathcal{G} = H \oplus_\alpha \mathcal{G}^\alpha$, where $\mathcal{G}^\alpha = \{x \in \mathcal{G}|\ ad_h(x) = \alpha(h)x\ \forall h \in H = \mathcal{G}_0\}$. (root space decomposition).

That is, for a given Cartan subalgebra $\mathcal{G}_0$ the remaining elements e_α of $\mathcal{G}$ in a basis of $\mathcal{G}$ can be chosen such that they are eigenvectors of $\mathcal{G}_0$: $[h_i, e_\alpha] = \alpha_i e_\alpha$.

We will denote by Φ the set of roots, that is, $\Phi = \{\alpha | \mathcal{G}^\alpha \neq 0\}$. So Φ is a subset of H^*, the dual space of H.

Definition 1.8. *A basis of the semisimple Lie algebra $\mathcal{G}$ of the form $\mathcal{B} = \{h_i | i = 1, \ldots, l\} \cup \{e_\alpha | \alpha \in \Phi\}$ is called a Cartan-Weyl basis of $\mathcal{G}$.*

If $\alpha, \beta \in H^*$, then $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$. For every $x \in L_\alpha$, $\alpha \neq 0$, adx is nilpotent and if we consider $\alpha, \beta \in H^*$ such that $\alpha + \beta \neq 0$, then L_α is orthogonal to L_β with respect to the Killing form κ. The restriction of κ to $\mathcal{G}_0 = H = C_{\mathcal{G}}(H)$ is non-degenerate. So we can identify H and H^*: For every $\phi \in H^*$ there is a unique $t_\phi \in H$ such that $\phi(h) = \kappa(t_\phi, h)$, $\forall h \in H$. In this way we can identify Φ with the subset of H that consists of t_α, $\alpha \in \Phi$.

Orthogonality properties

1. Φ spans H^*.

2. If $\alpha \in \Phi$, then $-\alpha \in \Phi$.

3. If $\alpha \in \Phi$, $x \in \mathcal{G}_\alpha$, $y \in \mathcal{G}_{-\alpha}$, then $[x, y] = \kappa(x, y)t_\alpha$. Hence $[\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$ is 1-dimensional with basis t_α.

4. For every $\alpha \in \Phi$, $\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$.

5. If $\alpha \in \Phi$ and x_α is any nonzero element of $\mathcal{G}_\alpha$, then there exists $y_\alpha \in \mathcal{G}_{-\alpha}$

such that $< x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha] >$ is a subalgebra of $\mathcal{G}$ isomorphic to $sl(2,F)$.

6. $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$; $h_\alpha = -h_{-\alpha}$.

Integrability properties

1. If $\alpha \in \Phi$, $\dim\mathcal{G}_\alpha = 1$ and if $H_\alpha = [L_\alpha, L_{-\alpha}]$, then $S_\alpha = L_\alpha + L_{-\alpha} + H_\alpha \simeq sl(2,F)$.
2. If $\alpha \in \Phi$ the only scalar multiples of α that are in Φ are α and $-\alpha$.
3. If $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) \in Z$ (Cartan integers) and $\beta - \beta(h_\alpha)\alpha \in \Phi$.
4. If $\alpha, \beta, \alpha+\beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.
5. If $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$, then there are r, q the largest integers for which $\beta - r\alpha$, $\beta + q\alpha$ are roots. Then $\beta + i\alpha \in \Phi$ $(-r \leq i \leq q)$ and $\beta(h_\alpha) = r - q$.
6. $\mathcal{G}$ is generated (as a Lie algebra) by the root spaces $\mathcal{G}_\alpha$.

Rationality properties

Since Φ spans H^*, we can choose a basis in H^* that consists of roots of Φ. Let us denote such a basis by $\{\alpha_1, \ldots, \alpha_n\}$. In particular, if $\beta \in \Phi$, $\beta = \sum_{i=1}^{l} c_i\alpha_i$, $c_i \in F$. It can be proved that the coefficients c_i belong to the rational field Q.

Let E_Q be the Q-subspace of H^* spaned by the roots. Then $dim_Q E_Q = l = dim_F H^* = dim_F H$. The Killing form on E_Q is positive definite.

Let $E = R \otimes_Q E_Q$ be the real vector space obtained by extending the base field from Q to R. Then E is a euclidean space and Φ contains a basis of E, so $dim_R E = l$.

In this way Φ is a root system of the euclidean space E, so that theory can be applied. The existence of a **base** $\Delta \subseteq \Phi$ can be proved. Δ is a basis of E and every root β can be written as $\beta = \sum k_\alpha \alpha$, $\alpha \in \Delta$ with integral coefficients k_α which are all nonnegative (Φ_+ will denote the set of roots with nonnegative coefficients, **positive roots**) or all nonpositive (Φ_- denotes the set of such roots, **negative roots**).

By the above mentioned properties of roots, we have $\Phi_- = -\Phi_+$ and $\Phi = \Phi_+ \cup \Phi_-$.

If $\mathcal{G}_+ = F(e_\alpha | \alpha \in \Phi_+)$, $\mathcal{G}_- = F(e_\alpha | \alpha \in \Phi_-)$ then $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_0 \oplus \mathcal{G}_-$ is a Cartan decomposition of $\mathcal{G}$ and $\mathcal{B}_\pm = \mathcal{G}_0 \oplus \mathcal{G}_\pm$ are Borel subalgebras of $\mathcal{G}$.

If we fix an order on the r elements of Δ, we can define the associated Cartan matrix.

Definition 1.9. *The Cartan matrix A of the root system Φ is the $l \times l$ matrix with elements $\alpha_{ij} =< \alpha_i, \alpha_j >= 2\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$.*

The elements $< \alpha_i, \alpha_j >$ are integer numbers (*Cartan integers*).

If $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ (dual root or coroot of α) we can writte $< \alpha_i, \alpha_j >= (\alpha_i, \alpha_j^\vee)$.

The Cartan matrix of Φ determines Φ up to isomorphism.

If α, β are positive roots, then $< \alpha, \beta >< \beta, \alpha >= 0, 1, 2$ or 3.

A root system Φ is called **irreducible** if it can not be expressed as the union of two proper subsets such that each root in one set is orthogonal to each root in the other. If Δ is a base of Φ, the same is true for Δ.

When the root system Φ is irreducible, not more than two root lengths can occur in Φ, so we can speak of short roots and long roots.

The **Coxeter graph** of Φ is defined as the graph with l vertices such that a vertex i is joined to a vertex j by $< \alpha_i, \alpha_j >< \alpha_j, \alpha_i >$ edges. The Coxeter graph determines the numbers $< \alpha_i, \alpha_j >$ in case all roots have equal length.

When a double or triple edge occurs in the Coxeter graph of Φ, we can add an arrow pointing to the shorter of the two roots. The resulting graph is called the **Dynkin diagram** of Φ.

It is clear that Φ is irreducible if and only if its Coxeter graph is connected.

In general Φ decomposes (uniquely) as the union of irreducible root systems Φ_i.

1.4. Clasification of simple Lie complex algebras

Let $\mathcal{G}$ be a semisimple complex Lie algebra, H a Cartan subalgebra and $\Phi \subset H^*$ the set of roots of $\mathcal{G}$ relative to H. If $\mathcal{G}$ is simple, then Φ is irreducible.

If $\mathcal{G}$ and $\bar{\mathcal{G}}$ are semisimple complex Lie algebras with repective Cartan subalgebras $\mathcal{G}_0$ and $\bar{\mathcal{G}}_0$ and root systems Φ and $\bar{\Phi}$, any isomorphism $\varphi : \mathcal{G}_0 \to \bar{\mathcal{G}}_0$ which induces a bijection of Φ onto $\bar{\Phi}$ can be extended to an isomorphism of Lie algebras of $\mathcal{G}$ onto $\bar{\mathcal{G}}$.

Hence, in order to classify simple Lie algebras up to isomorphism, we only need to classify root systems and the associated Cartan matrices.

The problem of characterizing semisimple Lie algebras by their root systems can be reduced to the problem of characterizing simple ones by their (irreducible) root systems. We can sumarize those results in the following Proposition and Theorem.

Proposition 1.1. *Let $\mathcal{G}$ be a semisimple Lie algebra with Cartan subalgebra H and root system Φ. If $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_t$ is the decomposition of $\mathcal{G}$ into simple ideals, then $H_i = H \cap \mathcal{G}_i$ is a Cartan subalgebra of $\mathcal{G}_i$ and the corresponding (irreducible) root system Φ_i may be regarded canonically as a subsystem of Φ in such a way that $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$ is the decomposition of Φ into its irreducible components.*

Theorem 1.2. *The semisimple Lie algebra $\mathcal{G}(\Phi_s)$ associated to a set $\Phi_s = \{\alpha_i | i = 1, \ldots, l\}$ of simple roots α_i is uniquely determined as follows:*

*i) There exist $3l$ generators $\{e_i^{\pm}, h_i | i = 1, \ldots, r\}$ such that $[h_i, h_j] = 0, [h_i, e_j^{\pm}] = \pm\alpha_{ij} e_j^{\pm}$, $[e_i^+, e_j^-] = \delta_{ij} h_i$. (**standard system of generators**)*

For every i, the subalgebra $< e_i^+, e_i^-, h_i >$ is isomorphic to $sl(2)$.

ii) These generators obey the Jacobi identity.

iii) $(ad_{e_i^{\pm}})^{1-\alpha_{ij}} e_j^{\pm} = 0$ if $i, j = 1, \ldots, r$, $i \neq j$.

If $A = (< \alpha_i, \alpha_j >)$ ($< \alpha_i, \alpha_j >= \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$) is the Cartan matrix associated to Φ_s, it is known that $< \alpha_i, \alpha_i >= 2$, $< \alpha_i, \alpha_j >= 0$ if and only if $< \alpha_j, \alpha_i >= 0$, if $i \neq j$ the elements $< \alpha_i, \alpha_i >$ are nonpositive integers and $\det A > 0$.

We can classify simple Lie algebras by using the associated Dynkin diagram. Remember that each vertex corresponds to a simple root and vertices associated to roots α_i and α_j are connected by $\max\{| < \alpha_i, \alpha_j > |, | < \alpha_j, \alpha_i > |\}$ edges.

In this way four families of simple complex Lie algebras are obtained: $A_r,\ r \geq 1$, $B_r,\ r \geq 2$, $C_r,\ r \geq 3$, $D_r,\ r \geq 4$ and also the five algebras E_6, E_7, E_8, F_4 and G_2. (See the corresponding Dynkin diagrams at the end of this survey).

2. Kac-Moody algebras

In physics, Lie algebras arise in the description of symmetries. Since many interesting systems possess infinitely many independent symmetries, infinite-dimensional Lie algebras are as important in physics as finite-dimensional ones.

At the present there is no general theory of infinite-dimensional Lie algebras and their representations. Some classes such as Lie algebras of vector fields, Lie algebras of operators in a Hilbert or Banach space or Kac-Moody algebras have been more intensively studied.

The Kac-Moody algebras have attracted a lot of attention in both mathematics and physics. Mathematically they are of interest because a large subclass of them (the affine Lie algebras) can be classified completely and the classification theory is very similar to the one of finite dimensional simple Lie algebras. Affine Lie algebras also possess an interesting representation theory that again reproduces the situation of finite dimensional simple Lie algebras. They have connections to other branches of mathematics such as number theory, topology, singularity theory or the theory of finite simple groups. In physics, Kac-Moody algebras have important applications in two-dimensional conformal field theory and in the theory of completely integrable systems.

As we have seen in the previous section, a finite dimensional simple Lie algebra is completely characterized by $3l$ generators $\{e_i, f_i, h_i | i = 1, \ldots, l\}$ obeying the Jacobi identity and the relations $[h_i, h_j] = 0$, $[h_i, e_j] = \alpha_{ij} e_j$, $[h_i, f_j] = -\alpha_{ij} f_j$, $[e_i, f_j] = \delta_{ij} h_j$ and $(ad_{e_i})^{1-\alpha_{ij}} e_j = 0$, $(ad_{f_i})^{1-\alpha_{ij}} f_j = 0$ if

$i \neq j$.

In other words, the simple Lie algebras are obtained by requiring that the $l \times l$ matrix $A = (\alpha_{ij})$ is a Cartan matrix, that is, its elements satisfy the following conditions: $\alpha_{ii} = 2$, $\alpha_{ij} \leq 0$ if $i \neq j$, $\alpha_{ij} = 0$ if and only if $\alpha_{ji} = 0$, the elements α_{ij} are integer and $det(A) > 0$.

The Kac-Moody algebras are obtained by weakening the conditions on the matrix A. When the condition $det(A) > 0$ is dropped, the class of Kac-Moody algebras is obtained.

In a more precise way, we have

Definition 2.1. *Let $A = (\alpha_{ij})$ be a* **generalized Cartan matrix***, that is, an integral $l \times l$ matrix such that $\alpha_{ii} = 2$, $\alpha_{ij} \leq 0$ if $i \neq j$ and $\alpha_{ij} = 0$ if and only if $\alpha_{ji} = 0$. The associated* **Kac-Moody algebra** *$\mathcal{G}'(A)$ is a complex Lie algebra on $3l$ generators $\{e_i, f_i, h_i | i = 1, \dots, l\}$ and the following defining relations:*

$$[h_i, h_j] = 0, \quad [e_i, f_i] = h_i, \quad [e_i, f_j] = 0 \text{ if } i \neq j$$

$$[h_i, e_j] = \alpha_{ij} e_j, \quad [h_i, f_j] = -\alpha_{ij} f_j,$$

$$(ad_{e_i})^{1-\alpha_{ij}} e_j = 0, \quad (ad_{f_i})^{1-\alpha_{ij}} f_j = 0 \text{ if } i \neq j.$$

The class of Kac-Moody algebras breaks up into three subclasses. We will assume that the matrix A is **indecomposable**, i.e., there is no partition of the set $\{1, \dots, l\}$ into two nonempty subsets so that $\alpha_{ij} = 0$ whenever i belongs to the first subset and j to the second. We do not lose generality in doing that since the direct sum of matrices corresponds to the direct sum of Kac-Moody algebras. Then the following three mutually exclusive possibilities appear:

a) There is a vector θ of positive integers such that the coordinates of the vector $A\theta$ are positive. In such case all the principal minors of the matrix A are positive and the Lie algebra $\mathcal{G}'(A)$ is finite-dimensional. The generalized Cartan matrix A is said to be of finite type.

b) There is a vector δ of positive integers such that $A\delta = 0$. In such case all the principal minors of the matrix A are nonnegative and $detA = 0$. Now the algebra $\mathcal{G}'(A)$ is infinite-dimensional (it has finite growth, that is, it admits a Z-gradation by subspaces whose dimensions are uniformly bounded). The Lie algebra $\mathcal{G}'(A)$ is called an **affine Lie algebra** and the Cartan matrix A is called of affine type.

c) There is a vector α of positive integers such that all the coordinates of the vector $A\theta$ are negative. In this case the Lie algebra $\mathcal{G}'(A)$ and the Cartan matrix A are called of indefined type.

To classify generalized affine matrices of affine type (the ones of finite type correspond to the simple finite dimensional algebras) Dynkin diagrams are considered. Let $A = (\alpha_{ij})$ be the $l \times l$ generalized Cartan matrix. The associated Dynkin diagram is the graph with l vertices such that vertices i, j are connected by $|\alpha_{ij}|$ if $\alpha_{ij}\alpha_{ji} \leq 4$ and $|\alpha_{ij}| \geq |\alpha_{ji}|$ and these lines are equipped with an arrow pointing toward i if $|\alpha_{ij}| > 1$. If $\alpha_{ij}\alpha_{ji} > 4$ the vertices i and j are connected by a bold-faced line equipped with an ordered pair of integers $|\alpha_{ij}|, |\alpha_{ji}|$.

Clearly A is indecomposable if and only if $S(A)$ is a connected graph. A is determined by the Dynkin diagram and some enumeration of its vertices. $S(A)$ is of finite, affine or indefinite type if A is of that type.

The classification of indecomposable generalized Cartan matrices can be given in the following

Proposition 2.1. *Let A be an indecomposable generalized Cartan matrix.*

a) A is of finite type if and only if all its principal minors are positive.

b) A is of affine type if and only if all its proper principal minors are positive and $detA = 0$.

c) If A is of finite or affine type, then any proper subdiagram of $S(A)$ is a union of (connected) Dynkin diagrams of finite type.

d) If A is of finite type, then $S(A)$ contains no cycles. If A is of affine type and $S(A)$ contains a cycle, then it is the cycle of $A_l^{(1)}$ (see the Dynkin diagrams at the end).

e) A is of affine type if and only if there exists $\delta > 0$ such that $A\delta = 0$. Such

δ is unique up to a constant factor

Applying Dynkin diagrams to the classification of affine algebras two types are obtained, "untwisted" ($A_r^{(1)}$, $B_r^{(1)}$, $C_r^{(1)}$, $D_r^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$) and "twisted" ($A_1^{(2)}$, $B_r^{(2)}$, $\tilde{B}_r^{(2)}$, $C_r^{(2)}$, $F_4^{(2)}$, $G_2^{(3)}$). (See diagrams at the end)

2.1. Loop algebras and central extensions

Now we will describe a concrete construction of affine algebras. All "untwisted" affine algebras can be realized in terms of an "underlying" simple finite dimensional Lie algebra.

Let $\mathcal{L} = C[t, t^{-1}]$ be the algebra of Laurent polynomials in t. If $P = \sum_{k \in Z} c_k t^k$ is a Laurent polynomial, its residue is defined by Res $P = c_{-1}$. This linear functional on $\mathcal{L}$ is defined by the properties:

$$Res\ t^{-1} = 1, \quad Res \frac{dP}{dt} = 0$$

and defines a C-bilinear function ϕ on $\mathcal{L}$ by

$$\phi(P, Q) = Res \frac{dP}{dt} Q.$$

The following two properties can be checked:

$$\phi(P, Q) = -\phi(Q, P),$$

$$\phi(PQ, R) + \phi(QR, P) + \phi(RP, Q) = 0, \ P, Q, R \in \mathcal{L}.$$

The affine algebra associated to a generalized Cartan matrix of type $X_l^{(1)}$ (see table at the end) is called a *nontwisted affine algebra.* Such a matrix of type $X_l^{(1)}$ (where $X = A, B, \ldots, G$) is the extended Cartan matrix of the simple finite dimensional Lie algebra $\dot{\mathcal{G}}$ whose Cartan matrix $\dot{A}$ is a matrix of finite type X_l obtained from A removing the 0th row and column.

Consider the loop algebra $\mathcal{L}(\dot{\mathcal{G}}) = \mathcal{L} \otimes_C \dot{\mathcal{G}}$. This is an infinite dimensional complex Lie algebra with the bracket $[,]$ defined by $[P \otimes x, Q \otimes y] = PQ \otimes [x, y]$, where $P, Q \in \mathcal{L}$ and $x, y \in \dot{\mathcal{G}}$.

If the set $\{a_i|i = 1, \ldots, d\}$ is a basis of $\mathcal{G}$, then $\{a_i^n = t^n \otimes a_i|i = 1, \ldots, d, n \in Z\}$ is a basis of $\mathcal{L}(\mathcal{G})$ and if $[a_i, a_j] = \lambda_{ijk}t_k$ then $[a_i^n, a_j^m] = \lambda_{ijk}a_k^{m+n}$.

For every Lie algebra $\mathcal{G}$ we can construct a central extension simply by adding r central generators $k_1, \ldots, k_r$ to the generators a_i and defining $[k_i, a_j] = 0$, $[a_i, a_j] = \lambda_{ijk}a_k + \mu_{ijl}k_l$ with the structure constants λ_{ijk} the ones appearing in the algebra $\mathcal{G}$ and μ_{ijl} choosen in a way that ensures the Jacobi identity is satisfied.

Let us consider the (nondegenerate invariant symmetric bilinear C-valued) Killing form κ on $\dot{\mathcal{G}}$. It can be extended by linearity to an $\mathcal{L}$-valued bilinear form κ_t on $\mathcal{L}(\dot{\mathcal{G}})$ by

$$\kappa_t(P \otimes x, Q \otimes y) = PQ\kappa(x, y).$$

We can also extend every derivation D of the algebra $\mathcal{L}$ to a derivation of the Lie algebra $\mathcal{L}(\dot{\mathcal{G}})$ by

$$D(P \otimes x) = D(P) \otimes x.$$

Let us define the bilinear C-valued function ψ on the Lie algebra $\mathcal{L}(\dot{\mathcal{G}})$ by

$$\psi(a, b) = Res\kappa_t(\frac{da}{dt}, b).$$

The function ψ satisfies the following two conditions:

$$\psi(a, b) = -\psi(a, b),$$

$$\psi(a, b) + \psi(a, b) + \psi(a, b) = 0.$$

Now we can use ψ to construct $\tilde{\mathcal{L}}(\dot{\mathcal{G}})$ a central extension of $\mathcal{L}(\dot{\mathcal{G}})$ by a 1-dimensional center. Explicitly, $\tilde{\mathcal{L}}(\dot{\mathcal{G}}) = \mathcal{L}(\dot{\mathcal{G}}) \otimes Ck$ (direct sum of vector spaces) and the bracket is given by

$$[a + \lambda k, b + \mu k] = [a, b] + \psi(a, b)k, \ a, b \in \mathcal{L}(\dot{\mathcal{G}}); \ \lambda, \mu \in C.$$

Finally we will denote $\hat{\mathcal{L}}(\dot{\mathcal{G}})$ the Lie algebra obtained by adjoining to $\tilde{\mathcal{L}}(\dot{\mathcal{G}})$ a derivation d that acts as $t\frac{d}{dt}$ on $\mathcal{L}(\dot{\mathcal{G}})$ and kills k. That is, $\hat{\mathcal{L}}(\dot{\mathcal{G}})$ is the complex vector space

$$\hat{\mathcal{L}}(\dot{\mathcal{G}}) = \mathcal{L}(\dot{\mathcal{G}}) \oplus Ck \oplus Cd$$

with the bracket

$$[t^m \otimes x \oplus \lambda k \oplus \mu d, t^n \otimes y \oplus \lambda_1 k \oplus \mu_1 d] = (t^{m+n} \otimes [x,y] + \mu n t^n \otimes y - \mu_1 m t^m \otimes x) \oplus m\delta_{m,-n}\kappa_t(x,y)k.$$

Then $\hat{\mathcal{L}}(\dot{\mathcal{G}})$ is an untwisted affine algebra associated to the generalized Cartan matrix of affine type $X_l^{(1)}$.

Hence affine algebras are infinite-dimensional.

Now we will do the same in the "twisted" case.

If ω is an automorphism of finite order m (i.e., $\omega^m = 1$) of the Lie algebra $\mathcal{G}$, ω induces a Z/mZ-gradation in $\mathcal{G}$. Indeed, since ω is diagonalizable, $\mathcal{G}$ splits in direct sum of eigenspaces of ω

$$\mathcal{G} = \oplus_{j=0}^{m-1} \mathcal{G}_j,$$

$\mathcal{G}_j = \{x \in \mathcal{G} | \omega(x) = e^{2\pi j/m} x\}$ and so $[\mathcal{G}_h, \mathcal{G}_j] \subseteq \mathcal{G}_{h+j}$, where $h+j$ is understood mod m (Let us notice that $\mathcal{G}_0$ is a subalgebra of $\mathcal{G}$).

Conversely, every Z/mZ- gradation in $\mathcal{G}$ defines an automorphism ω of order m by $\omega(x) = e^{2\pi/m} x$

If $\mathcal{G}$ is a simple finite dimensional Lie algebra, we can associate a subalgebra $\mathcal{L}(\mathcal{G}, \omega, m)$ of $\mathcal{L}(\mathcal{G})$ to the automorphism ω as follows

$$\mathcal{L}(\mathcal{G}, \omega, m) = \oplus_{j \in Z} \mathcal{L}(\mathcal{G}, \omega, m)_j,$$

where $\mathcal{L}(\mathcal{G}, \omega, m)_j = t^j \otimes \mathcal{G}_{\bar{j}}$, $\bar{j} = j$ mod m.

If $\hat{\mathcal{L}}(\dot{\mathcal{G}}) = \mathcal{L}(\dot{\mathcal{G}}) \oplus Ck \oplus Cd$ denotes the algebra that we have constructed before, we can consider its subalgebra

$$\hat{\mathcal{L}}(\dot{\mathcal{G}}, \omega, m) = \mathcal{L}(\dot{\mathcal{G}}, \omega, m) \oplus Ck \oplus Cd.$$

The following theorem expresses the concrete result that we are looking for:

Theorem 2.1. *Let $\mathcal{G}$ be a complex simple finite dimensional Lie algebra of type $X_N = D_{l+1},\ A_{2l-1},\ E_6,\ D_4$ or A_{2l} and $m = 2, 2, 2, 3,$ or 2, respectively. Let*

ω be an automorphism of $\mathcal{G}$ of order m induced by a diagram automorphism. Then the Lie algebra $\hat{\mathcal{L}}(\dot{\mathcal{G}}, \omega, m)$ is a twisted affine Kac-Moody algebra associated to the affine matrix of type $X_N^{(m)}$. (With the notation of [5]).

The analogue of the Cartan-Weyl basis can be constructed for affine Lie algebras.

Among the modules of an affine algebra, the highest weight modules are again the most interesting ones. A highest weight module can be constructed from some highest weight vector v_Λ by applying lowering operators. If the set of vectors obtained in this way (that can be described in terms of the universal enveloping algebra) is formally taken to be independent, then the module is called a Verma module. Thus the underlying vector space of a Verma module is not given a priori, but obtained by construction.

Verma modules are in general not irreducible. To get an irreducible module from some given Verma module V_Λ one must divide out any relations which exist among this set of vectors. Thus the irreducible quotient of V_Λ is obtained by setting an appropiate set of elements of V_Λ to zero. Such vectors, which are actually zero although formally they are not, are called null vectors. For any Verma module there is a unique irreducible highest weight module R_Λ that is obtained as a quotient of the Verma module V_Λ.

3. Hopf Algebras and Quantum Groups

One of the fundamental tasks in quantum field theory is to express the properties of the field theory in terms of an underlying symmetry object. For conformal field theories, this means that in particular one would like to understand the operator product algebra in terms of the symmetry algebra of the theory.

It seems natural to search for structures related to the symmetry algebra which do possess tensor products corresponding to the operator product algebras. This search leads to the concepts of Hopf algebras and quantum groups.

3.1. Hopf Algebras

Definition 3.1. *A* **Hopf algebra** *is a vector space* $\mathcal{A}$ *endowed with five linear operations:*

$\mu : \mathcal{A} \otimes A \longrightarrow \mathcal{A}$ *multiplication,*

$\eta : F \longrightarrow \mathcal{A}$ *unit map,*

$\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ *co-multiplication*

$\epsilon : \mathcal{A} \longrightarrow F$ *co-unit map,*

$\gamma : \mathcal{A} \longrightarrow \mathcal{A}$ *antipode,*

which possess the following properties:

(H.1) $\mu \cdot (id \otimes \mu) = \mu \otimes (\mu \otimes id)$ *associativity,*

(H.2) $\mu \cdot (id \otimes \eta) = id = \mu \cdot (\eta \otimes id)$ *unitary property,*

(H.3) $(id \otimes \Delta) \cdot \Delta = (\Delta \otimes id) \cdot \Delta$ *co-associativity,*

(H.4) $(\epsilon \otimes id) \cdot \Delta = id = (id \otimes \epsilon) \cdot \Delta$ *counitary property,*

(H.5) $\mu \cdot (id \otimes \gamma) \cdot \Delta = \eta \cdot \epsilon = \mu \cdot (\gamma \otimes id) \cdot \Delta,$

(H.6) The co-multiplication Δ *and co-unit* ϵ *are* F*-algebra morphisms, that is, preserve the multiplication.*

We will assume that the base field F is R or C.

An associative algebra with unit satisfies H.1 and H.2.

A vector space with a co-multiplication and a counit that satisfy (H.3) and (H.4) is called co-algebra. The theory of co-algebras is dual to the theory of co-algebras. The concepts of co-multiplication and co-unit arise naturally when one tries to define tensor products of representations of an algebra. It can not be defined for associative algebras which do not possess any further structure, for instance. To avoid this problem, one introduces an additional operation Δ such that for any two representations $R_i : A \longrightarrow gl(V_i)$, the map $R : A \longrightarrow gl(V_1 \otimes V_2)$, $x \rightarrow (R_1 \otimes R_2) \cdot \Delta(x)$ is a representation too.

If $\mathcal{A}$ is associative it is natural to require that the formation of tensor products is associative, what restricts the map Δ to being co-associative. If $\mathcal{A}$ is unital it is again natural to require the existence of a co-unit. An algebra with the four first operations and satisfying (H.1) to (H.4) and (H.6) is called bi-algebra.

Finally when one tries to connect the operations of product and co-product non-trivially we are lead naturally to the concept of antipode.

If we denote $\mu(x \otimes y) = x \cdot y$, the associativity property (H.1) reads $x \cdot (y \cdot z) = (x \cdot y) \cdot z \ \ \forall x, y, z \in \mathcal{A}$.

Similarly the existence of unit (H.2) means that there is an element $1 \in \mathcal{A}$ tal que $1 \cdot x = x = x \cdot 1 \ \ \forall x \in \mathcal{A}$.

The map η is then fiven by $\eta : \xi \to \xi 1 \ \ \forall \xi \in F$.

Examples

1. If G is a finite group, its group Hopf algebra is the vector space $F(G)$ (that is, the vector space over F with basis given by the elements of G) and the algebra structure defined by the multiplication and unit of G:

$(\sum_{g \in G} \alpha_g g) \cdot (\sum_{h \in G} \alpha_h h) = \sum_{g,h} \alpha_g \alpha_h (gh)$.

Now let us define:

$$\triangle : x \to x \otimes x$$
$$\epsilon : x \to 1 \ \ \forall x \in G$$
$$\gamma : x \to x^{-1}$$

for all $x \in \mathcal{G}$.

The co-algebra structure and the antipode are given by extending by linearity the previous definitions on G.

2. Let $U(\mathcal{G})$ be the universal enveloping algebra of a complex Lie algebra $\mathcal{G}$. Then $U(\mathcal{G})$ is a Hopf algebra by taking μ as the usual formal multiplication on $U(\mathcal{G})$ and defining the unit by $\eta(\xi) = \xi 1 \quad \forall \xi \in C$.

The co-multiplication, co-unit and antipode are given by:

$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad\qquad \Delta(1) = 1 \otimes 1$

$\epsilon(x) = 0 \qquad \forall x \in \mathcal{G} \qquad \epsilon(1) = 1$

$\gamma(x) = -x \qquad\qquad \gamma(1) = 1.$

Properties of a Hopf algebra

1. For a given multiplication and co-multiplication, the co-unit is unique.

2. If $(A, \mu, \eta, \Delta, \epsilon, \gamma)$ is a finite dimensional Hopf algebra, the dual vector space A^* inherits a Hopf algebra structure from A by "exchanging" μ, η with Δ and ϵ. That is, $(A^*, \mu^*, \eta^*, \Delta^*, \epsilon^*, \gamma^*)$ is a Hopf algebra with the operations defined by:

$(\mu^*(\varphi \otimes \psi))(x) = (\varphi \otimes \psi)(\Delta(x))$

$(\eta^*(\xi))(x) = \xi\epsilon(x)$

$(\Delta^*(\varphi))(x \otimes y) = \varphi(x \cdot y)$

$\epsilon^*(\varphi) = \varphi(\eta)$

$(\gamma^*(\varphi))(x) = \varphi(\gamma(x))$

for $x, y \in A, \quad \varphi, \psi \in A^*, \quad \xi \in F$.

In the infinite-dimensional case, in general $(A^*)^* \not\simeq A$.One possibility to define the dual Hopf algebra is via the previous definitions, but to restrict the vector space A^* to an appropiate subspace. In short, the dual of a Hopf algebra is the maximal Hopf algebra contained in the dual vector space A^*.

3. The multiplication μ and unit η are F-coalgebra morphisms.

4. The antipode is an antihomomorphism, that is, $\gamma(x \cdot y) = \gamma(y) \cdot \gamma(x)$ $\forall x, y \in A$ and $\gamma(1) = 1$.

5. The antipode is an anticohomomorphism, that is, $\epsilon \cdot \gamma = \epsilon$ and $(\gamma \otimes \gamma) \cdot \Delta = \pi \cdot \Delta \cdot \gamma$, where $\pi : A \otimes A \to A \otimes A$ is defined by $\pi(x \otimes y) = y \otimes x$.

6. The map $\Delta' = \pi \cdot \Delta$ is also a (co-associative) co-multiplication. So, together with γ' the inverse of the antipode γ, it provides another Hopf algebra structure for A. The Hopf algebra is called **co-commutative** if the co-multiplication satisfies $\pi \cdot \Delta = \Delta$. Examples 2 and 3 belong to this type. If A is commutative or co-commutative, then $\gamma^2 = 1$.

Commutative Hopf algebras are intimately related to compact topological groups. If for any such group G we consider the vector space $C(G)$ of continuous complex-valued functions on G, this space has a Hopf algebra structure via

$$(\mu(\varphi,\psi))(x) = \varphi(x)\psi(x)$$
$$\eta(\xi) = \xi\mathbf{1}$$
$$(\Delta(\varphi))(x\otimes y) = \varphi(xy)$$
$$\epsilon(\varphi) = \varphi(1)$$
$$(\gamma(\varphi))(x) = \varphi(x^{-1})$$

It can be easily verified that $C(G)$ is a commutative Hopf algebra. Moreover, it is co-commutative if and only if the group G is abelian.

In short, we can associate to any compact topological group a commutative Hopf algebra. And a commutative Hopf algebra is the ring of representative functions on an algebraic group G (see [6]).

It is possible to describe properties of compact topological groups G only in terms of the associated Hopf algebra of functions $C(G)$, without making explicit reference to G and its elements at all. If G is a Lie group, there exists a non-degenerate duality between $C(G)$ and the universal enveloping algebra $U(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G, that is, $C(G)$ is isomorphic to (a subspace of) the dual space $U(\mathcal{G})^*$.

Inspired by the previous fact, one is led to describe also non-commutative Hopf algebras as function spaces on appropiate objects. These objects can not be topological groups, and not much is known about their explicit structure. They are called **pseudogroups, non-commutative geometric grups** or **quantum groups**.

Remark. The term **quantum group** does not possess a generally accepted meaning. Often this name is used not only for the geometric objects introduced above, but generically for any Hopf algebra, or sometimes also for the quasitriangular Hopf algebras.

Quasitriangularity is a property that quantum universal enveloping algebras have and that we will define next.

Definition 3.2. *A* **quasitriangular Hopf algebra** *is a Hopf algebra for which the co-multiplications* Δ *and* Δ' *are related by conjugation, that is:*

(HT1) $\Delta'(x) = R \cdot \Delta(x) \cdot R^{-1} \quad \forall x \in A$ *for some element* R *of* $A \times A$ *which is invertible and satisfies*

(HT2) $(id \otimes \Delta)(R) = R_{13} \cdot R_{12}$

(HT3) $(\Delta \otimes id)(R) = R_{13} \cdot R_{23}$

(HT4) $(\gamma \otimes id)(R) = R^{-1}$.

In the above definition R_{13} is meant as the identity in the second factor of $A \otimes A \otimes A$ and as R in the first and third factors. (for instance, if $R = r_1 \otimes r_2$, then $R_{13} = r_1 \otimes 1 \otimes r_2$). Similarly for R_{12} and R_{23}.

The inverse R^{-1} of $R \in A \otimes A$ is the element of $A \otimes A$ which satisfies $R^{-1} \cdot R = 1 \otimes 1 = R \cdot R^{-1}$. In particular, $(\gamma \otimes \gamma)(R) = R$.

If $R = r_1 \otimes r_2$, then $R^{-1} = s_1 \otimes s_2$ with $s_1 = r_1^{-1}$, $s_2 = r_2^{-1}$.

A quasitriangular Hopf algebra is called **triangular** if $R_{12} \cdot R_{21} = 1 \otimes 1$.

Every co-commutative Hopf algebra is quasitriangular. Since $\Delta(1) = 1 \otimes 1$ and $\gamma(1) = 1$, it suffices to take $R = 1 \otimes 1$). In particular any group Hopf algebra is quasitriangular, and so is the universal enveloping algebra of any Lie algebra.

In general a quasitriangular Hopf algebra is neither commutative nor co-commuta- tive; however the non-cocommutativity is under control by the axiom (HT1). Every finite-dimensional Hopf algebra can be embedded in a suitable finite-dimensional quasitriangular Hopf algebra, its Drinfeld double.

An immediate consequence of the axiom (HT1) is:

$R_{12} \cdot (\Delta \otimes id)(x \otimes y) = (\Delta' \otimes id)(x \otimes y) \cdot R_{12} \quad \forall x \otimes y \in A \otimes A.$

Taking $R = x \otimes y$ and using (HT2) y (HT3) this yields

$R_{12} \cdot R_{13} \cdot R_{23} = R_{23} \cdot R_{13} \cdot R_{12}.$

This is the so-called **Yang-Baxter equation**, which plays a fundamental role in the theory of completely integrable systems. In this context, the quantity R is called a *universal R-matrix*, and this terminology has been adapted for the element R of an arbitrary quantum group.

Notice that in the simple case $R = r_1 \otimes r_2$, the Yang-Baxter equation simply reads

$$(r_1 \cdot r_1) \otimes (r_2 \cdot r_1) \otimes (r_2 \cdot r_2) = (r_1 \cdot r_1) \otimes (r_1 \cdot r_2) \otimes (r_2 \cdot r_2),$$

that is, $r_1 \cdot r_2 = r_2 \cdot r_1$.

3.2. Deformations of enveloping algebras

The quantum groups that turn out to be relevant in physics (for instansce in the so called WZW theories) are obtained as deformations of enveloping algebras of semisimple Lie algebras. The quantum universal enveloping algebra $U_q(\mathcal{G})$ can then be defined as a quantum deformation of the enveloping algebra. Deforming the relations of $U(\mathcal{G})$ means changing them in a manner depending on some formal parameter q such that the original algebra $U(\mathcal{G})$ is obtained in the limit $q \to 1$. The terms "quantum" and "classical" are used in this context because the q-deformations can be understood as a formal "quantization" procedure.

Definition 3.3. *The* **quantum universal enveloping algebra** $U_q(\mathcal{G})$ *is the algebra of power series in the* $3r+1$ *generators* $\{e^i_\pm, h^i | i = 1, \ldots, r\} \cup \{1\}$ *modulo the relations:*

(UQ1) $[h^i, h^j] = 0$,
(UQ2) $[h^i, e^j_\pm] = \pm \alpha_{ij} e^j_\pm$,
(UQ3) $[e^i_+, e^j_-] = \delta_{ij} \lfloor h^i \rfloor$,
(UQ4) $\sum_{p=0}^{1-\alpha_{ij}} (-1)^p \lfloor \begin{matrix} 1-\alpha_{ij} \\ p \end{matrix} \rfloor_i (e^i_\pm)^p (e^j_\pm)(e^i_\pm)^{1-\alpha_{ij}-p} = 0$ *for* $i \neq j$,
(UQ5) $1 \diamondsuit x = x = x \diamondsuit 1 \quad \forall x \in U_q(\mathcal{G})$.

The elements α_{ij} in this definition are the Cartan integers of the Lie algebra $\mathcal{G}$. Here brackets are to be undertood as commutators,

$$[x, y] \equiv x \diamondsuit y - y \diamondsuit x$$

and $x \diamondsuit y$ the formal product in $U(\mathcal{G})$. Also we have used the q-number symbol

$$\lfloor x \rfloor \equiv \lfloor x \rfloor_q := \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}$$

together with

$$\lfloor x \rfloor_i \equiv \lfloor x \rfloor_{q_i} \quad \text{with} \quad q_i := q^{(\alpha^{(i)},\alpha^{(i)})/(\theta,\theta)},$$

$$\lfloor n \rfloor ! = \prod_{m=1}^{n} \lfloor m \rfloor \quad \text{and} \quad \left\lfloor \begin{matrix} n \\ m \end{matrix} \right\rfloor = \frac{\lfloor n \rfloor !}{\lfloor m \rfloor ! \lfloor n-m \rfloor !}.$$

The exponential functions of generators appearing in the expressions $\lfloor h^i \rfloor$ are defined through the corresponding power series, that is,

$$e^{\xi h} = \sum_{h=0}^{\infty} \frac{\xi^n}{n!} h^n$$

where $h^n = h^{\Diamond n}$ is defined inductively, i.e., $h^{\Diamond n} = h \Diamond h^{\Diamond (n-1)}$. Note that, in particular, $e^{\xi h} \Diamond e^{-\xi h} = 1$.

By using this definition an explicit formula for the "R-matrix" can be given showing that $U_q(\mathcal{G})$ is quasitriangular.

Note that the appearance of $q^{\pm h^i/2}$ forces us to consider infinite power series in the h^i (contrary to the case of the enveloping algebra $U(\mathcal{G})$). As a consequence, usually this restriction is included in the definition of $U_q(\mathcal{G})$. Instead of allowing for infinite power series in the h^i, one can alternatively replace each generator h^i by a pair of generators $k^i_\pm \equiv q^{\pm h^i/2}$ and consider only finite power series in $k^i_\pm$ as well as in $e^i_\pm$. In terms of these new generators, the relations of $U_q(\mathcal{G})$ read

$$k^i_+ k^i_- = k^i_- k^i_+ = \mathbf{1},$$

$$[k^i_\pm, k^j_\pm] = [k^i_+ k^j_-] = 0,$$

$$k^i_+ e^j_\pm = q^{\pm h^i/2} e^j_\pm k^i_+,$$

$$k^i_- e^j_\pm = q^{\mp h^i/2} e^j_\pm k^i_-,$$

$$[e^i_+, e^i_-] = \delta^{ij} (q^{1/2} - q^{-1/2})^{-1} (k^i_+ - k^i_-).$$

We can define the quantum version of the map ad_x as a q-deformed commutator,

$$Ad_{e^\alpha}(e^\beta) \equiv [e^\alpha, e^\beta]_q := q^{-(\alpha,\beta)/4} e^\alpha \Diamond e^\beta - q^{-(\alpha,\beta)/4} e^\beta \Diamond e^\alpha.$$

Now the quantum analogue (UQ4) in Definition 3.3 of the Serre relations can also be written as

$$(Ad_{e^i_\pm})^{1-a_{ji}}(e^j_\pm) = 0.$$

Let us ilustrate the above mentioned with an example.

Example

Let us consider $\mathcal{G} = A_1 = sl(2)$. In this case there are no Serre relations, so the defining relations of $U_q(A_1)$ are

$$[h, e_\pm] = \pm 2e_\pm,$$

$$[e_+, e_-] = \lfloor h \rfloor.$$

By definition the enveloping algebra $U(A_1)$ is associative and its unit is given by the trivial power series 1. These properties are inhereted by $U_q(A_1)$. Moreover, it turns out that $U_q(A_1)$ is endowed with the structure of a quasitriangular Hopf algebra. The Hopf algebra structure arises via the following definitions on the generators h, $e_\pm$:

co-multiplication

$$\Delta(h) = h \otimes 1 + 1 \otimes h,$$

$$\Delta(e_\pm) = e_\pm \otimes q^{h/4} + q^{-h/4} \otimes e_\pm,$$

$$\Delta(1) = 1 \otimes 1,$$

co-unit

$$\epsilon(h) = \epsilon(e_\pm) = 0; \quad \epsilon(1) = 1,$$

antipode

$$\gamma(h) = -h, \quad \gamma(e_\pm) = -q^{\pm 1/2} e_\pm, \quad \gamma(1) = 1.$$

If we don't want to use infinite power series in h, we can replace h by two generators k_+ and k_-.(Notice that we are thinking of $k = e^h$, an exponential).

In this way $U_q(A_1)$ is generated by e_+, e_-, k_+, k_- and the product satisfies

$$k_+k_- = k_-k_+ = 1,\ k_+e_+k_- = q^2e_+,\ k_+e_-k_- = q^{-2}e_-,\ e_+e_- - e_-e_+ = \frac{k_+ - k_-}{q - q^{-1}}.$$

For a simple algebra $\mathcal{G}$ it can be shown that the representation theories of $\mathcal{G}$ and $U_q(\mathcal{G})$ are "very similar" . Namely, all finite-dimensional modules of $U_q(\mathcal{G})$ are fully reducible, and the irreducible finite- dimensional modules of $U_q(\mathcal{G})$ are parametrized by the dominant integral highest weights Λ of $\mathcal{G}$. Also the finite-dimensional modules can be decomposed into weight spaces in complete analogy to the semisimple Lie case:: $R = \oplus R_{(\lambda)}$ such that $R(h^i)v_\lambda = \lambda^i v_\lambda$ and $\dim R_{(\lambda)}(U_q(\mathcal{G})) = \dim R_{(\lambda)}(\mathcal{G})$.

3.3. Quantization

Quantum groups play a role for quantum mechanical systems that is analogous to the role of ordinary groups in classical mechanical systems. The Hopf algebras $U_q(\mathcal{G})$ are obtained as q-deformations of more familiar objects. The mathematical procedure of deformation is precisely what is employed in physics when the quantization of some theory is performed. It is therefore tempting to try to obtain quantum groups from more traditional objects by some appropiate (formal) *quantization procedure.*

We know an easy way of constructing noncommutative Hopf algebra by considering A^*, for A a commutative but not cocommutative algebra. In this way more or less all co-commutative noncommutative Hopf algebras are obtained. The most interesting and mysterious Hopf algebras are those which are neither commutative nor cocommutative. Though Hopf algebras have been intensively studied by algebraists, it seems that most of the examples of noncommutative non-cocommutative Hopf algebras invented independently of the integrable quantum system theory are counterexamples rather than "natural" examples. There is a general method for constructing such Hopf algebras based on the concept of quantization that was proposed under the influence of the QISM (quantum inverse scattering method) (see [2]).

A quantization of a commutative associative algebra A_0 over F is a (not necessarily commutative) deformation of A_0 depending on a parameter h (Plank's constant), that is, an associative algebra A over $F[[h]]$ such that $A_0 = A/hA$ and A is a topologically free $F[[h]]$-module.

Given A, we can define a new operation on A_0 (the Poisson bracket) by the formula

$$\{a \mod h, b \mod h\} = \frac{[a,b]}{h} \mod h.$$

Thus A_0 becomes a Poisson algebra (i.e., a Lie algebra with respect to $\{,\}$ and a commutative associative algebra with respect to multiplication, these two structures being compatible in the following sense: $\{a, bc\} = \{a, b\}c + b\{a, c\}$).

Definition 3.4. *A quantization of a Poisson algebra A_0 is an associative algebra deformation A of A_0 over $F[[h]]$ such that the Poisson bracket on A_0 defined as above is equal to the bracket given a priori.*

There is a Hopf algebra version of the above definition when A_0 is a Poisson-Hopf algebra (i.e., a Hopf algebra structure and a Poisson algebra structure on A_0 are given such that the multiplication is the same and the co-multiplication $A_0 \to A_0 \otimes A_0$ is a Poisson algebra homomorphism, the Poisson bracket on $A_0 \otimes A_0$ being defined by $\{a \otimes b, c \otimes d\} = ac \otimes \{b, d\} + \{a, c\} \otimes bd$), and A is a Hopf algebra deformation of A_0.

A classical dynamical system is defined by the fact that the dynamics is subject to the requirement $\partial f = \{h, f\}$ valid for any function f on the manifold in which the dynamics takes place. If A_0 denotes the space of these functions, A_0 is a commutative associative unital algebras and ∂ is a derivationon A_0, that is, $\partial(x \cdot y) = x \cdot \partial y + (\partial x) \cdot y$.

If the algebra A_0 is a Poisson-Hopf algebra, and there exists a quantization A of A_0, it is natural to interpret A as the space of functions on some topological object G_h, and to call G_h a quantization of the group G or for short quantum group.

The relation of this type of quantum group with the quantized universal enveloping algebras arises as follows. Take G a compact Lie group with Lie algebra $\mathcal{G}$. There is a pairing between the functions $C(G)$ on G and the dual $U(\mathcal{G})^*$ of the universal enveloping algebra $U(\mathcal{G})$. Thus one has $C(G) \simeq U(\mathcal{G})^*$. Moreover from $U(\mathcal{G})^*$ one can determine $U(\mathcal{G})$, so that we arrive at the following scheme of correspondences:

$$\begin{array}{ccccc} G & \rightleftharpoons & C(G) \simeq U(\mathcal{G})^* & \rightleftharpoons & U(\mathcal{G}) \\ group & & commutative & & co-commutative \\ & & Hopf\ algebra & & Hopf\ algebra \end{array}$$

The formal quantization procedure relates $C(G)$ to $C(G_h)$, but it is not possible to give a precise meaning to the quantum group G_h directly as a quantization of G. However this is not a real problem because any issue concerning G_h may be reformulated in terms of $C(G_h)$. But it is possible to translate the quantizaton procedure into the language of universal enveloping algebras. In this way we have the following picture of correspondences:

$$\begin{array}{ccccc} G & \rightleftharpoons & C(G) \simeq U(\mathcal{G})^* & \rightleftharpoons & U(\mathcal{G}) \\ group & & commutative & & co-commutative \\ & & Hopf\ algebra & & Hopf\ algebra \\ & & \downarrow\uparrow & & \downarrow\uparrow \\ G_h & \rightleftharpoons & C(G_h) \simeq (U(\mathcal{G})_h)^* & \rightleftharpoons & U(\mathcal{G})_h \\ quantum & & non-commutative & & non-cocommutative \\ group & & Hopf\ algebra & & Hopfalgebra \end{array}$$

Note that in this diagram (that is not precise and only gives some intuition into what is happening) there are no arrows directly connecting G and G_h. Although this does not prevent the study of the object G_h (in terms of the functions $C(G_h)$) it is still tempting to find a more direct connection. Indeed it exists via the so called matrix quantum groups which, in short, are matrix representations of suitable quantum groups G_h.

Example 3.1. Let us consider 2×2 matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are complex numbers, $ad \neq bc$. These matrices form the (non-semisimple) Lie group $GL_2(C)$.

If q is a complex number, the algebra generated by four elements $\{a, b, c, d\}$

satisfying the relations:

$$ab = q^{1/2}ba \qquad ac = q[1/2ca$$

$$bd = q^{1/2}db \qquad cd = q^{1/2}dc$$

$$bc = cb \qquad ad - da = (q^{1/2} - q^{-1/2})bc$$

is the quantum matrix group $(GL_2)_q(C)$.

This algebra can be seen as a quantization of $GL_2(C)$ by replacing those "matrices" by the coordenate functions on the matrices.

If we impose $ad - q^{1/2}bc = 1$, we obtain the quantum matrix group $(SL_2)_q)(C)$.

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