Matemática Contemporânea, Vol. 15, 169–186
http://doi.org/10.21711/231766361998/rmc1511

ON ONE PROBLEM OF TRANSONIC GAS DYNAMICS

Nikolai Larkin *

1. Introduction

We study the initial boundary value problem for the nonlinear evolution equation arising in the asymptotic theory of transonic gas dynamics

$$Lu = u_{xt} - \mu u_{xxx} + u_x u_{xx} - \Delta_y u = 0 \tag{1.1}$$

in $Q = D \times (0,T),\ D = \Omega \times (0,L);\ x \in (0,L),\ t \in (0,T),\ y \in \Omega \subset R^2,$

where Ω is a bounded domain with sufficiently smooth boundary $\partial\Omega$; $S_T = \partial\Omega \times (0,L) \times (0,T)$; L, T are positive numbers;

$$\frac{\partial u}{\partial \nu} |_{S_T} = 0,\ u |_{x=0} = 0,\ u_x |_{x=0,L} = 0, \tag{1.2}$$

$$u(x,y,0) = u_0(x). \tag{1.3}$$

Here ν is an outward normal vector on S_T, μ is a positive constant, $u(x,y,t)$ is the disturbance potential.

Equation (1.1) models nonstationary transonic flows around a thin body with effects of viscosity and heatconductivity when a velocity of a gas is close to the local speed of a sound. For more information about physical aspects of (1.1) see Napolitano and Ryzhov [1], Larkin [2]. If $\mu = 0$, we have the Lin-Reissner-Tsiegn equation which is hyperbolic for all values of $u_x(x,y,t)$. The presence of $-\mu u_{xxx}$ implies some dissipativeness of (1.1) that makes it possible to prove the global existence theorem. On the other hand, this dissipativeness

*Supported by CNPq.-Brasil as a visiting professor at the State University of Maringá

is not very strong since for the variables y we have only the Laplace operator. It means that the stationary part of (1.1) is anisotropic , has different properties in x and y variables. It can be noticed also that due to anisotropic properties of the stationary part, boundary conditions on ∂D are not uniform: we have the Neumann condition on $\partial\Omega$ and 3 conditions in points $x=0, x=L$.

Our approach in proving the existence theorem reflects this fact. First we consider the linear problem and use the Faedo-Galerkin procedure with the basis only in y variables. Unknown coefficients, depending on (x,t), we find resolving the initial boundary value problem for the parabolic equation. Then we exploit fixed point arguments and prove local solvability of (1.1)-(1.3) for arbitrary regular initial conditions (1.3). To prove global solvability, we use the dissipativeness of $-\mu u_{xxx}$ and assume sufficiently small appropriate norms of u_0. At last, we prove stability theorem.

In the sequel, we use mostly standard notations for the functional spaces, see [3], otherwise the necessary definitions will be given. Without loss of generality, we put $\mu=1$.

Assumptions.

$$1.\quad u_0\in H^6(D),\ \ u_0(x,0)=0,\ \ \frac{\partial u_{0x}}{\partial\nu}\ |_{\partial D}=0;$$

$$2.\quad \frac{\partial}{\partial\nu}(u_{0xxx}\ -u_{0x}u_{0xx}\ +\Delta_y u_0)\ |_{\partial D}=0.$$

2. Linear Problem

To solve (1.1)-(1.3) by fixed point arguments, we start from the linear problem. Let B be the set of functions $g(x,y,t,)$ with the following properties:

$$g, g_x, g_t\in L^\infty(0,T;H^2(D)),\ \ g_{tt}\in L^\infty(0,T;H^1(D)),$$

$$g_{xtt}\in L^2(0,T;H^1(D)),\ g_{xxxt}\in L^\infty(0,T;L^2(D)),$$

$$\frac{\partial g}{\partial\nu}\ |_{S_T}=0,\ \ g\ |_{x=0,L}=0,$$

$$g\ |_{t=0}=u_{0x},\ \ \partial_t^i g\ |_{t=0}=\partial_t^i u_x\ |_{x=0}\ \ (i=1,2,),$$

where $\partial_t^i u_x \mid_{t=0}$ are the formal derivatives at $t = 0$ calculated from (1.1)-(1.3). Denote

$$\|g\|_W = \|g\|_{L^\infty(0,T;H^2(D))} + \|g_x\|_{L^\infty(0,T;H^2(D))} +$$
$$\|g_t\|_{L^\infty(0,T;H^2(D))} + \|g_{tt}\|_{L^\infty(0,T;H^1(D))} +$$
$$\|g_{xtt}\|_{L^\infty(0,T;H^1(D))} + \|g_{xxxt}\|_{L^\infty(0,T;L^2(D))}.$$

The ball B_M is the set of functions $g(x,y,t)$ from B such that $\|g\|_W \leq M$. Clearly, B_M is a closed set. For any $g \in B_M$, $M > 0$, consider the following linear problem

$$L_g u = u_{xt} - \Delta_y u - u_{xxx} + \frac{1}{2}(g u_x)_x = 0, \tag{2.1}$$

$$u \mid_{x=0} = u_x \mid_{x=0} = u_x \mid_{x=L} = 0, \quad \frac{\partial u}{\partial \nu} \mid_{S_T} = 0, \tag{2.2}$$

$$u \mid_{t=0} = u_0. \tag{2.3}$$

Approximate solutions to (2.1)-(2.3) will be sought in the form

$$u^N(x,y,t) = \sum_{j=1}^N z_j^N(x,t) w_j(y), \tag{2.4}$$

where

$$\begin{aligned} &\Delta_y w_j + \lambda_j w_j = 0 \ \text{ in } \Omega, \\ &\frac{\partial w_j}{\partial \nu} \mid_{\partial\Omega} = 0, \ (j = 1, ..., N), \\ &(w_i, w_j) = \int_\Omega w_i w_j dy = \delta_{ij}. \end{aligned} \tag{2.5}$$

Unknown functions $z_j^N(x,t)$ are solutions to the following initial boundary value problem

$$\begin{aligned} &z_{jxt}^N - z_{jxxx}^N = -\lambda_j z_j^N - \frac{1}{2}(g u_x^N, w_j) \ \text{ in } (0,L) \times (0,T), \\ &z_j^N \mid_{x=0} = 0, \ z_{jx}^N \mid_{x=0} = z_{jx}^N \mid_{x=L} = 0, \\ &z_j^N \mid_{t=0} = (u_0, w_j), \ j = 1, ..., N. \end{aligned} \tag{2.6}$$

Observing that (2.6) is a linear parabolic problem for z_{jx}^N, one can prove

Lemma 2.1. *Let $g \in B_M$ and $u_0 \in H^6(0,L)$. Then there exists a unique* *solution to (2.6),* $z_j^N(x,t)$:

$$\partial_t^i z_j^N \in L^\infty(0,T;H^{6-2i}(0,L)) \cap L^2(0,T;H^{7-2i}(0,L)),\ (i=0,1,2,3).$$

To pass to the limit as $N \to \infty$, we have to prove a priori estimates for u^N which will allow us also to get results on solvability of the nonlinear problem (1.1)-(1.3).

3. Local solutions

Theorem 3.1 *Let* $u_0 \in H^6(D)$ *satisfy assumptions 1,2. Then there exist a number* T_0 *and a unique function* $u(x,t)$*, which is a solution to (1.1)-(1.3); and the following inequality holds*

$$\|u_x\|_W \leq M.$$

We prove this theorem in some steps. First, we obtain a priori estimates for the approximate solutions that allow us to pass to the limit in (2.6), as $N \to \infty$, and therewith to solve the linear problem (2.1)-(2.3). After that, using fixed point arguments, we come to the result of Theorem 3.1.

A priori estimates

We prove a priori estimates in some steps. One part of them can be obtained directly in the whole domain $Q_0 = D \times (0,T_0)$. To prove other estimates, we will use a partition of the interval $[0,L]$ and get at first estimates in subdomains $D' \subset D$ then in vicinities of the surfaces $x=0,\ x=L$. Combination of these estimates permits us to get necessary estimates in the whole domain Q. In the Lemma 3.1 we give estimates which are valid in $Q_0 = D \times (0,T_0)$.

Lemma 3.1. *For each* $M < \infty$*, there exists a number* $T_3 = T_3(M) > 0$ *such that for all* $t \in (0,T_3)$ *the inequality holds*

$$\|u_{xx}^N(t)\|_{H^1(D)}^2 + \|u^N(t)\|_{H^2(D)}^2 + \|u_{xt}^N(t)\|_{H^1(D)}^2 + \|u_{xtt}^N(t)\|^2 +$$

$$\int_0^t (\|u_{xx\tau\tau}^N(\tau)\|^2 + \|\Delta_y u_\tau^N(\tau)\|^2 + \|u_{xxx}^N(\tau)\|_{H^1(D)}^2) d\tau \leq C_1 \|u_0\|_{H^5(D)}^2, \quad (3.1)$$

where the constant C_1 does not depend on M, N, t.

Proof. We omit the index N in calculations that will be made for smooth solutions of (2.1)-(2.3) which, additionally to (2.2),(2.3), possess the property

$$\frac{\partial u}{\partial \nu} \mid_{S_T} = \frac{\partial}{\partial \nu} \Delta_y u \mid_{S_T} = 0.$$

It is posssible because of our way of construction of $u^N(x, y, t,)$, see (2.4)-(2.6). First, we consider the identity

$$2(L_g u, u_x)(t) = \frac{d}{dt} \|u_x(t)\|^2 + 2\|u_{xx}(t)\|^2 +$$

$$2(\nabla_y u, \nabla_y u_x)(t) + ((g u_x)_x, u_x)(t) = 0. \tag{3.2}$$

The last term can be estimated as follows

$$\mid (g u_x)_x, u_x) \mid = \mid (g u_x, u_{xx}) \mid \leq \|u_{xx}(t)\|^2 + C_1(M)\|u_x(t)\|^2.$$

Substituting this into (3.2), we obtain

$$\frac{d}{dt} \|u_x(t)\|^2 + \|u_{xx}(t)\|^2 \leq C_1(M)\|u_x(t)\|^2.$$

Integration over (0,t) gives

$$\|u_x(t)\|^2 + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \leq \|u_{0x}\|^2 + C_1(M) \int_0^t \|u_x(\tau)\|^2 d\tau. \tag{3.3}$$

From here

$$\|u_x(t)\|^2 \ \leq \ \|u_{0x}\|^2 + C_1(M) \int_0^t \|u_x(\tau)\|^2 d\tau.$$

By Gronwall's lemma

$$\|u_x(t)\|^2 \ \leq \ \|u_{0x}\|^2 e^{C_1(M)t}.$$

Choosing $T_1 > 0$ such that $0 < C_1(M)T_1 \leq 1$ and taking into account (3.3), we get for all $t \in (0, T_1)$

$$\|u_x(t)\|^2 + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \leq C\|u_{0x}\|^2, \tag{3.4}$$

where C does not depend on M, N, t.

Next, we consider the equality

$$-2(L_g u, (e^{-x}\Delta_y u_x + u_{xxx})) = 0.$$

Acting in the same manner as by proving (3.4) and choosing T_2 sufficiently small, we obtain for all $t \in (0, T_2)$

$$\|u_x(t)\|^2_{H^1(D)} + \int_0^t (\|\Delta_y u(\tau)\|^2 + \|u_{xx}(\tau)\|^2_{H^1(D)})\, d\tau \le C\|u_0\|^2_{H^2(D)}, \qquad (3.5)$$

where the constant C does not depend on M, N, t.

From the identity

$$2((L_g u)_t, u_{xt}) = \frac{d}{dt}\|u_{xt}(t)\|^2 + 2\|u_{xxt}(t)\|^2 - ((gu_x)_t, u_{xxt})(t) + 2(\nabla_y u_t, \nabla_y u_{xt})(t) = 0,$$

taking into account (3.5), we obtain for $T_3 > 0$ sufficiently small

$$\|u_{xt}(t)\|^2 + \int_0^t \|u_{xx\tau}(\tau)\|^2\, d\tau \le C\|u_0\|^2_{H^3(D)}; \qquad (3.6)$$

and from

$$-2((L_g u)_t, (e^{-x}\Delta_y u_{xt} + u_{xxxt})) = 0$$

for all $t \in (0, T_3)$

$$\|u_{xt}(t)\|^2_{H^1(D)} + \int_0^t (\|u_{xx\tau}(\tau)\|^2_{H^1(D)} + \|\Delta_y u_\tau(t)\|^2)\, d\tau \le C\|u_0\|^2_{H^4(D)}. \qquad (3.7)$$

Consider for a.e. $t \in (0, T_3)$ the stationary problem

$$u_{xxx} + \Delta_y u = u_{xt} + \frac{1}{2}(gu_x)_x = F(t),$$

$$\frac{\partial u}{\partial \nu}\mid_{S_T} = 0, \;\; u\mid_{x=0} = 0, \;\; u_x\mid_{x=0} = u_x\mid_{x=L} = 0.$$

Due to (3.7), $F(t) \in H^1(D)$. In this case, as was shown in [2],

$$\|u_{xx}(t)\|_{H^1(D)} + \|u(t)\|_{H^2(D)} \le C\|F(t)\|_{H^1(D)} \le C\|u_0\|_{H^4(D)}. \qquad (3.8)$$

Transforming the identity

$$2((L_g u)_{tt}, u_{xtt})(t) = \frac{d}{dt}\|u_{xtt}(t)\|^2 + 2\|u_{xxtt}(t)\|^2 -$$

$$((gu_t)_{tt}, u_{xxtt})(t) + 2(\nabla_y u_{tt}, \nabla_y u_{xtt})(t) = 0,$$

we obtain for $T_3 > 0$ sufficiently small

$$\|u_{xtt}(t)\|^2 + \int_0^t \|u_{xx\tau\tau}(\tau)\|^2 \, d\tau \le C\|u_0\|^2_{H^5(D)}, \ \ \forall t \in (0, T_3). \tag{3.9}$$

Combining (3.7)-(3.9), we prove Lemma 3.1.

□

In the next lemma, we give a priori estimates that are valid in the interior of Q. Let δ be a positive number such that $20\delta < L$. We define in (0,L) smooth nonnegative functions $\xi_i = \xi_i(x)$ and domains D_i as follows

$$\xi_i(x) = 1 \ \text{ if } \ x \in [(i+1)\delta, L - (i+1)\delta],$$

$$\xi_i(x) = 0 \ \text{ if } \ x \in [0, i\delta] \cup [L - i\delta, L], \quad D_i = \Omega \times ((i+1)\delta, L - (i+1)\delta); \ (i = 1, ..., 6).$$

Lemma 3.2. *For each $M < \infty$, there exists a number $T_4(M) > 0$ such that for all $t \in (0, T_4)$ the following inequality holds*

$$\|u^N(t)\|_{H^3(D_4)} + \|u^N_{xx}(t)\|_{H^2(D_4)} + \|\partial^5_x u^N(t)\|_{L^2(D_4)} + \|\partial^3_x u^N_t(t)\|_{L^2(D_4)} +$$
$$\|\Delta_y u^N_t(t)\|_{L^2(D_4)} \le C\|u_0\|_{H^5(D)},$$

where C does not depend on M, N, t.

Proof. We give here only ideas of the proof.
Considering the identity

$$-(\xi_1(L_g u)_t, u_{xxxt})(t) = -(\xi_1 u_{xtt}, u_{xxxt})(t) + (\xi_1, | \, u_{xxxt} \, |^2)(t) +$$
$$(\xi_1 \Delta_y u_t, u_{xxxt})(t) - \frac{1}{2}(\xi_1 (u_x g)_{xt}, u_{xxxt}) = 0$$

and choosing $T_4(M) > 0$ sufficiently small, we obtain

$$(\xi_1, | \, u_{xxxt} \, |^2)(t) \le C\|u_0\|^2_{H^5(D)}. \tag{3.10}$$

Analogously, from

$$-(\xi_1(L_g u)_t, \Delta_y u_t)(t) = 0$$

follows

$$(\xi_1, |\ \Delta_y u_t\ |^2)(t) \le C\|u_0\|^2_{H^5(D)}. \tag{3.11}$$

And from

$$-(\xi_1(L_g u)_x, \Delta u_x + u_{xxxx}) = 0$$

we get

$$(\xi_1, |\ u_{xxxx}\ |^2)(t) + (\xi_1, |\ \Delta_y u_x\ |^2)(t) \le C\|u_0\|^2_{H^5(d)}. \tag{3.12}$$

Now, from

$$(\xi_2 L_g u, (\Delta_y u_{xxx} + \Delta_y^2 u))(t) = 0$$

we come to

$$(\xi_2, (|\ \nabla_y u_{xxx}\ |^2 + |\ \nabla_y^3 u\ |^2))(t) \le C\|u_0\|^2_{H^5(D)}. \tag{3.13}$$

In order to estimate $\|u_{xx}(t)\|_{H^2(D_4)}$, we consider the identity

$$-(\xi_3(L_g u)_{xx}, \Delta_y u_{xx} + \partial_x^5 u)(t) = 0$$

and come to the inequality

$$(\xi_3, (|\ \Delta_y u_{xx}\ |^2 + |\ \partial_x^5 u\ |^2))(t) \ \le\ C\|u_0\|^2_{H^5(D)}. \tag{3.14}$$

Combining (3.10)-(3.14) with the estimate of Lemma 3.1, we obtain the assertion of Lemma 3.2.

□

The next lemma improves the results of Lemma 3.1 and gives estimates in the whole domain Q.

Lemma 3.3. *There is a small number $T_0(M) > 0$ such that for all $t \in (0, T_6)$ the inequality holds*

$$\|u_x^N(t)\|^2_{H^2(D)} + \|\partial_x^4 u^N(t)\|^2_{L^2(D)} + \|u_{xtt}^N(t)\|^2_{H^1(D)} +$$

$$\|\Delta_y u_{xt}^N(t)\|^2_{L^2(D)} + \int_0^t (\|\Delta_y u_{\tau\tau}^N(\tau)\|^2 + \|u_{xx\tau\tau}^N(\tau)\|^2_{H^1(D)})\, d\tau \le C\|u_0\|^2_{H^6(D)}.$$

The scheme of the proof. We define smooth nonnegative functions $s_i = s_i(x)$ as follows:

$$0 \leq s_i(x) \leq 1; \ \ s_i(x) = 1 \ \text{ if } \ x \in [0, i\delta] \cup [L - i\delta, L],$$

$$s_i(x) = 0 \ \text{ if } \ x \ \in [(i+1)\delta, L-(i+1)\delta], \ \ (i = 1, ..., 7).$$

The identity

$$2(s_6 L_g u, \Delta_y^2 u_x)(t) = 0$$

for T_6 sufficiently small can be reduced, taking into account Lemma 3.2, to the inequality

$$(s_6, (\Delta_y u_x)^2)(t) + \int_0^t (s_6, (\Delta_y u_{xx})^2)(\tau)\, d\tau \leq C\|u_0\|^2_{H^6(d)}.$$

Adding (3.11) and (3.8), we get

$$\|u_x(t)\|_{H^2(D)} \leq C\|u_0\|_{H^6(D)}. \tag{3.15}$$

Now, from the identity

$$-((L_g u)_x, u_{xxxx})(t) = 0,$$

we obtain

$$\|u_{xxxx}\| \leq C\|u_0\|_{H^6(D)}; \tag{3.16}$$

and from

$$-2((L_g u)_{tt}, (e^{-x}\Delta_y u_{xtt} + \partial_x^3 u_{tt}))(t) = 0$$

follows

$$\|u_{xtt}(t)\|^2_{H^1(D)} + \int_0^t (\|\Delta_y u_{\tau\tau}(\tau)\|^2 + \|u_{xx\tau\tau}(\tau)\|^2_{H^1(D)})\, d\tau \leq C\|u_0\|^2_{H^6(D)}. \tag{3.17}$$

At last, considering the identity

$$(\xi_5 (L_g u)_{xt}, \Delta_y u_{xt}) = 0,$$

we obtain

$$\|\Delta_y u_{xt}(t)\|_{L^2(D)} \leq C.$$

This and (3.15)-(3.17) imply the result of Lemma 3.3.

□

Now we are able to prove

Lemma 3.4. *Let $u_0 \in H^6(D)$ satisfy assumptions 1,2. Then for each fixed $M > 0$ there is $T_0 > 0$ such that for every $t \in (0, T_0)$ the approximate solutions to (2.1)-(2.3), u^N, satisfy the estimate*

$$\|u^N\|_{W^3} = \|u^N(t)\|_{H^3(D)} + \|u^N_{xx}(t)\|_{H^2(D)} + \|\partial_x^5 u^N(t)\| + \|u^N_{xt}\|_{H^2(D)} +$$

$$\|u^N_{xtt}(t)\|_{H^1(D)} + \|\partial_x^4 u^N_t(t)\| + (\int_0^t \|u^N_{xx\tau\tau}\|^2_{H^1(D)} d\tau)^{\frac{1}{2}} \leq C_0 \|u_0\|_{H^6(D)}, \quad (3.18)$$

where C_0 does not depend on M, N, t.

Proof. We start from the estimate $\|u_{xt}(t)\|_{H^2(D)}$. Since Lemma 3.2 gives this estimate in D_4, it is sufficient to prove it in vicinities of $x = 0$, $x = L$. The function $z = (1 - \xi_6)u$ satisfies the equation

$$L_g z = z_{xt} - z_{xxx} - \Delta_y z + \frac{1}{2}(g z_x)_x = -\xi_{6x}[u_t + \frac{1}{2}(gu)_x - 3u_{xx}] -$$

$$\frac{1}{2}(\xi_{6x} u g)_x + 3\xi_{6xx} u_x + \xi_{6xxx} u,$$

$$z = 0 \quad \text{when} \quad x \in [7\delta, L - 7\delta].$$

By the usual way, we show that

$$\|\Delta_y z_{xt}(t)\|)^2 + \int_0^t (\|\Delta_y z_{xx\tau}(\tau)\|^2 + \|\nabla_y^3 z_\tau(\tau)\|^2)\, d\tau \leq C\|u_0\|^2_{H^6(D)}.$$

Taking into account Lemma 3.2, we have

$$\|\nabla_y^3 u(t)\| + \|u_{xt}(t)\|_{H^2(D)} \leq C\|u_0\|_{H^6(D)}.$$

Now, from the identity

$$(L_g u, \Delta_y u_{xxx}) = 0$$

we come to

$$\|u_{xxx}(t)\|_{H^1(D)} \leq C\|u_0\|_{H^6(D)}.$$

Taking into account Lemmas 3.2, 3.3, we get

$$\|u_{xx}(t)\|_{H^2(D)} \leq C\|u_0\|_{H^6(D)}.$$

Considering

$$((L_g u)_{xx}, \partial_x^5 u)(t) = 0$$

and

$$((L_g u)_{xt}, \partial_x^4 u_t)(t) = 0,$$

we complete the proof of Lemma 3.4.

□

Now we are able to prove existence theorems. In fact, Lemma 3.4 allows us to pass to the limits in (2.6), as $N \to \infty$, hence the following assertion is valid.

Theorem 3.1. *Let $u_0 \in H^6(D)$ satisfy assumptions 1,2. Then for every $g \in B_M$ and $M < \infty$ there exists a unique solution to (2.1)-(2.3) satisfying (3.18). Moreover, for each fixed $M < \infty$, there is such $T_0 = T_0(M) > 0$ that the constant C_0 in (3.18) does not depend on M, $t \in (0, T_0)$.*

The proof is obvious, we drop it.

□

It follows from Theorem 3.1 that we can define the operator $P : u_x = Pg$.

Lemma 3.5. *Let M be sufficiently large and T_0 be sufficiently small. Then P maps B_M into B_M and is the contraction operator.*

Proof. Putting $M = 2C_0\|u_0\|_{H^6(D)}$, we can see that $\|u_x\|_W \leq M/2$. This proves the first part of Lemma 3.5. Defining $\rho(g_1, g_2) = \|g_1 - g_2\|_{L^\infty(0,T;L^2(D))}$, we obtain

$$\rho(u_{1x}, u_{2x}) \leq C(M)T^*\rho(g_1, g_2),$$

where $u_{ix} = P(g_i)$, $i = 1, 2$. Choosing for fixed M $T^* \in (0, \frac{1}{2C(M)})$, we complete the proof of Lemma 3.5.

□

It implies

Theorem 3.2. *Let $u_0 \in H^6(D)$ satisfy assumptions 1,2. Then there is such $T_0 > 0$ that in $G_0 = D \times (0, T_0)$ there exists a unique solution to (1.1)-(1.3); and (3.18) holds.*

4. Global Solutions

Existence of local in t solutions was proved without restrictions for a size of u_0. On the other hand, if the appropriate norm of u_0 is sufficiently small, it is possible to prove existence of global solutions. Let B be the set of functions $g(x, y, t)$ defined in $Q^+ = D \times R^+$ with the following properties:

$$g, g_x, g_t \in L^\infty(R^+; H^2(D)) \cap L^2(R^+; H^2(D));$$

$$g_{tt} \in L^\infty(R^+; H^1(D)) \cap L^2(R^+; H^1(D)),\ g_{xtt} \in L^2(R^+; H^1(D));$$

$$g_{xxxt} \in L^\infty(R^+; L^2(D)) \cap L^2(R^+; L^2(D));$$

$$\frac{\partial g}{\partial \nu} \mid_{S^+} = 0,\ g \mid_{x=0,L} = 0,\ S^+ = \partial\Omega \times (0, L) \times R^+,$$

$$g \mid_{t=0} = u_{0x},\ \partial_t^i g \mid_{t=0} = \partial_t^i u_x \mid_{t=0},\ (i = 1, 2).$$

Denote

$$\|g\|_W = \|g\|_{L^\infty(R^+;H^2(D))\cap L^2(R^+;H^2(D))} + \|g_x\|_{L^\infty(R^+;H^2(D))\cap L^2(R^+;H^2(D))} +$$

$$\|g_t\|_{L^\infty(R^+;H^2(D))\cap L^2(R^+;H^2(D))} + \|g_{tt}\|_{L^\infty(R^+;H^1(D))\cap L^2(R^+;H^1(D))} +$$

$$\|g_{xtt}\|_{L^2(R^+;H^1(D))} + \|g_{xxxt}\|_{L^\infty(R^+;L^2(D))\cap L^2(R^+;L^2(D))}.$$

The ball B_M is a closed set of $g(x, y, t)$ from B such that $\|g\|_W \leq M$.
As in section 3, we start from the linear problem

$$L_g u = u_{xt} - u_{xxx} - \Delta_y u + \frac{1}{2}(g u_x)_x = 0, \tag{4.1}$$

$$\frac{\partial u}{\partial \nu} |_{S^+} = 0, \ u |_{x=0} = u_x |_{x=0,L} = 0, \tag{4.2}$$

$$u |_{t=0} = u_0, \tag{4.3}$$

where g is an arbitrary function from B_M. To solve (4.1)-(4.3), we use the Faedo-Galerkin method. Having necessary a-priori estimates of solutions to (4.1)-(4.3), we can proceed as in section 3 and prove existence of global solutions.

Here we prove only the estimates in the whole domain $D \times R^+$ in order to give an idea how to use the small norm $\|g\|_W$.

Lemma 4.1. *Let $g \in B_M$, $\|u_0\|_{H^6(D)} \le \delta$; and assumptions 1,2 hold. If $M_0 >$ 0 is sufficiently small number and $0 < M \le M_0$, then for a.e. $t \in R^+$ regular solutions to (4.1)-(4.3) satisfy the inequality*

$$\|u(t)\|^2_{H^2(D))} + \|u_{xx}(t)\|^2_{H^1(D)} + \|u_{xt}(t)\|^2_{H^1(D)} +$$

$$\int_0^t (\|u(\tau)\|^2_{H^2(D)} + \|u_{xx}(\tau)\|^2_{H^1(D)} + \|u_x(\tau)\|^2_{H^2(D)} +$$

$$\|u_{x\tau\tau}(\tau)\|^2) d\tau \ \le \ C\|u_0\|^2_{H^4(D)},$$

where the constant C does not depend on t and on the choice of g .

Proof. First, we consider the integral

$$2(L_g u, u_x)(t) = \frac{d}{dt}\|u_x(t)\|^2 + 2\|u_{xx}(t)\|^2 +$$

$$2(\nabla_y u, \nabla_y u_x)(t) - (g u_x, u_{xx})(t) = 0. \tag{4.4}$$

We estimate the last term in (4.4) as follows

$$I = | (g u_x, u_{xx}) | \le max_{\overline{D}} | g | (\|u_{xx}\|^2 + \|u_x\|^2).$$

Since $\|u_x\|^2 \le L\|u_{xx}\|^2$, then

$$I \le C_D M(1+L)\|u_{xx}\|^2, \tag{4.5}$$

where C_D is the constant of embedding

$$sup_{Q^+} | g | \le C_D \|g\|_W \le C_D M.$$

Substituting (4.5) into (4.4), we obtain

$$\frac{d}{dt}\|u_x(t)\|^2 + (2 - C_D(1+L)M)\|u_{xx}(t)\|^2 \le 0.$$

Choosing M such that $2 - C_D(1+L)M = 1$ and integrating the result, we have

$$\|u_x(t)\|^2 + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \le \|u_{0x}\|^2, \ \ \forall t \in R^+. \tag{4.6}$$

On the next step, we consider the identity

$$-2(e^{-\lambda x}L_g u, \Delta_y u_x) = \frac{d}{dt}(e^{-\lambda x}, \mid \nabla_y u_x \mid^2)(t) +$$

$$2(e^{-\lambda x}\Delta_y u_x, u_{xxx})(t) + 2(e^{-\lambda x}\Delta_y u, \Delta_y u_x)(t) +$$

$$(e^{-\lambda x}\nabla_y (g u_x)_x, \nabla_y u_x)(t) = 0, \tag{4.7}$$

where λ is an arbitrary positive number.

We treat all the terms separately. Taking into account (4.2), we find

$$I_1 = 2(e^{-\lambda x}\Delta_y u, \Delta_y u_x) = \lambda(e^{-\lambda x}, \mid \Delta_y u \mid^2) + \int_\Omega e^{-\lambda x} \mid \Delta_y (y.L) \mid^2 dy; \tag{4.8}$$

$$I_2 = 2(e^{-\lambda x}\Delta_y u_x, u_{xxx}) = 2(e^{-\lambda x}, \mid \nabla_y u_{xx} \mid^2) - \lambda^2(e^{-\lambda x}, \mid \nabla_y u_x \mid^2). \tag{4.9}$$

If $\lambda > 0$ is sufficiently small, then direct calculations give

$$(e^{-\lambda x}, u_x^2) \le \frac{2L^2}{2 - \lambda L^2}(e^{-\lambda x}, u_{xx}^2). \tag{4.10}$$

Substituting (4.10) into (4.9), we have

$$I_2 \ge 2(1 - \frac{\lambda^2 L^2}{2 - \lambda L^2})(e^{-\lambda x}, \mid \nabla_y u_{xx} \mid^2). \tag{4.11}$$

The last term in (4.7) we transform to the form:

$$I_3 = (e^{-\lambda x}\nabla_y (g u_x)_x, \nabla_y u_x) = \lambda(e^{-\lambda x}(u_x \nabla_y g + g\nabla_y u_x), \nabla_y u_x) -$$

$$(e^{-\lambda x}(u_x \nabla_y g + g\nabla_y u_x), \nabla_y u_{xx}). \tag{4.12}$$

The first term in (4.12) can be estimated as follows

$$I_{31} = \mid \lambda((u_x \nabla_y g + g\nabla_y u_x), e^{-\lambda x}\nabla_y u_x) \mid \le \lambda max_{\overline{D}} \mid g \mid (e^{-\lambda x}, \mid \nabla_y u_x \mid^2) +$$

$$\lambda\|e^{-\frac{\lambda x}{2}}\nabla_y u_x\|_{L^2(D)}\|\nabla_y g\|_{L^4(D)}\|e^{-\frac{\lambda x}{2}}u_x\|_{L^4(D)} \leq \lambda max_{\overline{D}} \mid g \mid (e^{-\lambda x}, \mid \nabla_y u_x \mid^2)+$$

$$\lambda C_D\|g\|_{H^2(D)}(\|e^{-\frac{\lambda x}{2}}u_x\|^2_{H^1(D)} + \|e^{-\frac{\lambda x}{2}}\nabla_y u_x\|^2_{L^2(D)}) \leq \lambda C_D M(e^{-\lambda x}, \mid \nabla_y u_x \mid^2)+$$

$$\lambda C_D M(\|e^{-\frac{\lambda x}{2}}u_x\|^2_{L^2(D)} + \|e^{-\frac{\lambda x}{2}}u_{xx}\|^2_{L^2(D)} + \|e^{-\frac{\lambda x}{2}}\nabla_y u_x\|^2_{L^2(D)}),$$

where C_D depends only on D.

Using (4.10), we obtain

$$I_{31} \leq \lambda C_D M(\|e^{-\frac{\lambda x}{2}}u_x\|^2_{L^2(D)} + \|e^{-\frac{\lambda x}{2}}u_{xx}\|^2_{L^2(D)} + \|e^{-\frac{\lambda x}{2}}\nabla_y u_{xx}\|^2_{L^2(D)}). \quad (4.13)$$

Analogously

$$I_{32} = \mid ((u_x\nabla_y g + g\nabla_y u_x), e^{-\lambda x}\nabla_y u_{xx}) \mid \leq$$

$$C_D M(\|e^{-\frac{\lambda x}{2}}u_x\|^2_{L^2(D)} + \|e^{-\frac{\lambda x}{2}}u_{xx}\|^2_{L^2(D)} + \|e^{-\frac{\lambda x}{2}}\nabla_y u_{xx}\|^2_{L^2(D)}). \quad (4.14)$$

Substituting (4.13), (4.14) into (4.12) and taking into account (4.8)-(4.11), we reduce (4.7) to the inequality

$$\frac{d}{dt}(e^{-\lambda x}, \mid \nabla_y u_x \mid^2)(t) + (2 - \frac{2\lambda^2 L^2}{2-\lambda L^2} - C_D M)(e^{-\lambda x}, \mid \nabla_y u_{xx} \mid^2)(t)+$$

$$\lambda(e^{-\lambda x}, \mid \Delta_y u \mid^2)(t) \leq C_D M\|u_{xx}\|^2(t).$$

Choosing $\lambda > 0, M$ sufficiently small and using (4.6), we get

$$\|\nabla_y u_x(t)\|^2_{L^2(D)} + C_0\int_0^t(\|\nabla_y u_{xx}(\tau)\|^2 + \|\Delta_y u(\tau)\|^2)d\tau \leq C_2\|u_0\|^2_{H^2(D)}, \quad (4.15)$$

where the constants C_0, C_2 do not depend on t, M and on the choice of $g \in B_M$.

Acting in the same manner, we obtain from the identity

$$(L_g u, u_{xxx}) = 0$$

the estimate

$$\|u_{xx}(t)\|^2 + \int_0^t \|u_{xxx}(\tau)\|^2 d\tau \leq C\|u_0\|^2_{H^2(D)}, \quad (4.16)$$

and from the identities

$$((L_g u)_t, u_{xt})(t) = 0,$$

$$(e^{-\lambda x}(L_g u)_t, \Delta_y u_{xt}) = 0,$$

$$((L_g u)_t, u_{xxxt}) = 0,$$

choosing $\lambda > 0, M > 0$ sufficiently small, we get

$$\|u_{xt}(t)\|^2_{H^1(D)} + \int_0^t (\|u_{xx\tau}(\tau)\|^2_{H^1(D)} + \|\Delta_y u_\tau(\tau)\|^2) d\tau \le C\|u_0\|^2_{H^4(D)}. \quad (4.17)$$

All the constants in (4.15)-(4.17) do not depend on t, M.

At last, considering for $a.e.\, t \in R^+$ the stationary problem

$$u_{xxx} + \Delta_y u = \frac{1}{2}(g u_x)_x + u_{xt},$$

$$\frac{\partial u}{\partial \nu} \mid_{S^+} = 0,$$

$$u \mid_{x=0} = 0, \; u_x \mid_{x=0,L} = 0,$$

we obtain

$$\|u_{xx}(t)\|^2_{H^1(D)} + \|u(t)\|^2_{H^2(D)} \le \|u_0\|^2_{H^4(D)}.$$

This completes the proof of Lemma 4.1.

□

Acting in the same manner as in section 3, we can estimate the derivatives of a higher order and to prove at first solvability of(4.1)-(4.3) then solvability of (1.1)-(1.3).

Theorem 4.1. *Let* $u_0 \in H^6(D)$ *satisfy assumptions 1,2 and* $\|u_0\|_{H^6(D)} \le \delta$. *Then there is* $\delta_0 > 0$ *such that for all* $\delta \in (0, \delta_0)$ *there exists a unique solution to* (1.1)-(1.3), $u(x, y, t)$:

$$u_x, u_{xx}, u_{xt} \in L^\infty(R^+; H^2(D)) \cap L^2(R^+; H^2(D)),$$

$$u_{xtt} \in L^\infty(R^+; H^1(D)) \cap L^2(R^+; H^1(D)), \;\; u_{xxtt} \in L^2(R^+; H^1(D)),$$

$$\partial_x^5 u, \partial_x^4 u_t \in L^\infty(R^+; L^2(D)) \cap L^2(R^+; L^2(D)).$$

The proof is similar to the proof of Theorem 3.1, but here we use the dissipativeness of u_{xxx} and choose $M > 0$ sufficiently small instead of small T.

5. Stability

The presence of the dissipation u_{xxx} in (1.1) along with the global existence theorem also permits us to prove stability of small solutions.

For every $T \in (0,\infty)$, $S_T = \partial D \times (0,T)$, let $u(x,y,t)$ be a unique regular solution to the nonstationary problem

$$\begin{aligned} u_{xt} - u_{xxx} - \Delta_y u + u_x u_{xx} &= f(x,y), \\ \frac{\partial u}{\partial \nu} \mid_{S_T} = 0, \ u \mid_{x=0} &= 0, \\ u \mid_{t=0} = u_0(x,y); & \end{aligned} \tag{5.1}$$

and let $v(x,y)$ be a unique solution in D to the stationary problem

$$\begin{aligned} -v_{xxx} - \Delta_y v + v_x v_{xx} &= f(x,y), \\ \frac{\partial v}{\partial \nu} \mid_{\partial D} = 0, \ v \mid_{x=0} &= 0. \end{aligned} \tag{5.2}$$

Theorem 5.1. *Let $u(x,y,t)$ and $v(x,y)$ be unique regular solutions to (5.1) and (5.2) respectively. If $\|f\|_{H^3(D)}$ is sufficiently small, then the following inequality holds*

$$\|(u_x - v_x)(t)\| \leq C\|u_{0x} - v_x\| e^{-\alpha t},$$

where α is a positive constant.

Proof. For $z = u - v$ we have the following problem

$$\begin{aligned} Lx = z_{xt} - z_{xxx} - \Delta_y z + \frac{1}{2}((z_x + 2v_x)z_x)_x &= 0, \\ \frac{\partial z}{\partial \nu} \mid_{S_T} = 0, \ z \mid_{x=0} &= 0, \\ z \mid_{t=0} = u_0 - v. & \end{aligned}$$

Considering the identity

$$2(Lz, z_x)(t) = \frac{d}{dt}\|z_x(t)\|^2 + 2\|z_{xx}(t)\|^2 +$$

$$2(\nabla_y z, \nabla_y z_x)(t) - ((z_x + 2v_x)z_x, z_{xx})(t) = 0$$

and taking into account that $\|f\|_{H^3(D)}$ and, consequently, $\max_{\bar{D}} \mid v_{xx} \mid$, see [2], are sufficiently small, we come to the inequality

$$\frac{d}{dt}\|z_x(t)\| + \frac{1}{2L}\|z_x(t)\| \leq 0.$$

This implies the assertion of Theorem 5.1.

□

Remark 5.1. We consider homogeneous boundary conditions (1.2) only for technical reasons. Nonhomogeneous conditions and the right-hand side of (1.1) also can be treated.

References

[1] Napolitano, D.; Ryzhov, O., *On analogy between nonequilibrium and viscous inertial flows with transonic velocities*, J. Comput. Math. Math. Phys. 1, (1971), 1229-1261.

[2] Larkin, N.A., *Smooth Solutions for the Equations of Transonic Gas Dynamics*, Novosibirsk: Nauka, (1991).

[3] Lions, J.-L.; Magenes E., *Problemes aux Limites non Homogenes, Aplications*, Dunod: Paris, (1969).

The Institute of Theoretical and Applied Mechanics
Novosibirsk-90, 630090, Russia.
mail address:
Depart. de Matemática, UEM,
Ag. Postal UEM
87020-900, Maringá, Pr., Brasil
e-mail: nalarkin@gauss.dma.uem.br