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DIRECT DECOMPOSITIONS IN ARTINIAN MODULES OVER FC-HYPERCENTRAL GROUPS

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Abstract

In this article the existence of $\mathcal{X}$- decomposition in artinian $\mathbb{Z}G$-modules is established for some kinds of formations $\mathcal{X}$, where G is a locally soluble FC-hypercentral group.

1. Introduction.

As in Finite Group Theory a class $\mathcal{X}$ of groups is called a formation if it satisfies the following conditions:

if $G \in \mathcal{X}, H$ is a normal subgroup of G, then $G/H \in \mathcal{X}$;

if H_1, H_2 are normal subgroups of G such that $G/H_1, G/H_2 \in \mathcal{X}$, then $G/H_1 \cap H_2 \in \mathcal{X}$.

Let R be a ring, G a group, A an RG-module, B_1, B_2 RG-submodules of A, $B_1 \leq B_2$, and let $\mathcal{X}$ be a class of groups. The factor B_1/B_2 is called $\mathcal{X}$-central (respectively $\mathcal{X}$-eccentric) if $G/C_G(B_2/B_1) \in \mathcal{X}$ (respectively $G/C_G(B_2/B_1) \notin \mathcal{X}$).

Let $a \in A$. We say that a is an $\mathcal{X}C$-element if $G/C_G(aRG) \in \mathcal{X}$.

Put

$$\mathcal{X}C_{RG}(A) = \{a \in A | a \text{ is an } \mathcal{X}C\text{-element of } A\}.$$

It is easy to see that $\mathcal{X}C_{RG}(A)$ is a RG-submodule of A in the case when $\mathcal{X}$ is a formation. The submodule $\mathcal{X}C_{RG}(A)$ is called the $\mathcal{X}C$-center of A (more

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precisely: the $\mathcal{X}C$-RG-center). Starting from the $\mathcal{X}C$-center, we can construct the upper $\mathcal{X}C$-central series of the module A. It is the following series

$$\langle 0\rangle = A_0 \leq A_1 \leq \ldots A_\alpha \leq A_{\alpha+1} \leq \ldots A_\gamma,$$

where $A_1 = \mathcal{X}C_{RG}(A), A_{\alpha+1}/A_\alpha = \mathcal{X}C_{RG}(A/A_\alpha), \alpha < \gamma, \mathcal{X}C_{RG}(A/A_\gamma) = \langle 0\rangle$.

The last term A_γ of this series is called the upper $\mathcal{X}C$-hypercenter of the module A (more precisely, the $\mathcal{X}C$-RG-hypercenter), and is denoted by $\mathcal{X}C^{\infty}_{RG}(A)$; the terms A_α of this series are called the upper $\mathcal{X}C$-hypercenters.

If $A = A_\gamma$ then the module A is called $\mathcal{X}C$-hypercentral; if γ is finite, then A is called $\mathcal{X}C$-nilpotent.

If $\mathcal{X} = \mathcal{G}$ is the class of all identity groups then we come to the concept of RG-hypercentral (or hypertrivial) module.

If $\mathcal{X} = \mathcal{F}$ is the class of all finite groups then we will obtain the concept of FC-hypercentral module.

Let R be a ring, G a group, A an RG-module, and let $\mathcal{X}$ be a formation of groups. We say that A has the $\mathcal{X}$-decomposition (more precisely, $\mathcal{X}$-RG-decomposition) if $A = \mathcal{X}C^{\infty}_{RG}(A) \oplus \mathcal{X}C^{*}_{RG}(A)$ where $\mathcal{X}C^{*}_{RG}(A)$ is an RG-submodule of A such that every of its non-zero RG-factor is $\mathcal{X}$-eccentric.

If $\mathcal{X} = \mathcal{G}$ then we obtain the Z-decomposition. This themes began from the famous Fitting lemma which is very useful in Group Theory. In the Infinite Group Theory first results on the existence of Z-decomposition were obtained by B. Hartley and M.J.Tomkinson [3].

Artinian modules are very good extensions of finite modules. In his paper [9] D.I. Zaitsev proved that every artinian $\mathbb{Z}G$-module has the Z-decomposition for every hypercentral group G.

The next natural formation is the class $\mathcal{F}$ of all finite groups. The first result about the $\mathcal{F}$-decomposition was obtained by D.I.Zaitsev [10], who proved that every artinian $\mathbb{Z}G$-module over locally soluble hyperfinite group G has the $\mathcal{F}$-decomposition. The condition of local solubility is not necessary (Z.Duan [2]). But for non-torsion FC-hypercentral groups we have only some initial results (D.I.Zaitsev [11], Z.Duan [1]).

In this paper we consider the question about the existence of $\mathcal{X}$-decomposition not only for the formation $\mathcal{F}$ but for some extension of $\mathcal{F}$.

We say that the formation $\mathcal{X}$ is overfinite if $\mathcal{X}$ satisfies the following conditions:

(1) $\mathcal{F} \leq \mathcal{X}$;

(2) if $G \in \mathcal{X}, H$ is a normal subgroup of finite index, then $H \in \mathcal{X}$;

(3) if G is a group, H is a normal subgroup of G such that $|G : H|$ is finite and $H \in \mathcal{X}$, then $G \in \mathcal{X}$.

Our main result is the following theorem.

Theorem. *Let G be a locally soluble FC-hypercentral group, and let A be an artinian $\mathbb{Z}G$-module. If $\mathcal{X}$ is an overfinite formation of groups then A has the $\mathcal{X}$-decomposition.*

Corollary. *Let G be a locally soluble FC-hypercentral group, A be an artinian $\mathbb{Z}G$-module. Then A has the $\mathcal{X}$-decomposition for the following formations $\mathcal{X}$:*

(1) $\mathcal{X} = \mathcal{F}$, *the formation of all finite groups;*

(2) $\mathcal{X} = \mathcal{LF}$, *the formation of all polycyclic-by-finite groups;*

(3) $\mathcal{X} = \mathcal{C}$, *the formation of all Chernikov groups;*

(4) $\mathcal{X} = \mathcal{S}_2\mathcal{F}$, *the formation of all soluble-by-finite minimax groups;*

(5) $\mathcal{X} = \hat{\mathcal{S}}\mathcal{F}$, *the formation of all soluble-by-finite groups of finite special* (Mal'cev-Prufer) rank;

(6) $\mathcal{X} = \mathcal{S}_0\mathcal{F}$, *the formation of all soluble-by-finite groups of finite section rank.*

2. Some preliminary results.

The first two lemmas and their corollaries are almost obvious and we omit their proofs.

Lemma 1. *Let R be a ring, G a group, A an RG-module, B an RG-submodule of A, and let $\mathcal{X}$ be a formation of groups. Then $\mathcal{X}^{\infty}_{RG}(B) \leq \mathcal{X}^{\infty}_{RG}(A)$.*

Lemma 2. *Let R be a ring, G a group, A an RG-module, B_1 and B_2 RG-submodules of A, $B_1 \leq B_2$, and let $\mathcal{X}$ be a formation of groups. If every non-zero RG-factor of B_1 and of B_2/B_1 is $\mathcal{X}$-eccentric, then every non-zero RG-factor of B_2 is $\mathcal{X}$-eccentric.*

Corollary 1. *Let $\{B_\alpha | \alpha \leq \gamma\}$ be an ascending chain of RG-submodules of A satisfying the following condition:*

if E, C are RG-submodules such that $B_\alpha \leq C < E \leq B_{\alpha+1}, C \neq E, \alpha < \gamma$, then E/C is $\mathcal{X}$-eccentric.

Then every non-zero RG-factor of the submodule B_γ is $\mathcal{X}$-eccentric.

Corollary 2. *Let $\{B_\lambda | \lambda \in \Lambda\}$ be a family of RG-submodules of A such that every non-zero RG-factor of B_λ is $\mathcal{X}$-eccentric for any $\lambda \in \Lambda$, and $B = \sum_{\lambda\in\Lambda} B_\lambda$. Then every non-zero RG-factor of B is $\mathcal{X}$-eccentric.*

Corollary 3. *Let $\{B_\lambda | \lambda \in \Lambda\}$ be a family of RG-submodules of A such B_λ has the $\mathcal{X}$-RG-decomposition for any $\lambda \in \Lambda$, and $B = \sum_{\lambda\in\Lambda} B_\lambda$. Then B has the $\mathcal{X}$-RG-decomposition.*

Corollary 4. *The module A has the largest RG-submodule having the $\mathcal{X}$-RG-decomposition.*

Lemma 3. *Let R be a ring, G a group, A a RG-module, H a normal subgroup of G such that G/H is finite, and let B be a RH-submodule of A such that $A = BRG$. If $\mathcal{X}$ is an overfinite formation of groups and B has the $\mathcal{X}$-RH-decomposition then A has the $\mathcal{X}$-RG-decomposition.*

Proof. Let $\{g_1, \dots, g_n\}$ be a transversal for H in G. Then $A = Bg_1 + \dots + Bg_n$. If C, D are RH-submodules, $C \geq D, L = C_H(C/D), g \in G$, then $C_H(Cg/Dg) = g^{-1}Lg$. Therefore $H/C_H(Cg/Dg) = H/g^{-1}Lg \cong H/L$. It follows that if the RH-factor C/D is $\mathcal{X}$-central (respectively, $\mathcal{X}$-eccentric) then the RH-factor Cy/Dy is $\mathcal{X}$-central (respectively, $\mathcal{X}$-eccentric) too. This means that the RH-submodule Bg has the $\mathcal{X}$-RH-decomposition.

Let $a \in \mathcal{X}C_{RH}(A), A_0 = aRH, A_1 = aRG$. Then $A = A_0g_1 + \dots + A_0g_n$. Put $U = C_H(A_0)$, then $H/U \in \mathcal{X}$. It follows that $H/C_H(A_0g) \in \mathcal{X}$ for each $g \in G$, so that $ag \in \mathcal{X}C_{RH}(A)$. In particular, $\mathcal{X}\ C_{RH}(A)$ is an RG-submodule of A. Further, $C_H(A_1) = g_1^{-1}Ug_1 \cap \dots \cap g_n^{-1}Ug_n$, hence $H/C_H(A_1) \in \mathcal{X}$. From the definition of overfinite formation we obtain that $G/C_G(A_1) \in \mathcal{X}$, i.e. $\mathcal{X}C_{RH}(A) \leq \mathcal{X}C_{RG}(A)$. The converse inclusion is also valid, so that $\mathcal{X}C_{RH}(A) = \mathcal{X}C_{RG}(A)$. It follows from transfinite induction that $\mathcal{X}C^{\infty}_{RH}(A) = \mathcal{X}C^{\infty}_{RG}(A)$.

Let $B = B_1 \oplus B_2$ where $B_1 = \mathcal{X}C^{\infty}_{RH}(B), B_2 = \mathcal{X}C_{RH}{}^*(B)$. Then $B_2g_1 + \dots + B_2g_n$ is an RG-submodule of A. Corollary 2 of Lemma 2 implies that $B_2g_1 + \dots + B_2g_n \leq \mathcal{X}C_{RH}{}^*(A)$. Lemma 1 yields that $B_1g_1 + \dots + B_1g_n \leq \mathcal{X}C^{\infty}_{RH}(A)$, hence $\mathcal{X}C_{RH}{}^*(A) = B_2g_1 + \dots + B_2g_n$, in particular, $\mathcal{X}C_{RH}{}^*(A)$ is an RG-submodule of A. Let U and V be RG-submodules of $\mathcal{X}C_{RH}{}^*(A)$ such that $U \geq V$ and $U \neq V$. Then U/V is a non-zero RH-factor of $\mathcal{X}C_{RH}{}^*(A)$, so that $H/C_H(U/V) \notin \mathcal{X}$. From the definition of overfinite formation we obtain that $G/C_G(U/V) \notin \mathcal{X}$. This proves the equation $\mathcal{X}C_{RH}{}^*(A) = \mathcal{X}C_{RG}{}^*(A)$.

The following lemma is well-known.

Lemma 4. *Let G be a FC-hypercentral group, L be a finitely generated subgroup of G. Then L is nilpotent-by-finite. In particular, G is a locally (polycyclic-by-finite) group.*

Lemma 5. *Let G be a polycyclic-by-finite group, $1 \neq g \in \zeta(G)$, and let A be a finitely generated $\mathbb{Z}G$-module. If A is a monolithic module with the monolith M and $M(g-1) = \langle 0 \rangle$, then $A(g-1)^m = \langle 0 \rangle$ for some $m \in \mathbb{N}$.*

Proof. Since M is a simple $\mathbb{Z}G$-submodule then $pM = \langle 0 \rangle$ for some prime p ([6], theorem 9.55). Let T be the torsion part of A. Since A is a monolithic module then T is a p-subgroup. Since $\mathbb{Z}G$ is a noetherian ring and A is a finitely generated module then A is a noetherian $\mathbb{Z}G$-module. In particular, T is finitely generated. It follows that there exists a number $n \in \mathbb{N}$ such that $p^n T = \langle 0 \rangle$. Then $p^n A$ is torsion-free, therefore, $M \cap p^n A = \langle 0 \rangle$. This means that $p^n A = \langle 0 \rangle$, i.e. $A = T$. In other words A is a p-group and $p^n A = \langle 0 \rangle$.

We can consider the submodule $A_1 = \Omega_1(A)$ as $\mathbb{F}_p G$-module. Let $R = \mathbb{F}_p\langle x \rangle$ be the group algebra of an infinite cyclic group $\langle x \rangle$ over $\mathbb{F}_p$. Put $ax = ag$ for each $a \in A_1$. Then A is an RG-module and R is a principal ideal domain. Since $M(g-1) = M(x-1) = \langle 0 \rangle$ then the $(x-1)$-component of A_1 is non-zero. Using the same arguments we obtain that A_1 coincides with its $(x-1)$-component. Since A is a noetherian RG-module then this implies that $A_1(x-1)^{m_1} = \langle 0 \rangle$ for some $m_1 \in \mathbb{N}$.

The mapping $\varphi : \Omega_2(A) \to \Omega_1(A)$ defining by the rule $a\varphi = pa, a \in \Omega_2(A)$, is a $\mathbb{Z}G$-homomorphism, so $Im\varphi$ and $\mathrm{Ker}\varphi = \Omega_1(A)$ are $\mathbb{Z}G$-submodules. Since $Im\varphi \leq A_1$ then $Im\varphi(x-1)^{m_1} = \langle 0 \rangle$. Similarly $\mathrm{Ker}\varphi(x-1)^{m_1} = \langle 0 \rangle$. Therefore $\Omega_2(A)(x-1)^{2m_1} = \langle 0 \rangle$.

Using a simple induction and the equality $A = \Omega_n(A)$, we obtain that $A(x-1)^m = \langle 0 \rangle$ where $m = nm_1$. Thus $A(g-1)^m = \langle 0 \rangle$.

Corollary. *Let G be a polycyclic-by-finite group, $1 \neq g \in \zeta(G), A$ be a finitely generated $\mathbb{Z}G$-module. If $C_A(g) \neq \langle 0 \rangle$ then $A \neq A(g-1)$.*

Proof. Let $0 \neq a \in C_A(g)$. Since $g \in \zeta(G)$ then $C_A(g)$ is a $\mathbb{Z}G$-submodule. Let Ba be a maximal $\mathbb{Z}G$-submodule of A with the property $a \notin Ba$. Then A/Ba is a monolithic $\mathbb{Z}G$-module with the monolith $a\mathbb{Z}G + Ba/Ba = M/Ba$. Then $(M/Ba)(g-1) = \langle 0 \rangle$, so we can apply lemma 5 to the module A/Ba.

Lemma 6. *Let G be a locally (polycyclic-by-finite) group, $1 \neq g \in \zeta(G), A$ a* *finitely generated $\mathbb{Z}G$-module. If $C_A(g) \neq \langle 0 \rangle$ then $A \neq A(g-1)$.*

Proof. Let $A = a_1\mathbb{Z}G + \ldots + a_n\mathbb{Z}G$. Assume that $A = A(g-1)$. Then there are elements $b_1, \ldots, b_n \in A$ such that $a_i = b_i(g-1), 1 \leq i \leq n$. Let $0 \neq c \in C_A(g)$. Choose in G a finitely generated subgroup H with the properties $g \in H, c, b_1, \ldots, b_n \in a_1\, \mathbb{Z}H + \ldots + a_n\, \mathbb{Z}H = B$. Let $b \in B$ then $b = a_1x_1 + \ldots + a_nx_n = b_1(g-1)x_1 + \ldots + b_n(g-1)x_n = (b_1x_1 + \ldots + b_nx_n)(g-1)$.

It follows that $B = B(g-1)$. On the other hand, $c \in C_A(g) \cap = C_B(g)$, in particular, $C_B(g) \neq \langle 0 \rangle$. Corollary to lemma 5 implies that $B \neq B(g-1)$. This is a contradiction.

3. Proof of the main theorem.

If $G \in \mathcal{X}$ then $A = \mathcal{X}C^{\infty}_{\mathbb{Z}G}(A)$. Therefore we can assume that $G \notin \mathcal{X}$.

Suppose that A does not have the $\mathcal{X}$-$\mathbb{Z}G$-decomposition. Put $\mathcal{M} = \{B | B$ is a $\mathbb{Z}G$-submodule such that B does not have the $\mathcal{X}$-$\mathbb{Z}G$-decomposition$\}$. Then $A \in \mathcal{M}$, in particular, $\mathcal{M} \neq \emptyset$. Since A is an artinian $\mathbb{Z}G$-module, then $\mathcal{M}$ has a minimal element C. Corollary 4 of lemma 2 implies that C includes the largest $\mathbb{Z}G$-submodule M having the $\mathcal{X}$-$\mathbb{Z}G$-decomposition . From the choice of C we obtain that M includes every proper $\mathbb{Z}G$-submodule of C, in particular, M is a maximal $\mathbb{Z}G$-submodule of C.

Let $M = M_1 \oplus M_2$ where $M_1 = \mathcal{X}C^{\infty}_{\mathbb{Z}G}(M), M_2 = \mathcal{X}C^{*}{}_{\mathbb{Z}G}(M)$. Assume first that $G/C_G(C/M) \notin \mathcal{X}$, and consider the factor-module C/M_2. In other words, we can assume that $M = \mathcal{X}C^{\infty}_{\mathbb{Z}G}(M)$. We can assume also that $C_G(C) = \langle 1 \rangle$.

Put $S = Soc\ _{\mathbb{Z}G}(C)$. Since S has the $\mathcal{X}$-$\mathbb{Z}G$-decomposition then $S \leq M$. It follows that $G/C_G(S) \in \mathcal{X}$, in particular, $C_G(S) \neq \langle 1 \rangle$. Since G is a FC-hypercentral group then $C_G(S) \cap FC(G) \neq \langle 1 \rangle$ [4, lemma 3]. Let $1 \neq x \in C_G(S) \cap FC(G)$, then $\langle x \rangle^G$ is central-by-finite and the index $|G : C_G(\langle x \rangle^G)|$ is finite. Therefore either $\langle x \rangle^G$ includes a finite minimal G-invariant subgroup X

or a G-invariant torsion-free finitely generated subgroup X. If X is finite then X is also abelian, because G is locally soluble. Put $H = C_G(X)$. In these both cases $|G : H|$ is finite and $X \leq \zeta(H)$.

Since C/M is a simple $\mathbb{Z}G$-module then $C/M = \oplus_{1 \leq i \leq n}(B/M)g_i$ where B/M is a simple $\mathbb{Z}H$-module, $g_1, \ldots, g_n \in G$ [8, lemma]. If we assume that $H/C_H(B/M) \in \mathcal{X}$ then from the equation $H/C_H((B/M)g) = H/g^{-1}C_H(B/M)g \cong H/C_H(B/M)$ we obtain that $H/C_H((B/M)g_i) \in \mathcal{X}$ for any $i, 1 \leq i \leq n$. It follows that $H/C_H(C/M) \in \mathcal{X}$. This contradiction shows that $H/C_H(B/M) \notin \mathcal{X}$. Since $B \not\leq M^G$ then $B\ \mathbb{Z}G = C$. If we assume that B has the $\mathcal{X}$-$\mathbb{Z}G$-decomposition then $C = B\ \mathbb{Z}G$ has the $\mathcal{X}$-$\mathbb{Z}G$-decomposition by lemma 3, a contradiction. Hence B does not have the $\mathcal{X}$-$\mathbb{Z}G$-decomposition.

Put $\mathcal{C} = \{Q | Q$ is a $\mathbb{Z}G$-submodule of B such that Q does not have the $\mathcal{X}$-$\mathbb{Z}G$-decomposition$\}$. Since $B \in \mathcal{C}$ then $\mathcal{C} \neq \emptyset$. By theorem A of paper [7] C is an artinian $\mathbb{Z}G$-module. Thus $\mathcal{C}$ has a minimal element E. Lemma 3 yields that $E \not\leq M$, so that $B = E + M$. Corollary 4 of lemma 2 shows that E includes the largest $\mathbb{Z}H$-submodule E_1 having the $\mathcal{X}$-$\mathbb{Z}G$-decomposition. Lemma 3 shows that $E_1\mathbb{Z}G \leq M$, i.e. $E_1 \leq M$. Since B/M is a simple $\mathbb{Z}H$-module then $E_1 = E \cap M$. Moreover, $E/E_1 = E/E \cap M \cong_{DH} E + M/M = B/M$, so that $H/C_H(E/E_1) \notin \mathcal{X}$.

Let $S_1 = Soc_{\mathbb{Z}G}(C)$. Since C is an artinian $\mathbb{Z}H$-module then $S_1 = L_1 \oplus \ldots \oplus L_S$ for some simple $\mathbb{Z}H$-submodules $L_i, 1 \leq i \leq s$. Since $X \leq \zeta(H)$ then $L_i(\omega\mathbb{Z}X)$ is a $\mathbb{Z}H$-submodule of $L_i, 1 \leq i \leq s$, here $\omega\mathbb{Z}X$ is the augmentation ideal of $\mathbb{Z}X$. This means that either $L_i(\omega\mathbb{Z}X) = L_i$ or $L_i(\omega\mathbb{Z}X) = \langle 0 \rangle$ because L_i is a simple $\mathbb{Z}G$-module of $C, 1 \leq i \leq s$. Consequently, $S_1 = C_{S1}(X) \oplus S_1(\omega\mathbb{Z}X)$. Since S_1 is a $\mathbb{Z}G$-submodule of C and X is a normal subgroup then $S_1(\omega\mathbb{Z}X)$ is a $\mathbb{Z}G$-submodule of C. If we assume that $S_1(\omega\mathbb{Z}X) \neq \langle 0 \rangle$ then $S_1(\omega\mathbb{Z}X) \cap Soc_{\mathbb{Z}G}(C) \neq \langle 0 \rangle$. On the other hand, $X \leq C_G(Soc_{\mathbb{Z}G}(C))$, so that $S_1(\omega\mathbb{Z}X) \cap Soc_{\mathbb{Z}G}(C) \leq S_1(\omega\mathbb{Z}X) \cap C_{S1}(X) = \langle 0 \rangle$. This contradiction shows that $S_1 \leq C_G(X)$. Hence $E \cap C_C(X) \neq \langle 0 \rangle$. If $e \in E \backslash E_1$ then $e\mathbb{Z}H \not\leq E_1$, so

$e\mathbb{Z}H = E$. In particular, E is a finitely generated $\mathbb{Z}H$-module. It follows from lemmas 4 and 6 that $E(g-1) \le E_1$ for each $g \in X$. As in lemma 3 we can prove that $E_1 = \mathcal{X}C^{\infty}_{\mathbb{Z}H}(E)$. Consider the mapping $\theta : a+E_1 \to a(g-1)+E_1(g-1), a \in E$. If $E(g-1) \neq E_1(g-1)$ then from $E(g-1) \le E_1 = \mathcal{X}C^{\infty}_{ZH}(E)$ we obtain that $H/C_H(E(g-1)/E_1(g-1)) \in \mathcal{X}$. However $E(g-1)/E_1(g-1) \cong_{ZH} E/E_1$, in particular, $C_H(E/E_1) = C_H(E(g-1)/E_1(g-1))$. But $H/C_H(E/E_1) \notin \mathcal{X}$. This contradiction shows that $E(g-1) = E_1(g-1))$. This means that $E = C_E(g) + E_1$. It follows from the choice of E that $E = C_E(g)$ because $C_E(g)$ is a $\mathbb{Z}H$-submodule of E. Since this is true for each $g \in X$ then $E \le C_C(X)$, in particular, $C_C(X) \not\le M$. Since X is a normal subgroup of G then $C_C(X)$ is a $\mathbb{Z}G$-submodule. Hence $C = C_C(X)$, i.e. $X \le C_G(C) = \langle 1 \rangle$. This is a contradiction.

Now consider the case when $G/C_G(C/M) \in \mathcal{X}$. We will consider the factor-module C/M_1. In other words, we can assume that $M = \mathcal{X}C^{*}_{\mathbb{Z}G}(M)$. Again we can assume that $C_G(C) = \langle 1 \rangle$.

Since $G/C_G(C/M) \in \mathcal{X}$ then $C_G(C/M) \neq \langle 1 \rangle$. Therefore $C_G(C/M) \cap FC(G) \neq \langle 1 \rangle$.

Let $1 \neq y \in C_G(C/M) \cap FC(G)$ then $Y = \langle y \rangle^G$ is central-by-finite and $|G:R|$ is finite where $R = C_G(Y)$. Again we can assume that Y is abelian, i.e. $Y \le \zeta(R)$.

Let $\mathcal{C}_1 = \{Q | Q$ is a $\mathbb{Z}R$-submodule of C such that $Q \not\le M\}$. Since $C \in \mathcal{C}_1$ then $\mathcal{C}_1 \neq \emptyset$. By theorem A of paper [7] C is an artinian $\mathbb{Z}R$-module. Hence the set $\mathcal{C}_1$ has a minimal element U. Since $Y \le \zeta(R)$ then $U(g-1)$ and $C_U(g)$ are $\mathbb{Z}R$-submodules for any $g \in Y$. Moreover, $U(g-1) \cong U/C_U(g)$. As in lemma 3 we can prove that $M = \mathcal{X}C^{*}_{\mathbb{Z}R}(M)$. Since $g \in C_G(C/M)$ then $U(g-1) \le M$. If we assume that $C_U(g) \le M$ then $U/C_U(g)$ has one non-zero $\mathcal{X}$-central $\mathbb{Z}R$-factor. On the other hand, from $U(g-1) \cong_{\mathbb{Z}R} U/C_U(g)$ and $U(g-1) \le \mathcal{X}C^{*}_{\mathbb{Z}R}(M)$ we obtain that every non-zero $\mathbb{Z}R$-factor of $U/C_U(g)$

is $\mathcal{X}$-eccentric. This means that $C_U(g) \not\leq M$. It follows from the choice of U that $C_U(g) = U$. It is valid for every $g \in Y$. Therefore $U \leq C_C(Y)$. In particular, $C_C(Y) \not\leq M$. Since Y is normal in G then $C_C(Y)$ is a $\mathbb{Z}G$-submodule. Then $C = C_C(Y)$ because M includes every proper $\mathbb{Z}G$-submodule of C. Consequently, $Y \leq C_G(C) = \langle 1 \rangle$, so that in this case we also obtain a contradiction. This final contradiction completes the proof.

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