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ON THE COMPRESSIBLE EULER EQUATIONS IN THERMOELASTICITY

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Abstract

We discuss some recent developments in studying global entropy solutions with large initial data of the compressible Euler equations in thermoelasticity, for a class of constitutive relations, which is compatible with the first law of thermodynamics. The asymptotic decay of large entropy solutions without bounded variation is shown. The uniqueness and stability of Riemann solutions in a class of large entropy solutions in L^∞ or BV are analyzed.

Resumo

Discutimos desenvolvimentos recentes no estudo de soluções globais entrópicas das equações de Euler em termoelasticidade, para uma classe de relações constitutivas que é compatível com a primeira lei da termodinâmica. O decaimento assintótico de soluções entrópicas de grande amplitude sem variação limitada é mostrado. A unicidade e estabilidade das soluções de Riemann numa classe de soluções entrópicas com grande amplitude em L^∞ ou BV são analisadas.

1. Introduction

The balance laws of mass, momentum, and energy for one-dimensional elastic media that do not conduct heat are expressed, in Lagrangian coordinates, by the equations

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t \left(e + \frac{1}{2}u^2\right) + \partial_x(up) = 0, \end{cases} \tag{1.1}$$

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where v, u, p, and e denote respectively deformation gradient (specific volume for fluids, strain for solids), velocity, pressure, and internal energy. Other relevant fields are the entropy s and the temperature θ. The system (1.1) is equivalent to the system (5.1) in the Euler coordinates, and hence we focus on (1.1) for notational convenience throughout the paper (see Remark 2 in Section 5).

The above system of conservation laws is complemented by the Clausius inequality

$$\partial_t s \geq 0, \tag{1.2}$$

which expresses the second law of thermodynamics.

Selecting (v, u, s) as the state vector, we write the constitutive equations

$$e = \hat{e}(v, s), \quad p = \hat{p}(v, s), \quad \theta = \hat{\theta}(v, s) \tag{1.3}$$

satisfying the first law of thermodynamics: $\theta ds = de + pdv$, that is,

$$\hat{p} = -\hat{e}_v, \quad \hat{\theta} = \hat{e}_s, \tag{1.4}$$

whose role is to ensure that (1.2) holds automatically (as an equality) for any smooth solution of (1.1), that is, to make system (1.1) close.

The system (1.1) is a prototype of hyperbolic systems of conservation laws (see [21]):

$$\partial_t U + \partial_x F(U) = 0. \tag{1.5}$$

Under the standard assumptions $\hat{p}_v < 0, \hat{\theta} > 0$, the system (1.1) is strictly hyperbolic. It is well known that solutions of the initial value problem, starting out from smooth initial data, generally develop discontinuities that propagate as shock waves. Thus only a theory of weak solutions in the large may be feasible. As usual, we say that $U(x,t) = (v(x,t), u(x,t), E(x,t)), E = e + \frac{1}{2}u^2$ (the total energy), is a weak solution of (1.1) in Π_T with initial data

$$U|_{t=0} = U_0(x), \tag{1.6}$$

if, for all $\phi \in C^1(\Pi_T)$ with compact support in Π_T, one has

$$\int\int_{\Pi_T} \{U\phi_t + F(U)\phi_x\}\, dx\, dt + \int_{\mathbb{R}} U_0(x)\phi(0, x)\, dx = 0, \tag{1.7}$$

where $F(U) = (-u, p(v,s), up(v,s))$. Since the mapping from (v,u,E) to (w,u,s) is one-to-one, we will not distinguish these two coordinates in terms of solutions. In the context of weak solutions, (1.2) is no longer a consequence of (1.1), (1.3), and (1.4), but rather an independent condition identifying physically admissible solutions of (1.1).

When the initial data have small total variation, a global BV solution of (1.1) can be constructed by the random choice method of Glimm [14]. For large L^∞ initial data, the situation is much more complicated. In this paper we discuss some recent developments in studying large entropy solutions for the compressible Euler equations with the following class of constitutive relations, proposed by Chen-Dafermos [3],

$$v = w + \alpha s, \;\; p = h(w), \;\; e = H(w) + \beta s, \;\; \theta = \alpha h(w) + \beta, \tag{1.8}$$

where $h(w) > 0$ is a function with $h'(w) < 0$, α and β are positive constants, and

$$H(w) \equiv -\int^{w} h(\omega) d\omega. \tag{1.9}$$

Of course, (1.8) can be written in the form (1.3) as

$$e = H(v - \alpha s) + \beta s, \;\; p = h(v - \alpha s), \;\; \theta = \alpha h(v - \alpha s) + \beta. \tag{1.10}$$

Observe that the relations (1.10) are compatible with (1.4).

The model (1.8) is special. Even so, when one is dealing with solutions in which the entropy does not deviate much from some constant value $\bar{s}$ (i.e. relatively weak shocks), one may obtain a reasonable approximation of general constitutive equations (1.3) by the equations of the form (1.10) upon choosing

$$\alpha = -\hat{p}_s(\bar{v},\bar{s})/\hat{p}_v(\bar{v},\bar{s}), \quad \beta = \hat{\theta}(\bar{v},\bar{s}) - \alpha\hat{p}(\bar{v},\bar{s}), \quad h(w) = \hat{p}(w + \alpha\bar{s}, \bar{s}), \tag{1.11}$$

so that $\hat{e}(v,\bar{s})$ and $\hat{p}(v,\bar{s})$ are matched for all v and the values of $\hat{\theta}, \hat{p}_s$, and $\hat{\theta}_v$ are matched at a particular point $(\bar{v},\bar{s})$ with $\bar{v} > 0$ and $\bar{s} > 0$. For the polytropic gas, $\alpha > 0$ and $\beta > 0$.

In Section 2, we recall the results of the existence and compactness of entropy solution operators established in [3].

In Section 3, we show the asymptotic decay of periodic entropy solutions. An interesting feature here is that, because of the linear degeneracy of the second characteristic family, one can not expect the decay of all components of the solutions. However, it is proved that some important quantities such as the velocity, the pressure, and the temperature do decay as $t \to \infty$.

In Section 4, we discuss the uniqueness and stability of Riemann solutions in the class of entropy solutions in L^∞ or BV with $L^1 \cap L^\infty$ or $L^1 \cap BV$ initial perturbation. A framework is analyzed to ensure that the uniqueness of Riemann solutions in the class of L^∞ or BV entropy solutions and the compactness of the self-similar scaling sequence imply the stability of Riemann solutions in a certain sense. Then we discuss the uniqueness of Riemann solutions in the class of L^∞ or BV entropy solutions. The first is the uniqueness of Riemann solutions in the class of L^∞ entropy solutions, provided that the initial left and right states of the Riemann data are connected only by rarefaction curves of the first and third families and, possibly, a contact discontinuity curve of the second family. No assumption of small oscillation is required here. Some basic facts of divergence-measure fields in [7] are used. Combining this uniqueness result with a compactness result in Section 2 yields the asymptotic stability of shock-free Riemann solutions with respect to the initial perturbation (with the entropy function $s(x,t)$ in a weaker sense). The second is the uniqueness of general Riemann solutions in the class of BV solutions. Again, no assumption of small oscillation is required for this case. This result together with the compactness of bounded sets in BV implies the asymptotic stability of Riemann solutions in the class of BV entropy solutions with $O(T_0)$ growth of its total variation over $[0,T_0] \times \mathbb{R}$.

Finally, we comment on some essential differences between our asymptotic results and earlier results on related problems. First, there has been a large literature on the asymptotic stability of viscous shock profiles and rarefaction waves (see, *e.g.*, [16, 19, 20, 27, 23, 30] and references cited therein). In general, their analysis is based on energy estimates and gives more precise information about the large-time behavior of the solutions, besides implying the asymptotic stability in the sense of (4.1). However, they are suitable only for viscous equations and, as far as we know, it has not been possible to treat general large perturbation of Riemann data with both shock and rarefaction waves for such systems by a similar analysis. There is also an important analysis of large-time behavior of Glimm solutions of hyperbolic conservation laws introduced by Liu (see [25, 26]), which is designed specifically for solutions obtained from the Glimm method. In his analysis the asymptotic approach to the Riemann solution is obtained in terms of a norm, which is equivalent to the total variation for small initial data. It is not difficult to see that the results obtained for 2×2 systems in [23] imply the asymptotic stability of the Riemann solution in the class of solutions, obtained from the Glimm method, in the sense of (4.1). The main motivation of our program is to develop some approaches that are applicable to general large entropy solutions, constructed by any method, for hyperbolic systems of conservation laws.

2. Existence and Compactness of Entropy Solutions

In this section we recall some results proved in [3], which will be referred later in Sections 3-4. We refer to [3] for the proofs of both theorems of this section. Consider the Cauchy problem for (1.1) and (1.8) with initial data

$$(w, u, s)|_{t=0} = (w_0(x), u_0(x), s_0(x)), \tag{2.1}$$

satisfying

$$|w_0(x)| \leq C_0, \qquad |v_0(x)| \leq C_0, \qquad s_0(x) \in \mathcal{M}_{loc}(\mathbb{R}), \tag{2.2}$$

and

$$(w_0(x), u_0(x)) \in \Sigma_{C_1} \equiv \{(w, u) \,|\, -C_1 \leq u \pm \int_{\hat{w}}^{w} \sqrt{-h'(\omega)} d\omega \leq C_1\}, \tag{2.3}$$

which contains only physical admissible states. In particular, if $(w, u) \in \Sigma_{C_1}$, then $\theta = \alpha h(w) + \beta > 0$.

For concreteness, in this section we assume that $h(w)$ is a smooth function with $h'(w) < 0$ satisfying

$$h''(w) - 4\frac{\alpha h'(w)^2}{\alpha h(w)+\beta} \begin{cases} > 0, & \text{if} \quad w < \hat{w}, \\ < 0, & \text{if} \quad w > \hat{w}, \end{cases} \tag{2.4}$$

and α and β are positive constants. Then we have

Theorem 2.1 *There exists a global distributional solution* $(w(x,t), u(x,t), s(x,t))$ *for the Cauchy problem* (1.1), (1.8), (2.4), *and* (2.1)-(2.2), *satisfying*

$$(w,u) \in L^\infty(\mathbb{R}^2_+), \quad (s, s_t) \in \mathcal{M}_{loc}(\mathbb{R}^2_+), \quad \theta(w) \geq 0, \tag{2.5}$$

$$|s|\{[-cT_0, cT_0] \times [0, T_0]\} \leq CT_0^2, \tag{2.6}$$

for any $c > 0, T_0 > 0$, *with* $C > 0$ *independent of* T_0, *where* $|s|$ *denotes the variation measure associated with the signed measure* s. *Moreover,* $(w(x,t), u(x,t), s(x,t))$ *satisfies the entropy condition:*

$$\partial_t \eta(w,u) + \partial_x q(w,u) \leq 0, \qquad s_t \geq 0, \tag{2.7}$$

in the sense of distributions for any C^2 *entropy pair* $(\eta(w,u), q(w,u))$ *of the prototypical system*

$$\partial_t w - \partial_x u = 0, \qquad \partial_t u + \partial_x h(w) = 0, \tag{2.8}$$

for which the strong convexity condition holds:

$$\begin{aligned} &\theta\eta_{ww} - \alpha h'(w)\eta_w \geq 0, \quad \theta\eta_{uu} + \alpha\eta_w \geq 0 \\ &\qquad (\theta\eta_{ww} - \alpha h'(w)\eta_w)(\theta\eta_{uu} + \alpha\eta_w) - \eta_{wu}^2 \geq 0. \end{aligned} \tag{2.9}$$

Furthermore, if the initial data (2.1) *are periodic, then there exists a global periodic distributional solution satisfying* (2.5)-(2.7).

Observe that

$$\eta_*(w,u) = H(w) + \frac{1}{2}u^2, \qquad q_*(w,u) = uh(w) \tag{2.10}$$

is a strictly convex entropy-entropy pair satisfying (2.9) (with strict inequalities).

Theorem 2.2 *Assume that the sequence* $(w^T(x,t), u^T(x,t))$ *satisfies the following:*
(i) *There exists a constant* $C > 0$ *such that*

$$\|(w^T, u^T)\|_{L^\infty} \leq C;$$

(ii) *The sequence*

$$\partial_t \eta(w^T, u^T) + \partial_x q(w^T, u^T) \leq 0 \qquad \text{in the sense of distributions,}$$

for any C^2 *entropy pair* $(\eta(w,u),\ q(w,u))$ *of the system* (2.8) *satisfying* (2.9).

Then the sequence $(w^T(x,t), u^T(x,t))$ *is compact in* $L^1_{loc}(\mathbb{R}^2_+)$.

3. Decay of Entropy Solutions Without Bounded Variation

This section is concerned with the asymptotic decay of periodic entropy solutions without bounded variation of (1.1), (1.8)-(1.9), and (2.1). Our main result is the following.

Theorem 3.1. *Let* $(v(x,t), u(x,t), s(x,t))$, $|u(x,t)| + |v(x,t) - \alpha s(x,t)| \leq C$, *be a periodic entropy solution of* (1.1), (1.8)-(1.9), *and* (2.1) *with period* P *satisfying* (2.1)-(2.7). *Then the velocity* $u(x,t)$ *asymptotically decays to* $\bar{u} = \frac{1}{|P|}\int_P u_0(x)dx$ *in* $L^p(P)$, $1 \leq p < \infty$. *Moreover, the pressure* $p(w(x,t))$ *and the temperature* $\theta(w(x,t))$ *decay to*

$$\tilde{p} = p(\Theta^{-1}(\frac{1}{|P|}\int_P \Theta(w_0(x))dx)), \text{ and } \quad \tilde{\theta} = \theta(\Theta^{-1}(\frac{1}{|P|}\int_P \Theta(w_0(x))dx),$$

in $L^p(P), 1 \leq p < \infty$, *respectively, where* $\Theta(w) = \beta w + \alpha \int_0^w h(\omega)d\omega$.

Sketch of the proof. We give a sketch of the proof; see [5] for more details. Let $(v^T(x,t), u^T(x,t), s^T(x,t)) = (v(Tx,Tt), u(Tx,Tt), s(Tx,Tt))$ be the scaling sequence associated with the periodic solution $(v(x,t), u(x,t), s(x,t))$, where now the scaling of $s(x,t)$ must be taken in the sense of distributions. Using rescaling arguments, it is not difficult to verify that the condition (2.6) is also satisfied by s^T with the same constant $C > 0$. Theorem 2.2 implies $(w^T(x,t), u^T(x,t))$ is compact in $L^1_{loc}(\mathbb{R}^2_+)$. From the uniform boundedness of (w^T, u^T), we have that there exists a subsequence $\{T_k\}_{k=1}^{\infty}$, $T_k \to \infty$ as $k \to \infty$, such that

$$(w^{T_k}(x,t), u^{T_k}(x,t)) \to (w(x,t), u(x,t)), \quad a.e. \quad k \to \infty.$$

Then we conclude (*cf.* [5]) that the function $(w(x,t), u(x,t))$ depends only on t. Using the conservation of momentum

$$\partial_t u + \partial_x h(w) = 0$$

in the sense of distributions, we conclude that

$$u = \bar{u} \equiv \frac{1}{|P|}\int_P u_0(x)dx.$$

We now return to the equations in (1.1). For the limits in the sense of distributions of (v^T, p^T, e^T), $(\bar{v}, \bar{p}, \bar{e})$, we get

$$\partial_t \bar{v} = 0, \quad \partial_x \bar{p} = 0, \quad \partial_t \bar{e} = 0.$$

This implies

$$\partial_t(\beta w - \alpha H(w)) = \partial_t(\beta \bar{v} - \alpha \bar{e}) = 0$$

in the sense of distributions. Hence, the function $\Theta(w(x,t)) \equiv \beta w(x,t) - \alpha H(w(x,t))$ does not depend on t either. We obtain

$$\Theta(w(x,t)) \equiv \frac{1}{|P|}\int_P \Theta(w_0(x))dx.$$

Since $\Theta'(w) = \theta(w) > 0$, $\Theta(w)$ is a monotone function. Therefore, $w(x,t)$ also does not depend on t. In fact, one has

$$w(x,t) = \tilde{w} \equiv \Theta^{-1}(\frac{1}{|P|}\int_P \Theta(w_0(x))dx).$$

The entropy inequality (2.7) with $\eta = \eta_*(w, u)$, given in (2.10), implies (*cf.* [5]) that there exists a set $\mathcal{T} \subset (0, \infty)$, with $meas((0,\infty) - \mathcal{T}) = 0$, such that

$$\int_P |(u(x,t) - \bar{u}, w(x,t) - \tilde{w})|^p dx \to 0, \quad t \to \infty,\ t \in \mathcal{T}, \quad \text{for any } 1 \le p < \infty.$$

The decay of $p(w)$ and $\theta(w)$ follows from that of w.

□

4. Uniqueness and Stability of Entropy Solutions

First we have the following observation.

Theorem 4.1. *Let $\mathcal{S}(\mathbb{R}^2_+)$ denote a class of functions defined on $\mathbb{R}^2_+$. Assume that the Cauchy problem (1.5)-(1.6) satisfies the following.*

(i) *The Riemann solution $R(x/t)$ is unique in the class $\mathcal{S}(\mathbb{R}^2_+)$;*

(ii) *Given an entropy solution of (1.5)-(1.6), $U \in \mathcal{S}(\mathbb{R}^2_+)$, the sequence $U^T(x,t) \equiv U(Tx, Tt)$ is compact in $L^1_{loc}(\mathbb{R}^2_+)$, and any limit function of its subsequences is still in $L^1_{loc} \cap \mathcal{S}(\mathbb{R}^2_+)$.*

Then the Riemann solution $R(x/t)$ with Riemann data $R_0(x)$ is asymptotically stable in $\mathcal{S}(\mathbb{R}^2_+)$ with respect to the corresponding initial perturbation $P_0(x) \in L^1(\mathbb{R})$ in the following sense:

$$\frac{1}{T}\int_0^T |U(t\xi, t) - R(\xi)| dt \to 0, \quad \text{in } L^1_{loc}(\mathbb{R}), T \to \infty, \tag{4.1}$$

where $U(x,t)$ is an entropy solution of (1.5)-(1.6) taking $U_0(x) = R_0(x) + P_0(x)$.

In this section the class $\mathcal{S}(\mathbb{R}^2_+)$ will be always either a subset of $BV_{loc}(\mathbb{R}^2_+) \cap L^\infty(\mathbb{R}^2_+)$, or a subset of $L^\infty(\mathbb{R}^2_+)$.

Theorem 4.1 indicates that the compactness of scaling sequences and the uniqueness of Riemann solutions imply the asymptotic stability of Riemann solutions in the sense of (4.1).

For BV solutions, the compactness of the scaling sequence is obtained through the following observation.

Lemma 4.1 *Assume* $U(x,t) \in BV_{loc}(\mathbb{R}^n_+ \times (0,\infty))$ *satisfies*

$$|\nabla_{(x,t)}U|\{\{|x| \leq cT_0\} \times (0,T_0)\} \leq CT_0^n, \tag{4.3}$$

for any $c > 0$, $T_0 > 0$, *and some* $C > 0$ *independent of* T_0, *where* $|\nabla_{(x,t)}U|$ *is the variation measure associated with the signed measure* $\nabla_{(x,t)}U$. *Then* $U^T(x,t)$ *also satisfies* (4.2) *with the same constant* C. *In particular, for* $U \in L^\infty(\mathbb{R}^{n+1}_+)$, *the sequence* U^T *is compact in* $L^1_{loc}(\mathbb{R}^{n+1}_+)$.

This condition is satisfied by the entropy solutions possessing total variation in x uniformly bounded for all $t > 0$, which is the case for the solutions constructed by Glimm's method (see [14, 15]). Hence, the compactness follows from Helly's theorem for bounded sets in BV.

For L^∞ solutions of the systems considered here, the method of compensated compactness has been applied successfully and yields the compactness of uniformly bounded sequences of entropy solutions. See Section 2.

The uniqueness of Riemann solutions in the class of BV solutions for the 2×2 systems is due to DiPerna [13]. In this section we discuss uniqueness theorems for Riemann solutions for (i) L^∞ solutions with large oscillation and initial Riemann states connected only by rarefaction wave curves of the first and third families, and, possibly, a contact discontinuity curve of the second family; (ii) BV solutions with large oscillation and general Riemann initial states.

As we indicated above, once we have the compactness of the scaling sequence, the asymptotic problem reduces to the uniqueness problem of Riemann solutions of (1.1) and

$$(v,u,E)|_{t=0} = R_0(x) \equiv \begin{cases} (v_L, u_L, E_L), & x < 0, \\ (v_R, u_R, E_R), & x > 0. \end{cases} \tag{4.3}$$

Therefore, in what follows, we mainly study the uniqueness problem with the aid of entropy analysis.

Throughout the following, we assume that $h(w)$ in (1.8) satisfies

$$h \in C^3, \quad h(w) > 0, \quad h'(w) < 0, \quad \text{and} \quad h''(w) > 0, \tag{4.4}$$

for w in the region of interest.

4.1. 3×3 Euler Equations: Shock-Free Riemann Solutions.

We first prove the uniqueness of large Riemann solutions in the class of L^∞ solutions, when the Riemann solutions do not contain shock waves. In this case, besides rarefaction waves of the first and third families, it may contain a contact discontinuity of the second family.

Theorem 4.2. *Let $R(x/t)$ be the classical shock-free Riemann solution of* (1.1) *and* (4.3). *Let $U(x,t) = (v(x,t), u(x,t), E(x,t)) \in L^\infty(\Pi_T;\mathbb{R}^3)$ be any weak solution of* (1.1) *and* (2.1) *in Π_T, satisfying* (2.7) *in the sense of distributions. Assume* (4.4) *holds. Then $U(x,t) = R(x/t)$, a.e. in Π_T.*

Sketch of the proof. We sketch the proof as follows; see [6] for further details. Let $W(x,t)$ and $\bar{W}(x,t)$ be the projections of $U(x,t)$ and $R(x/t)$ on the w–u plane. We notice that $\bar{W}$ is a Lipschitz solution of (2.8) for $t > 0$. Indeed, by assumption, $R(x/t)$ does not contain any shock discontinuities, and s is constant along rarefaction wave curves, while u and p (hence, also w) are constant along the contact discontinuity wave curves. We consider the strictly convex entropy pair $(\eta_*(w,u), q_*(w,u))$ in (2.10) for (2.8). Then $W(x,t)$ is a weak solution of

$$\partial_t w - \partial_x u = -\alpha \partial_t s = k(\partial_t \eta_*(w,u) + \partial_x q_*(w,u)), \quad \partial_t u + \partial_x p(w) = 0, \tag{4.5}$$

from (1.1) and (1.10), where $k = \alpha/\beta$.

Next, we consider the family of quadratic entropy pairs, parameterized by $\bar{W} = (\bar{w}, \bar{u})$, given by

$$\begin{aligned}
\alpha(W,\bar{W}) &= \eta_*(W) - \eta_*(\bar{W}) - \nabla\eta_*(\bar{W})(W - \bar{W}),\\
\beta(W,\bar{W}) &= q_*(W) - q_*(\bar{W}) - \nabla\eta_*(\bar{W})(f(W) - f(\bar{W})),
\end{aligned}$$

where $f(W) = (-u, h(w))$. We use Theorem 3 in [7] for the divergence-measure fields to conclude

$$\left(\alpha(W(x,t), \bar{W}(x,t)), \beta(W(x,t), \bar{W}(x,t))\right) \in \mathcal{DM}(\Pi_T),$$

and the validity of the product rule, since $\bar{W}(x,t)$ is locally Lipschitz in Π_T. Consider the measures

$$\begin{aligned} \theta &= \partial_t \eta_*(W(x,t)) + \partial_x q_*(W(x,t)), \\ \gamma &= \partial_t W(x,t), \bar{W}(x,t)) + \partial_x \beta(W(x,t), \bar{W}(x,t)), \end{aligned}$$

where the fact that θ is a nonpositive measure is granted by the entropy condition (2.7). We have

$$\begin{aligned} \gamma \ &= \partial_t \alpha(W, \bar{W}) + \partial_x \beta(W, \bar{W}) \\ &= \partial_t \eta_*(W) + \partial_x q_*(W) - \kappa \partial_w \eta_*(\bar{W})(\partial_t \eta_*(W) + \partial_x q_*(W)) \\ &\quad -\nabla^2 \eta_*(\bar{W}) \left(\bar{W}_t(W - \bar{W}) + \bar{W}_x(f(W) - f(\bar{W}))\right) \\ &\leq \theta - \nabla^2 \eta_*(\bar{W}) \left(f(W) - f(\bar{W}) - \nabla f(\bar{W})(W - \bar{W})\right) \bar{W}_x, \end{aligned}$$

where we used again the fact that $\nabla^2 \eta_* \nabla f$ is symmetric and that $\partial_w \eta_*$ is negative. Therefore, we conclude that $W(x,t) = \bar{W}(x,t)$. To conclude the proof, we notice that, by the first equation in (4.5), we must have $\partial_t(s(x,t) - \bar{s}(x,t)) = 0$, a.e. in Π_T. It then follows that $s(x,t) = \bar{s}(x,t)$, a.e. in Π_T, from $s(x,0) = \bar{s}(x,0)$, $x \in \mathbb{R}$. Hence we obtain $U(x,t) = \bar{U}(x,t)$, a.e. in Π_T, as desired.

□

Although we have assumed $U \in L^\infty(\Pi_T; \mathbb{R}^3)$ through the proof of Theorem 4.2, we only used the property $(w, u) \in L^\infty(\Pi_T; \mathbb{R}^2)$. Hence the same proof gives the uniqueness of Riemann solutions in the class of weak solutions satisfying (2.6)-(2.7), with $(w, u) \in L^\infty(\Pi_T; \mathbb{R}^2)$ and $s \in \mathcal{M}(\Pi_T)$, where the definition of weak solution should be adapted in an obvious way.

Therefore, combining Theorem 2.2 with Theorem 2.2 yields the following result.

Theorem 4.3. *Suppose* $U(x,t)$ *is a weak solution of* (1.1), (2.1), *and* (4.5) *such that* $(w, u) \in L^\infty(\mathbb{R}^2_+)$, $(s, s_t) \in \mathcal{M}_{loc}(\mathbb{R}^2_+)$, *satisfying* (2.1)-(2.7), *and*

(w_0, u_0, s_0) satisfies (2.2)-(2.3) and $P_0(x) \in L^1(\mathbb{R})$. Assume that U_L is connected to U_R by a Riemann solution $(W, S)(x, t)$ consisting of only rarefaction waves of the first and third families and possibly a contact discontinuity of the second linearly degenerate family. Then $(w, u)(x, t)$ asymptotically tends to $W(x/t)$ in the sense of (4.1). Moreover, for any $\phi \in C_0^\infty(\Omega)$,

$$< s^T, \phi > \longrightarrow < S, \phi > .$$

The Riemann solution (W, S) is the unique attractor.

Sketch of the proof. The only thing to be observed is that if (w^T, u^T, s^T) is the scaling sequence associated with the weak solution (w, u, s), where the scaling of s must be taken in the sense of distributions, then s^T also satisfies (2.6) with the same constant $C > 0$. Hence, the theorem follows from Theorem 2.2 and the straightforward extension of Theorem 4.2 to the case where $(w, u) \in L^\infty(\Pi_T; \mathbb{R}^2)$ and $s \in \mathcal{M}(\Pi_T)$.

□

4.2. 3×3 Euler Equations: General Riemann Solutions

We now investigate the uniqueness of general Riemann solutions in the class of BV solutions. The existence of BV solutions can be obtained by the Glimm scheme for BV initial data with moderate oscillation. The idea is to prove first the uniqueness of solutions of the corresponding Cauchy problem for the subsystem (2.8). The difficulty is now that the projection of any Riemann solution in the w–u plane no longer satisfies the entropy identity: $\partial_t \eta_*(\bar{W}) + \partial_x q_*(\bar{W}) = 0$ in the sense of distributions. Therefore, more careful analysis is needed.

Theorem 4.4. *Let $U(x, t) = (v(x, t), u(x, t), E(x, t)) \in BV(\Pi_T; \mathbb{R}^3)$ be a weak solution of (1.1), (2.1), and (4.4) in Π_T, satisfying the entropy condition (2.7)* *in the sense of distributions. Then* $U(x,t) = R(x/t)$*, a.e. in* Π_T*.*

Sketch of the proof. The strategy is to consider first the subsystem (2.8) to get the coincidence of the projections on the w–u plane, and then to conclude immediately the coincidence of $U(x,t)$ and the Riemann solution $R(x/t)$ a.e.. For concreteness, we consider only a generic Riemann solution $\bar{U}(x,t)$ consisting of the constant state $\bar{U}_L$ connected on the right by a 1-shock to the constant state $\bar{U}_M$, a stationary contact discontinuity connecting $\bar{U}_M$ to $\bar{U}_N$ on the right, and a rarefaction wave connecting $\bar{U}_N$ on the right to $\bar{U}_R$. Using DiPerna's method in [13], we consider the auxiliary function

$$\tilde{U}(x,t) = \begin{cases} \bar{U}_L, & x < x(t),\ 0 \le t < T, \\ \bar{U}_M, & x(t) < x < \max\{x(t), \sigma t\},\ 0 < t < T, \\ \bar{U}(x,t), & x > \max\{x(t), \sigma t\},\ 0 < t \le T, \end{cases}$$

where $x(t)$ is the minimal 1-characteristic of $U(x,t)$, and $x = \sigma t$ is the line of 1-shock discontinuity in $R(x/t)$. We then consider the measure

$$\tilde{\gamma} = \partial_t \alpha(W(x,t), \tilde{W}(x,t)) + \partial_x \beta(W(x,t), \tilde{W}(x,t)),$$

where $\tilde{W}$ is the projection of $\tilde{U}$ over the w–u plane, and $\alpha(W, \tilde{W})$ and $\beta(W, \tilde{W})$ are defined above. Our problem essentially reduces to analyzing the measure $\tilde{\gamma}$ over the region where the Riemann solution experiments a rarefaction wave and over the curve $x(t)$, which for simplicity may be taken as the jump set of $\tilde{W}(x,t)$.

By the Gauss-Green formula for BV functions and the finiteness of propagation speeds of the solutions, we have

$$\tilde{\gamma}\{\Pi_t\} = \int_{-\infty}^{+\infty} W(x,t), \tilde{W}(x,t))\, dx. \tag{4.6}$$

On the other hand,

$$\tilde{\gamma}\{\Pi_t\} = \tilde{\gamma}\{L(t)\} + \gamma\{\bar{\Omega}_2(t)\} + \theta\{\Pi_t - (L(t) \cup \bar{\Omega}_2(t))\}, \tag{4.7}$$

where $L(t) = \{(s, x(s)) \,|\, 0 < s < t\}$, since $\tilde{\gamma}$ reduces to the measure θ on the open sets where $\tilde{W}$ is a constant, and $\tilde{W}(x,t) = \bar{W}(x,t)$ over $\bar{\Omega}_2$. Hence, if one shows that

$$\tilde{\gamma}\{L(t)\} \leq 0, \tag{4.8}$$

the problem reduces to the same verification as in the shock-free case. Thus, we consider the functional

$$D(\sigma, W_-, W_+, \bar{W}_-, \bar{W}_+) = \sigma[\alpha(W, \bar{W})] - [\beta(W, \bar{W})],$$

where the square bracket denotes the left limit minus the right limit of shock wave curve in the (x, t)-plane for the function inside the bracket. We conclude that

$$D(\sigma, W_-, W_+, \bar{W}_-, \bar{W}_+) \leq 0, \tag{4.9}$$

if W_-, W_+ are projections over the w–u plane of states U_-, U_+, respectively, which are connected by a 1-shock of speed σ, and $\bar{W}_-$, $\bar{W}_+$ are projections over the same plane of states $\bar{U}_-$, $\bar{U}_+$, respectively, which are connected by a 1-shock of speed $\bar{\sigma}$, and also either $U_- = \bar{U}_-$ or $U_+ = \bar{U}_+$. Using Theorem 4.4 in [6] for the 2×2 case, it is then clear that (4.9) immediately implies (4.8).

As we already said, from (4.9) and the arguments in the shock-free case, we get that $W(x,t) = \tilde{W}(x,t)$, a.e. in Π_T. From the last equality and the Rankine-Hugoniot relations for (4.5), we conclude that $W(x,t) = \bar{W}(x,t)$, a.e. in Π_T. Now, by the same arguments in the proof of Theorem 4.2, we conclude $U(x,t) = \bar{U}(x,t)$, a.e. in Π_T. This leads to Theorem 4.4.

□

Again, as an immediate consequence of Theorem 4.4 and the L^1_{loc}-compactness of bounded sets in BV, we have the following theorem.

Theorem 4.5. *Suppose that* $U(x,t) \in BV_{loc}(\mathbb{R} \times (0,\infty); \mathbb{R}^3)$ *is an entropy solution of* (1.1), (2.1), *and* (4.4), *satisfying* (2.1)-(2.6) *and the entropy condition* (2.7) *in the sense of distributions. Then* $U(x,t)$ *asymptotically tends to the Riemann solution of* (1.1) *and* (4.3), *the unique attractor.*

5. Final Remarks

1. Applying Theorem 2.3 in [6], we conclude, from the convergence in time-average given in (4.1), the convergence in the usual sense, that is, $|U(t,t\xi)-R(\xi)|\to 0$, as $t\to\infty$, in $L^1_{\rm loc}(\mathbb{R})$. The rationale of the referred result is the same as in [4, 5, 29]. It may be summarized in the sentence: "Convergence in average plus entropy inequality implies convergence in the usual sense".

2. Using the transformation between the Lagrange coordinates and the Euler coordinates (see [33]), one can transform all results presented in this paper to the corresponding results for the following system (1.1) in the Euler coordinates:

$$\begin{cases} \partial_t\rho+\partial_x(\rho u)=0, \\ \partial_t(\rho u)+\partial_x(\rho u^2+p)=0, \\ \partial_t\left(\rho e+\frac{1}{2}\rho u^2\right)+\partial_x(u(\rho e+p+\frac{1}{2}\rho u^2))=0, \end{cases} \tag{5.1}$$

where $\rho=1/v$ denote the density. Related two special cases of (5.1) were also discussed in [1].

3. The assumption on $h(w)$ in (2.4) can be easily relaxed in Theorems 2.1-2.2 by following the arguments as in [18].

4. In the above sections, we discuss the class of constitutive relations (1.8) to study the existence and compactness of the entropy solutions of the Cauchy problem with arbitrarily large initial data. It would be interesting to explore some approaches to deal with other classes of constitutive relations. For polytropic gases, the global existence of entropy solutions of the Euler system for small initial data has been established (see [23] and [32]). Also see [8] for more complicated physical situations.

5. We also discuss the class of constitutive relations (1.9) to study the asymptotic behavior (decay and stability) of the entropy solutions of the Cauchy problem with arbitrarily large initial data. It would be interesting to explore some approaches to deal with such problems for other classes of constitutive relations. For polytropic gases, the asymptotic behavior of entropy solutions, obtained from the Glimm scheme, of the Euler system for small initial data has been studied (see [25, 26]).

6. The approach of Section 4 can also be applied to proving the asymptotic stability of Riemann solutions for the degenerate 4×4 system of Maxwell equations for plane waves in electromagnetism and the $m \times m$ systems with symmetry as models for magnetohydrodynamics and elastic strings (e.g. [2]). It can also be applied to studying the large-time behavior of solutions of hyperbolic systems with relaxation for the same type of initial data. For these and other correlated results, see [6].

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