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# MULTIPLE SOLUTIONS FOR A CLASS OF STRONGLY INDEFINITE PROBLEMS

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### Abstract

We consider variations problems associated with differential equations $(P)$ of the form

$$\mathcal{L}u = \nabla\mathcal{F}(u) \quad ,$$

where $\mathcal{L} : D(\mathcal{L}) \subset H \longrightarrow H$ is an unbounded, selfadjoint operator on a closed subspace $H$ of $L^2(\Omega, \mathbb{R}^N), \Omega \subset \mathbb{R}^N$ is a bounded domain and $\mathcal{F} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a $C^1$ function. If $\nabla\mathcal{F}(u)$ satisfies suitable growth conditions, then the weak solutions of $(P)$ are precisely the critical points of a related functional $J = q - N$ on an appropriate Hilbert space $E \subset H$, where $q$ is the quadratic form on $E$ corresponding to the operator $\mathcal{L}$ and $N(u) = \int_\Omega \mathcal{F}(u)dx$. We are interested in situations where $J$ is strongly indefinite in the sense that it is neither bounded from above nor from below, even modulo subspaces of finite dimension or codimension.

Assuming that $N$ is *nonquadratic at infinity* and that the functional $J$ is *invariant* under a given $\boldsymbol{Z}_2$ or $S^1$ action, we prove multiplicity results on critical points of $J$ which depend only on the behavior of the quotient $2\mathcal{F}(u)/|u|^2$ as $|u|$ varies from 0 to $\infty$. Such results, which apply to Hamiltonian Systems, Nonlinear Wave Equations and Noncooperative Elliptic Systems, partially extend and complement many other results in the literature.

### Resumo

Consideramos problemas variacionais associados a equações diferenciais $(P)$ da forma

$$\mathcal{L}u = \nabla\mathcal{F}(u) \quad ,$$

onde $\mathcal{L} : D(\mathcal{L}) \subset H \longrightarrow H$ é um operador autoadjunto, não-limitado, num subespaço fechado $H$ de $L^2(\Omega, \mathrm{R}^n)$ , $\Omega \subset \mathrm{R}^N$ é um domínio limitado e $\mathcal{F} : \mathrm{R}^n \longrightarrow \mathrm{R}$ é uma função de classe $C^1$. Se $\nabla\mathcal{F}(u)$ satisfaz uma condição adequada de crescimento então as soluções fracas de $(P)$ são precisamente os pontos críticos de um funcional associado $J = q - N$ num espaço de Hilbert apropriado $E \subset H$, onde $q$ é a forma quadrática correspondente ao operador $\mathcal{L}$ e $N(u) = \int_\Omega \mathcal{F}(u)dx$. Estamos interessados em situações onde $J$ é fortemente indefinida no sentido de que o funcional $J$ é ilimitado tanto inferiormente quanto superiormente, mesmo módulo subespacos de dimensão ou codimensão finita.

Supondo que $N$ é *não-quadrático no infinito* e que o funcional $J$ é invariante sob uma dada ação de $\boldsymbol{Z}_2$ ou de $S^1$ no espaço $E$, provamos resultados de multiplicidade para os pontos críticos de $J$, os quais dependem apenas do comportamento do quociente $2\mathcal{F}(u)/|u|^2$ quando $|u|$ varia de 0 a $\infty$. Tais resultados, os quais podem ser aplicados a Sistemas Hamiltonianos, Equações Não-Lineares de Onda e Sistemas Elípticos Não-Cooperativos, estendem parcialmente e complementam muitos outros resultados anteriores na literatura.

## 1. Introduction

This paper is concerned with multiplicity results for a class of differential equation problems of the form

$$\mathcal{L}u = \nabla\mathcal{F}(u) \ . \tag{P}$$

Here, $\mathcal{L} : D(\mathcal{L}) \subset H \longrightarrow H$ is an unbounded, selfadjoint operator on a closed subspace $H$ of $L^2(\Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $\mathcal{F} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a $C^1$ function. If $\nabla\mathcal{F}(u)$ satisfies suitable growth conditions, it turns out that the weak solutions of $(P)$ are precisely the critical points of a related functional $J : E \longrightarrow \mathbb{R}$ of the form

$$J(u) = q(u) - \int_\Omega \mathcal{F}(u)\ dx \ ,$$

on an appropriate Hilbert space $E \hookrightarrow H$, where $q$ is the quadratic form on $E$ corresponding to the operator $\mathcal{L}$.

We are interested in situations where the functional $J$ is strongly indefinite in the sense that it is neither bounded from above nor from below, even modulo subspaces of finite dimension or codimension. In particular, both the quadratic form $q$ and its negative have infinite Morse indices. Such situations cover Hamiltonian Systems, Nonlinear Wave Equations and Noncooperative Elliptic Systems (cf. [8]).

Roughly speaking, our basic hypotheses say that the functional $N(u) = \int_\Omega \mathcal{F}(u)dx$ is *nonquadratic at infinity* (cf. [8, 9]) and that the quotient $2\mathcal{F}(u)/|u|^2$

*crosses* $m$ points of the discrete spectrum of $\mathcal{L}$ as $|u|$ varies from 0 to $\infty$. Then, assuming that the functional $J$ is invariant under a given $\mathbb{Z}_2$ or $S^1$ action on $E$, our main result says that problem $(P)$ has at least $m$ nontrivial solutions. This complements the results in [8] and extends some of the results in [11].

It should be mentioned that the basic common underlying structure of problems on Hamiltonian systems, nonlinear wave equations and elliptic systems has been previously noticed by Amann [1] and Benci-Rabinowitz [4], among others, and that there is a vast literature on such problems when the assumptions are placed on the nonlinearity $\nabla\mathcal{F}(u)$ itself (see the basic references [13, 18, 21]). Moreover, the situations where $\nabla\mathcal{F}(u)$ is sublinear or superlinear are handled separately (see e.g. [5, 7, 12, 15, 16, 17, 20, 19]).

In Section 2 we present the abstract framework to which our main result applies, while Section 3 is reserved to the proof of that result and to applications. We consider applications to the problem of finding multiple time periodic solutions for the nonlinear wave equation

$$u_{tt} - u_{xx} + f(u) = 0 \quad , \quad 0 < x < \pi \; , \tag{$(WE)_1$}$$

under Dirichlet boundary condition

$$u(t,0) = u(t,\pi) = 0 \; , \tag{$(WE)_2$}$$

as well as to the problem of finding multiple solutions of noncooperative elliptic systems

$$\begin{cases} -\Delta z &= F_z(z,w) \;\text{ in }\; \Omega \\ \Delta w &= F_w(z,w) \;\text{ in }\; \Omega \\ \quad z &= w \;=\; 0 \;\text{ on }\; \partial\Omega \; , \end{cases} \tag{$(ES)$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. Typical results are given by the following theorems which complement results in [8, 10].

**Theorem 1.1** *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function satisfying* $f(0) = 0$, $F(u) = \int_0^u f(s)ds \geq 0$ *and*

$$|f(u)| \leq A|u|^r + B \;\; \forall u \in \mathbb{R} \; , \tag{$(f_1)$}$$

*for some $r > 1$. Assume further that*

$$\liminf_{|u|\to\infty} \frac{uf(u) - 2F(u)}{|u|^\mu} > 0 \ , \qquad (f_2)$$

$$\limsup_{u\to 0} \frac{2F(u)}{u^2} < \lambda_l \leq \lambda_k < \liminf_{|u|\to\infty} \frac{2F(u)}{u^2} \ , \qquad (f_3)$$

*where $\mu > r - 1$ and $0 < \lambda_l \leq \lambda_k$ are eigenvalues of $\mathcal{L} = \partial_x^2 - \partial_t^2$ in the space $H = \{ \ u \in L^2((0,\pi) \times (0,\pi)) \mid u(t, \pi - x) = u(t,x) \ a. \ e. \ \}$. Then, problem $(WE)_1, (WE)_2$ has at least $m = k - l + 1$ (t dependent) solutions which are $\pi$-periodic in $t$ and* geometrically distinct *(that is, do not differ by a time translation).*

**Theorem 1.2** *Assume that $F \in C^1(\mathbb{R}^2, \mathbb{R})$ is* even, *$\nabla F(0,0) = (0,0)$ and*

$$|\nabla F(u)| \leq C|u|^r + D \quad \forall u = (z,w) \in \mathbb{R}^2 \ , \qquad (h_1)$$

*where $1 < r < (N+2)/(N-2)$, $N \geq 3$. Assume further that*

$$\liminf_{|u|\to\infty} \frac{u \cdot \nabla F(u) - 2F(u)}{|u|^\mu} > 0 \ , \qquad (h_2)$$

$$\limsup_{|u|\to 0} \frac{2F(u)}{|u|^2} < \lambda_l \leq \lambda_k < \liminf_{|u|\to\infty} \frac{2F(u)}{|u|^2} \ , \qquad (h_3)$$

*where $\mu > N(r-1)/2 > 0$ and $\lambda_l \leq \lambda_k$ are eigenvalues of the operator $\mathcal{L} = diag(-\Delta, \Delta)$ on $L^2(\Omega) \times L^2(\Omega)$. Then, problem $(ES)$ has at least $m = k - l + 1$ pairs of nontrivial solutions $u = (z,w) \in H_0^1(\Omega) \times H_0^1(\Omega)$.*

## 2. The Abstract Framework

We shall start by recalling some definitions as well as an abstract multiplicity result due to Benci-Capozzi-Fortunato [3].

Let $E$ be a real Hilbert space with inner product $\langle \cdot \ , \cdot \rangle$ and norm $\| \cdot \|$ and assume that $\{T(g) | g \in G\}$ is a *unitary representation* of a compact topological (abelian) group $G$ on the space $E$ (see [13]). In other words, for each $g \in G$ the mapping $T(g) : E \longrightarrow E$ is a linear isometry and the following properties are satisfied:

(i) $T(0) = Identity$;

(ii) $T(g_1)T(g_2) = T(g_1 + g_2)$ for all $g_1, g_2 \in G$;

(iii) $(g, u) \mapsto T(g)u$ is continuous on $G \times E$.

In particular, each $T(g)$ is a *unitary operator* with inverse $T(-g)$.

The *orbit* of an element $u \in E$ is the set $\mathcal{O}(u) = \{T(g)u | g \in G\}$. A given functional $J : E \longrightarrow \mathbb{R}$ is said to be *invariant* if $J(T(g)u) = J(u)$ for all $(g, u) \in G \times E$, and a subset $A \subset E$ is called *invariant* if $T(g)A = A$ for all $g \in G$. The *fixed set* of the representation $\{T(g)\}$ is the (closed) subspace

$$Fix(G) = \{u \in E \mid T(g)u = u \; \forall g \in G\}$$

consisting of the *most symmetric elements* of E. Finally, we say that a functional $J \in C^1(E, \mathbb{R})$ satisfies the Cerami compactness condition at the level $c \in \mathbb{R}$ if the following holds (cf. [6]):

$(Ce)_c$ *Any sequence* $\{u_n\} \subset E$ *such that* $J(u_n) \longrightarrow c$ *and* $(1+\|u_n\|)\|\nabla J(u_n)\| \longrightarrow 0$ *possesses a convergent subsequence.*

This condition, which was considered by Cerami in [6] and allows for rather general minimax results (cf. [2]), is a variant of the famous compactness condition introduced by Palais-Smale in [14]:

$(PS)_c$ *Any sequence* $\{u_n\} \subset E$ *such that* $J(u_n) \longrightarrow c$ *and* $\|\nabla J(u_n)\| \longrightarrow 0$ *possesses a convergent subsequence.*

The following abstract multiplicity result was proved by Benci-Capozzi-Fortunato in [3]:

**Theorem 2.1** *Suppose that a unitary representation of a group G acts on a real*

*Hilbert space* $E$, *where* $G = \mathbb{Z}_2$ *or* $G = S^1$. *Let* $J \in C^1(E, \mathbb{R})$ *be an invariant functional of the form*

$$J(u) = \frac{1}{2}\langle Lu, u\rangle - N(u) ,$$

*where* $L : E \longrightarrow E$ *is a bounded selfadjoint operator which does not contain* 0 *in its essential spectrum and* $\nabla N : E \longrightarrow E$ *is a compact operator. Moreover, assume that there exist constants* $0 < a < b$, $\rho > 0$ *and closed, invariant subspaces* $V, W \subset E$ *with* $dim(V \cap W) < \infty$, $codim(V + W) < \infty$ *such that*

*(i)* $J(u) \leq b \ \ \forall \ u \in V$;

*(ii)* $J(u) \geq a \ \ if \ \ \|u\| = \rho \ , \ u \in W$;

*(iii)* $Fix(G) \subset V \ \ or \ \ Fix(G) \subset W$;

*(iv)* $J(u) < a \ \ if \ \ u \in Fix(G) \ , \ \nabla J(u) = 0$ .

*Then, the functional* $J$ *possesses at least* $m$ *orbits of critical points with critical values in* $[a, b]$, *where*

$$m = \delta[dim(V \cap W) - codim(V + W)]$$

*and* $\delta = 1$ *if* $G = \mathbb{Z}_2$ *or* $\delta = \frac{1}{2}$ *if* $G = S^1$.

Next, we describe the differential equation scenario to which the above theorem will apply. Following [8], we start by introducing some *structural hypotheses* ($(\mathcal{L}_1) - (\mathcal{L}_3)$, $(\mathcal{I})$ below) involving the underlying spaces as well as the linear part $\mathcal{L}u$ of the equation in $(P)$.

Let $H$ be a closed subspace of $L^2(\Omega, \mathbb{R}^n)$ endowed with its usual inner product $(\cdot|\cdot)$ and norm $|\cdot|$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain, and let $\mathcal{L} : D(\mathcal{L}) \subset H \longrightarrow H$ be an unbounded, selfadjoint operator having a discrete pure point spectrum and no essential spectrum, that is, the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ consists solely of isolated eigenvalues with finite multiplicities

$$\ldots \leq \lambda_{-2} \leq \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \ldots \qquad (2.1)$$

with corresponding eigenfunctions forming an orthonormal basis for $H$ ($\lambda_0 = 0$ may or may not be an eigenvalue). We assume that

$$D(\mathcal{L}) \subset E \subset H \ , \ |u| \leq C_1\|u\| \ \ \forall u \in E \ , \qquad (\mathcal{L}_1)$$

$$(\mathcal{L}u|v) \leq C_2\|u\|\|v\| \ \ \forall u, v \in D(\mathcal{L}) \ , \qquad (\mathcal{L}_2)$$

for some constants $C_1, C_2 > 0$, where we recall that $\|\cdot\|$ denotes the norm in the Hilbert space $E$. It follows from $(\mathcal{L}_1), (\mathcal{L}_2)$ and the Riesz-Fréchet theorem that the bilinear, symmetric form $(u,v) \mapsto (\mathcal{L}u|v)$ defined on $D(\mathcal{L}) \times D(\mathcal{L})$ extends to a bounded, bilinear, symmetric form $a : E \times E \longrightarrow \mathbb{R}$ given by

$$a(u,v) = \langle Lu, v\rangle \ , \qquad (2.2)$$

where $L : E \longrightarrow E$ is a bounded, selfadjoint operator. Similarly, we have that

$$(u|v) = \langle Tu, v\rangle \ \ \forall u, v \in E \ , \qquad (2.3)$$

where $T : E \longrightarrow E$ is a bounded, positive, selfadjoint operator. In addition, we assume that

$$E \hookrightarrow H \text{ is compact } \textit{and } LTu = TLu \ \ \forall u \in E \ . \qquad (\mathcal{L}_3)$$

It follows that $LT$ is compact, selfadjoint and, in view of the spectral theorem, the operators $L$ and $T$ can be *simultaneously diagonalized*, that is, there exists an orthonormal basis $\{\phi_j | j \in \mathbb{Z}\}$ for $E$ and sequences $\{\mu_j\}, \{\nu_j\} \subset \mathbb{R}$ such that

$$L\phi_j = \mu_j\phi_j \ , \ T\phi_j = \nu_j\phi_j \ \ \forall j \in \mathbb{Z} \ ,$$

where we may assume, after suitable relabelling, that $j\mu_j \geq 0$ and that $\mu_j/\nu_j = \lambda_j$ are the eigenvalues of the operator $\mathcal{L}$ listed in (2.1) (or, equivalently, of the eigenvalue problem $Lu = \lambda Tu$ for the operator $L$ with respect to $T$).

Now, given $\alpha \in \mathbb{R}$, let us consider the quadratic form

$$q_a(u) = \frac{1}{2}\langle Lu, u\rangle - \frac{1}{2}\alpha|u|^2 = q(u) - \frac{1}{2}\alpha|u|^2 \ , \quad u \in E \ , \qquad (2.4)$$

and the subspaces

$$\begin{aligned} V_\alpha &= \overline{span}\{\phi_j | \lambda_j < \alpha\} \ , \\ Z_\alpha &= span\{\phi_j | \lambda_j = \alpha\} \ , \\ W_\alpha &= \overline{span}\{\phi_j | \lambda_j > \alpha\} \ , \end{aligned} \qquad (2.5)$$

where $\overline{span}$ denotes the closed span in the space $E$. We observe that $V_\alpha, Z_\alpha$ and $W_\alpha$ are orthogonal, invariant subspaces under both operators $L$ and $T$ and $E = V_\alpha \oplus Z_\alpha \oplus W_\alpha$.

Finally, as our last *structural hypothesis*, we assume that there exists a Banach $X$, with norm $|\cdot|_X$, such that $E \hookrightarrow X$ and the following *interpolation* type inequalities hold,

$$\begin{array}{ll} (i) & |u|_X \leq \psi(u)^{1-t}\|u\|^t \quad \forall u \in E \ , \\ (ii) & |u| \leq C|u|_X^{1-s}\|u\|^s \quad \forall u \in E \ , \end{array} \tag{$\mathcal{I}$}$$

where $C > 0$, $t, s \in (0,1)$ and $\psi : X \longrightarrow \mathbb{R}_+$ is a positive homogeneous function of degree 1.

On the other hand, concerning the $C^1$ functional $N : E \longrightarrow \mathbb{R}$, we assume that $\nabla N$ satisfies the growth condition

$$\|\nabla N(u)\| \leq b|u|_X^r + d \quad \forall u \in E \ , \tag{$N_1$}$$

for some $r, b, d \geq 0$, and that $N$ is *nonquadratic at infinity* in the sense that either

$$\langle \nabla N(u), u\rangle - 2N(u) \geq a\psi(u)^\mu - c \quad \forall u \in E \ , \tag{$N_2^+$}$$

or

$$\langle \nabla\ N(u), u\rangle - 2N(u) \leq -a\psi(u)^\mu + c \quad \forall u \in E \ , \tag{$N_2^-$}$$

holds true for some constants $a, c > 0$ and some $\mu > 0$. Then, the following results can be proved (cf. Proposition 2.1 and 2.6 in [8]):

**Proposition 2.1** *Assume $(\mathcal{L}_1) - (\mathcal{L}_3)$ and that $\mu = 0$ is at most an isolated eigenvalue with finite multiplicity of the bounded operator $L$ (representing $\mathcal{L}$ in $E$). Then, given $\alpha \in \mathbb{R}$, there exists $\nu = \nu_\alpha > 0$ such that*

$$\begin{array}{lll} q_\alpha(u) \leq - & \nu\|u\|^2 & \forall u \in V_\alpha \ , \\ q_\alpha(u) = & 0 & \forall u \in Z_\alpha \ , \\ q_\alpha(u) \geq & \nu\|u\|^2 & \forall u \in W_\alpha \ . \end{array} \tag{2.6}$$

**Proposition 2.2** *Let conditions $(\mathcal{L}_1) - (\mathcal{L}_3)$ and $(\mathcal{I})$ hold true and assume $N \in C^1(E, \mathbb{R})$ satisfies $(N_1), (N_2^+)$ (or $(N_2^-)$) with $\nabla N : E \longrightarrow E$ compact. Then the functional $J(u) = q(u) - N(u)$ satisfies the compactness condition $(Ce)_c$ for all $c \in \mathbb{R}$ provided that $tr < 1$.*

## 3. Main Result and Applications

We are now ready to state and prove our main result.

**Theorem 3.1** *Under the conditions of Proposition 2.2, assume that the operator $\mathcal{L}$ and the functional $N \in C^1(E, \mathbb{R})$ are invariant under a unitary representation $\{T(g)\}$ of $G$ on $E$, where $G = \mathbb{Z}_2$ or $G = S^1$. Further, assume that $N(u)$ is bounded whenever $|u|$ is bounded,*

$$\limsup_{|u|\to 0} \frac{2N(u)}{|u|^2} < \lambda_l \leq \lambda_k < \liminf_{|u|\to\infty} \frac{2N(u)}{|u|^2} \ , \tag{$N_3$}$$

*where $\lambda_l \leq \lambda_k$ are eigenvalues of $\mathcal{L}$, and that*

$$\sigma(\mathcal{L}_0) \subset \{\lambda_j | j \leq k\} \quad or \quad \sigma(\mathcal{L}_0) \subset \{\lambda_j | j \geq l\} \ , \tag{$*$}$$

$$J(u) \leq 0 \quad if \quad u \in Fix(G) \ , \ \nabla J(u) = 0 \ , \tag{$**$}$$

*where $\mathcal{L}_0$ is the restriction of $\mathcal{L}$ to the subspace $Fix(G)$. Then, $J$ has at least $m$ orbits of critical points outside $Fix(G)$, where $m = \delta \sum_{j=l}^{k} dim\ ker(\mathcal{L} - \lambda_j)$ and $\delta = 1$ if $G = \mathbb{Z}_2$ or $\delta = \frac{1}{2}$ if $G = S^1$.*

**Proof.** Recalling the definitions (2.5), let us define the subspaces

$$\begin{aligned} V &= V_{\lambda_k} \oplus Z_{\lambda_k} = \overline{span}\{\phi_j | \lambda_j \leq \lambda_k\} \ , \\ W &= Z_{\lambda_l} \oplus W_{\lambda_l} = \overline{span}\{\phi_j | \lambda_j \geq \lambda_l\} \ . \end{aligned} \tag{3.1}$$

Then, using $(N_3)$ and the fact that $N(u)$ is bounded for $|u|$ bounded, we have

$$N(u) \geq \frac{1}{2}\beta|u|^2 - K \quad \forall u \in E \ , \tag{3.2}$$

$$N(u) \leq \frac{1}{2}\alpha|u|^2 \quad \forall |u| \leq \epsilon \ , \tag{3.3}$$

for some $\epsilon, K > 0$ and $\alpha < \lambda_l \leq \lambda_k < \beta$. Also, $(N_1)$ and the continuous embedding $E \hookrightarrow X$ imply that

$$\frac{N(u)}{\|u\|^{r+1}} \leq b_1 + \frac{d_1}{\|u\|^r} \leq M \quad \forall |u| \geq \epsilon \ , \tag{3.4}$$

for some $b_1,\ d_1,\ M > 0$, since $|u| \leq C_1\|u\| \ \ \forall u \in E$.

Therefore, using (3.2) and Proposition 2.1 we obtain the estimate

$$J(u) \leq q(u) - \frac{1}{2}\beta|u|^2 + K = q_\beta(u) + K \leq -\nu_\beta\|u\|^2 + K \quad \forall\, u \in V\ , \tag{3.5}$$

so that we have

$$J(u) \leq b \ \ \forall\, u \in V\ , \tag{$i$}$$

for some $b = K > 0$. On the other hand, (3.3) and (3.4) give

$$N(u) \leq \frac{1}{2}\alpha|u|^2 + M\|u\|^{r+1} \quad \forall u \in E\ ,$$

and hence

$$\begin{aligned} J(u) \ &\geq q(u) - \tfrac{1}{2}\alpha|u|^2 - M\|u\|^{r+1} = q_\alpha(u) - M\|u\|^{r+1} \\ &\geq \nu_\alpha\|u\|^2 - M\|u\|^{r+1} = (\nu_\alpha - M\|u\|^{r-1})\|u\|^2 \quad \forall\, u \in W\ . \end{aligned} \tag{3.6}$$

Since $r > 1$, we can find $0 < a < b$ and $\rho > 0$ such that

$$J(u) \geq a \ \ if \ \ \|u\| = \rho\ ,\ u \in V\ . \tag{$ii$}$$

Finally, $(**)$ obviously gives condition $(iv)$ of Theorem 2.1, whereas condition $(iii)$ of that theorem follows from $(*)$.

Therefore, since $V + W = E$ and $V \cap W = Z_{\lambda_l} \oplus \ldots \oplus Z_{\lambda_k} = span\{\phi_j | \lambda_l \leq \lambda_j \leq \lambda_k\}$, we can use Theorem 2.1 to conclude that $J$ possesses at least $m = \delta \sum_{j=l}^{k} dim\ ker(\mathcal{L} - \lambda_j)$ orbits of critical points with critical values in $[a, b]$, hence outside $Fix(G)$ by $(iv)$. The proof of Theorem 3.1 is complete.

□

**I. Semilinear Wave Equations**

Next, as our first application, we consider the problem of finding multiple time-periodic solutions with period $T$ for the wave equation

$$u_{tt} - u_{xx} + f(u) = 0 \ \ ,\ \ 0 < x < \pi\ , \tag{$WE$)$_1$}$$

under Dirichlet boundary condition

$$u(t, 0) = u(t, \pi) = 0\ . \tag{$WE$)$_2$}$$

In order to avoid problems with *small divisors* the period $T$ should be a rational multiple of the length $\pi$ of the space domain.

Taking $T = \pi$ for simplicity and letting $\Omega = (0, \pi) \times (0, \pi)$, we consider the usual Hilbert space $L^2(\Omega)$ of square integrable functions $u : \Omega \longrightarrow \mathbb{R}$, endowed with the inner product

$$(u|v) = \sum_{k \in N, j \in Z} u_{k,j} \overline{v_{k,j}}$$

where $u = \sum u_{k,j} sin(kx) e^{2ijt}$, $v = \sum v_{k,j} sin(kx) e^{2ijt}$ with $u_{k,j}, v_{k,j} \in \mathbb{C}$, $u_{k,-j} = \overline{u_{k,j}}$, $v_{k,-j} = \overline{v_{k,j}}$, for $k \in I\!N$ and $j \in \mathbb{Z}$. From now on, unless stated otherwise, all summation signs are taken over $k \in I\!N$ and $j \in \mathbb{Z}$. Also for $u \in L^2(\Omega)$ as above, we let

$$\|u\|^2 := \sum (1 + |k^2 - 4j^2|) |u_{k,j}|^2$$

and define the Sobolev space

$$W^{1,2}_{0,per} := \{u \in L^2(\Omega) \mid \|u\| < \infty\} ,$$

whose norm $\| \cdot \|$ clearly derives from the inner product

$$\langle u, v \rangle := \sum (1 + |k^2 - 4j^2|) u_{k,j} \overline{v_{k,j}} \ .$$

Now, let $\Box$ be the selfadjoint *d'Alembertian* operator $\Box = \partial_x^2 - \partial_t^2$ on $L^2(\Omega)$ with domain

$$D(\Box) = W^{2,2}_{0,per} := \{u \in L^2(\Omega) \mid \sum (1 + |k^2 - 4j^2|^2) |u_{k,j}|^2 < \infty\}$$

and defined by $\Box u := \sum (4j^2 - k^2) u_{k,j} sin(kx) e^{2ijt}$ if $u = \sum u_{k,j} sin(kx) e^{2ijt} \in D(\Box)$.

Also, consider the closed subspace $H = \{ u \in \mathrm{L}^2(\Omega) \mid u(t, \pi - x) = u(t, x) \ a.\ e. \}$ of $L^2(\Omega)$ and the restriction $\mathcal{L} = \Box|_H$ with domain $D(\mathcal{L}) = D(\Box) \cap H$. Clearly, the space $H$ is invariant under $\Box$ (that is, $\Box u \in H$ if $u \in D(\Box) \cap H$) and $\mathcal{L} : D(\mathcal{L}) \subset H \longrightarrow H$ is a selfadjoint operator.

Finally, we define the Sobolev space $E = W^{1,2}_{0,per} \cap H$ and consider the *unitary representation* of the group $S^1 = \mathbb{R}/\pi\mathbb{Z}$ on the space $E$ given by $T(\theta)u(t, x) =$

$u(t+\theta,x)$, $\theta \in [0,\pi)$. Then, the *fixed set* of this representation consists of those functions $u(t,x)=u(x)$ of $E$ which are $t$-independent.

We point out the fact that subspaces of $L^2(\Omega)$ such as $H$ above have the important property of being *transversal* (cf. [7]) to the infinite-dimensional kernel, $ker(\Box)=\{u=\sum u_{k,j} sin(kx)e^{2ijt} \in D(\Box) \mid k=2|j|\ \}$, of the *d'Alembertian*; in particular, we have $u_{k,j}=0$ if $k=2|j|$ and $u\in H$, so that $ker(\mathcal{L})=\{0\}$ and $\mathcal{L}$ (unlike $\Box$) has no essential spectrum.

From the above definitions it is clear that $D(\mathcal{L})\subset E\subset H$ with $|u|\leq \|u\|\ \ \forall u\in E$. Also, using the *transversality* property of $H$ and the fact that

$$\|sin(kx)e^{2ijt}\|^2=(1+|4j^2-k^2|)|sin(kx)e^{2ijt}|^2\ ,$$

it is not hard to show that the embedding $E\hookrightarrow H$ is compact. In fact, one can show (see [8]) that all the *structural hypotheses* $(\mathcal{L}_1)-(\mathcal{L}_3)$, $(\mathcal{I})$ assumed in Section 2 hold true in this case, with $\psi(u)=|u|_\mu$, $X=L^{r_1}(\Omega)$ for suitable $r_1>\mu$, and where the operators $L,T:E\longrightarrow E$ are given by

$$Lu=\sum\frac{4j^2-k^2}{1+|4j^2-k^2|}u_{k,j}sin(kx)e^{2ijt}\ , \tag{3.7}$$

$$Tu=\sum\frac{u_{k,j}}{1+|4j^2-k^2|}sin(kx)e^{2ijt}\ . \tag{3.8}$$

Finally, we note that the operator $\mathcal{L}$ is invariant under the representation $\{T(\theta)\}$ and its spectrum consists of the eigenvalues $\lambda_{kj}=4j^2-k^2$, where $j=0,1,2,\ldots$ and $k=1,3,5,\ldots$ ($k$ *odd*). Moreover, the spectrum of the restriction $\mathcal{L}_0=\mathcal{L}|_{Fix(S^1)}$ is given by

$$\sigma(\mathcal{L}_0)=\{-k^2\mid k=1,3,5,\ldots\}\ . \tag{3.9}$$

Next, letting $F(u)=\int_0^u f(s)\,ds$, we are going to apply Theorem 3.1 to the functional

$$J(u)=\frac{1}{2}\langle Lu,u\rangle-\int_0^\pi\int_0^\pi F(u)\ dtdx\ . \tag{3.10}$$

First of all, we note (cf. [7]) that $E=W^{1,2}_{0,per}\cap H$ is continuously embedded in $L^q\cap H$ for all $q\geq 1$. From this, it is not hard to show that $J$ is well-defined and of class $C^1$ on the space $E$ provided that $f(u)$ satisfies the growth condition

$$|f(u)| \leq A|u|^r + B \quad \forall u \in \mathbb{R} \;, \tag{$f_1$}$$

for some $A, B > 0$ and $r > 1$. In fact, since the Nemytskii operator $u(t,x) \longmapsto f(u(t,x))$ obviously leaves the subspace $H \subset L^2(\Omega)$ invariant, it follows that

$$\langle \nabla J(u), h\rangle = \langle Lu, h\rangle - \int_0^\pi \int_0^\pi f(u)h \; dtdx \quad \forall u, h \in E \;,$$

so that the critical points of $J$ are the *weak* solutions $u \in E$ of the problem $(WE)$. Finally, since $E = W^{1,2}_{0,per} \cap H \hookrightarrow L^q \cap H$ for all $q \geq 1$ and the embedding $E \hookrightarrow H$ is compact, we conclude that $\nabla N : E \longrightarrow E$, defined by

$$\langle \nabla N(u), h\rangle = \int_0^\pi \int_0^\pi f(u)h \; dtdx \quad \forall u, h \in E \;,$$

is a compact mapping.

**Proof of Theorem 1.1** We have seen above that all the basic hypotheses involving the underlying spaces and the operator $\mathcal{L}$ hold true in this present situation. In addition, it is not hard to check that $(f_1)$, $(f_2)$ and $(f_3)$ imply $(N_1)$, $(N_2^+)$ and $(N_3)$, respectively.

On the other hand, since (3.9) gives that $\sigma(\mathcal{L}_0) \subset (-\infty, 0]$ and $0 < \lambda_l \leq \lambda_k$, we have that condition $(*)$ is satisfied. Moreover, since (3.7) implies that $\langle Lu, u\rangle \leq 0$ for all $u \in Fix(S^1)$ and since $F(u) \geq 0$ by assumption, we have that $J(u) \leq 0$ for all $u \in Fix(S^1)$, so that $(**)$ also holds true.

Therefore, since $ker(\mathcal{L} - \lambda)$ is *even-dimensional* for any eigenvalue $\lambda$, it follows from Theorem 3.1 that $J$ has at least $m = k - l + 1$ orbits of critical points outside $Fix(S^1)$. In other words, problem $(WE)_1$, $(WE)_2$ has at least $m = k - l + 1$ ($t$-dependent) weak solutions which are $\pi$-periodic in $t$ and geometrically distinct.

□

## II. Noncooperative Elliptic Systems

Now, as our second application, we consider the problem of finding multiple solutions of noncooperative elliptic systems

$$\begin{cases} -\Delta z &= F_z(z,w) \text{ in } \Omega \\ \Delta w &= F_w(z,w) \text{ in } \Omega \\ z &= w = 0 \text{ on } \partial\Omega \ , \end{cases} \quad , \tag{ES}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain.

Letting $H = L^2(\Omega, \mathbb{R}^2)$ and denoting $u = (z, w)$ , $\nabla F = (F_z, F_w)$ and

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \ \vec{\Delta} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} ,$$

problem $(ES)$ can be rewritten as

$$\mathcal{L}u = \nabla F(u) \ ,$$

where $\mathcal{L} : D(\mathcal{L}) \subset H \longrightarrow H$ is the selfadjoint operator given by $\mathcal{L}u = -\vec{\Delta}Ru$ with domain

$$D(\mathcal{L}) = W^{2,2}(\Omega, \mathbb{R}^2) \cap W_0^{1,2}(\Omega, \mathbb{R}^2) \ .$$

Choosing the norm $\|\cdot\|$ in $E := W_0^{1,2}(\Omega, \mathbb{R}^2)$ induced by the inner product

$$\langle (z, w), (\varphi, \psi) \rangle := \int_\Omega \nabla z \cdot \nabla \varphi dx + \int_\Omega \nabla w \cdot \nabla \psi dx \ ,$$

standard arguments show that the embedding $E \hookrightarrow H$ is compact and that hypotheses $(\mathcal{L}_1)$ and $(\mathcal{L}_2)$ are satisfied. Also, considering the $Z_2$-representation $\{T(0), T(1)\}$ on $E$ given by $T(0) = Identity$, $T(1) = -Identity$, and recalling the $F$ is an even function, it is clear that both the operator $\mathcal{L}$ and the functional $N(u) = \int_\Omega F(u)dx$ are invariant under this representation and that $Fix(Z_2) = \{0\}$. Finally, one can show (cf. [8]) that the remaining structural hypotheses $(\mathcal{L}_3)$ and $(\mathcal{I})$ are satisfied with $\psi(u) = |u|_\mu$, $X = L^{r_1}(\Omega, \mathbb{R}^2)$ for suitable $\mu < r_1 < 2N/(N-2)$, and where $L, T : E \longrightarrow E$ are given by $Lu = Ru$ and $\langle Tu, v \rangle = \int_\Omega u \cdot v dx$. In particular, the eigenvalues of $\mathcal{L}$ (that is, of the problem $Lu = \lambda Tu$) are given by $\alpha_j, -\alpha_j$, with corresponding eigenfunctions $(\varphi_j, 0), (0, \varphi_j)$, where

$\{\alpha_j\}$ and $\{\varphi_j\}$, $j \in I\!N$, denote the eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$.

**Proof of Theorem 1.2.** As before, we are going to apply Theorem 3.1 to the pertinent functional

$$J(u) = \frac{1}{2}\langle Lu, u\rangle - \int_0^\pi \int_0^\pi F(u)\, dtdx = q(u) - N(u)\ . \tag{3.11}$$

It is well-known that, under the growth condition

$$|\nabla F(u)| \leq C|u|^r + D \quad \forall u \in \mathbb{R}^2\ , \tag{$h_1$}$$

where $1 \leq r < (N+2)/(N-2)$, $N \geq 3$, the functional $J$ is of class $C^1$ on $E$ and its critical points are the weak solutions of $(ES)$. And the compact embedding $E \hookrightarrow H$ implies that $\nabla N : E \longrightarrow E$ is a compact mapping.

On the other hand, using integration and Holder's inequality, conditions $(N_1)$, $(N_2^+)$ and $(N_3)$ of Theorem 3.1 follow from $(h_1)$, $(h_2)$ and $(h_3)$, respectively.

Finally, in view of the fact that $Fix(Z_2) = \{0\}$ and $J(0) = 0$, conditions $(*)$ and $(**)$ are clearly satisfied.

Therefore, Theorem 3.1 implies that $J$ has at least $m = k - l + 1$ pairs of nontrivial critical points. In other words, problem $(ES)$ has at least $m = k-l+1$ pairs of nontrivial (weak) solutions $u = (z, w) \in H_0^1(\Omega) \times H_0^1(\Omega)$ .

□

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