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ON A POSSIBLE NONSIMPLICITY CRITERION FOR FINITE FACTORIZED GROUPS

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Let G be a finite group. G is factorized if G is written as $G = HK$ with H and K subgroups of G. Numerous papers have been written on various aspects of such a group. There are a number of results which allow to conclude the non-simplicity of G from suitable conditions on H and K (see for instance [1]).

A well known theorem of R. Maier and H. Wielandt (see [9] and [10]), establishes the following:

Theorem. *Let $G = HK$ be a group such that H and K are subgroups of G. If $X \leq H \cap K$ is a subnormal subgroup of H and K, then X is subnormal in G.*

In [8] O. Kegel proved a very useful criterion for the non-simplicity of a finite group (which he exploited to demonstrate that any product $G = HK$ of finite nilpotent groups H, K is always solvable).

Theorem. (Kegel [8]) (a special formulation) *Let $G = HK$ be a group, H and K subgroups of G. Let p be a prime number such that H and K contain non-trivial normal Sylow-p-subgroups. Then G cannot be a nonabelian simple group.*

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In view of these results the following is a natural conjecture:

Conjecture. Let $G = HK$ be a finite group such that H and K are subgroups of G and let p be a prime number. If $\mathbf{O}_p(H) \neq 1 \neq \mathbf{O}_p(K)$, then G cannot be a nonabelian simple group. (Here $\mathbf{O}_p(H)$ denotes the largest normal p-subgroup of H).

When $\mathbf{O}_p(H) \cap \mathbf{O}_p(K) \neq 1$ the Conjecture is true by Maier-Wielandt's Theorem and when $\mathbf{O}_p(H) \in Syl_p(H)$ and $\mathbf{O}_p(K) \in Syl_p(K)$ it is true by Kegel's Result.

Using classification theorems of simple groups we prove a particular case of our Conjecture.

Theorem. *Let $G = HK$ be a finite group such that H and K are solvable subgroups of G and let p be a prime number. If $\mathbf{O}_p(H) \neq 1 \neq \mathbf{O}_p(K)$, then G cannot be a nonabelian simple group.*

2. Preliminary Results

In this section, we collect some of the results that are needed.

If G is the product of two solvable subgroups, it is known that G is not necessarily non-simple. Particular cases of finite groups factorizable by two solvable subgroups were studied by many authors, Kazarin [7] studied the general case and obtained the following result:

Lemma 2.1. (Kazarin [7]) *Let $G = HK$ be a group with H and K solvable subgroups of G. If all composition factors of G are known groups, then the nonabelian simple composition factors of G belongs to the following* *list of groups:*

(a) $\mathbf{PSL}(2, q)$ *with* $q > 3$

(b) $\mathbf{M}_{11}$

(c) $\mathbf{PSL}(3, q)$ *with* $q < 9$

(d) $\mathbf{PSp}(4, 3)$

(e) $\mathbf{PSU}(3, 8)$

(f) $\mathbf{PSL}(4, 2)$

Remark 1. Let $G = \mathbf{PSL}(2, q)$ with $q = p^l$ and p a prime number. The following properties of G are well known:

(a) A Sylow-p-subgroup P of G is elementary abelian of order $q = p^l$ and P is disjoint from its conjugates.

(b) If r is a prime distinct from p or 2, then a Sylow-r-subgroup of G is cyclic.

(c) If p is odd, then a Sylow-2-subgroup of G is dihedral.

For a proof see: [5], p. 191, Satz 8.2/8.3/8.4.

Lemma 2.2. *Let* G *be a group,* H *and* K *subgroups of* G*. If* H *permutes with every conjugate of* K *in* G*, then* $H^K \cap K^H$ *is subnormal in* G*.*

For a proof see: Wielandt [11].

Proposition 2.3. Let $G = HK$ be a group, H and K subgroups of G. Let p be a prime number such that $\mathbf{O}_p(H) \neq 1 \neq \mathbf{O}_p(K)$. If $\mathbf{O}_p(H)\mathbf{O}_p(K) = \mathbf{O}_p(K)\mathbf{O}_p(H)$, then G cannot be a nonabelian simple group.

Note that this proposition contains Kegel's result as a particular case, if $\mathbf{O}_p(H) \in Syl_p(H)$ and $\mathbf{O}_p(K) \in Syl_p(K)$.

Proof: Suppose G be a simple group. Put $X = \mathbf{O}_p(H)$ and $Y = \mathbf{O}_p(K)$. Let $g = hk \in G$ with $h \in H$ and $k \in K$. We have:

$$X^g Y = X^{hk} Y = X^k Y = (XY^{k^{-1}})^k = (XY)^k = (YX)^k = YX^{hk} = YX^g$$

So, by Lemma 2.2 we have $X^Y \cap Y^X$ is subnormal in G. Since XY is a p-subgroup, we conclude that $X^Y \cap Y^X = 1$. Hence the commutator subgroup $[X^Y, Y^X] \leq X^Y \cap Y^X = 1$. Let $g = hk \in G$, $x \in X$ and $y \in Y$. Since $X \trianglelefteq H$ we have that $x^g = x^{hk} = x_1^k$, with $x_1 \in X$ and $y^{x^g} = y^{x_1^k} = k^{-1}x_1^{-1}kyk^{-1}x_1k = y$. So $G = X^G \leq \mathbf{C}_G(Y)$, a contradiction.

Proposition 2.4. Let $G = HK$ be a group, H and K subgroups of G. Let p be a prime number such that $\mathbf{O}_p(H) \neq 1 \neq \mathbf{O}_p(K)$. Then each of the following conditions implies the non-simplicity of G:

(a) $\mathbf{O}_p(H) \cap \mathbf{O}_p(K) \neq 1$

(b) G has only one conjugacy class of elements of order p.

(c) $P \in Syl_p(G)$ is disjoint from its conjugates.

Proof: (a) This is a particular case of Maier-Wielandt's theorem [9] and [10]. Also see Lemma 1.4 of Kazarin [6]

(b) Suppose G be a simple group. Let $x \in \mathbf{O}_p(H)$ and $y \in \mathbf{O}_p(K)$ with $|x| = |y| = p$ and $g = hk \in G$ such that $x^g = x^{hk} = y$. We have $x^h = y^{k^{-1}} \in \mathbf{O}_p(H) \cap \mathbf{O}_p(K)$ in contradiction to (a).

(c) This item is clear.

3. Proof of our theorem

Suppose that the theorem is false and let G be a counterexample. We will apply the Lemma 2.1 to G :

(I) Assume $G \cong \mathbf{PSL}(2,q)$ as in (a).
If $q = 2^n$, then every Sylow-subgroup of G is abelian, hence by proposition 2.3 we have that a contradiction.
If q is odd, then for every prime $r \neq 2$ a Sylow-r-subgroup is abelian. Let $S \in Syl_2(G)$. By Remark 1, we have that S is a dihedral group. By proposition 2.3 , we have that $|S| > 4$. So by [2], p. 262, Th. 7.3, we have that G has one conjugate class of involutions, in contradiction to proposition 2.4 (b).

Since $\mathbf{M}_{11}$ has one conjugate class of elements of order two and one of elements of order three, we have that G is not isomorphic to $\mathbf{M}_{11}$.

(II) Assume $G \cong \mathbf{PSL}(3,q)$ as in (c).
In this case G has one conjugate class of involutions. Hence p is odd and by proposition 2.3 the only possibility is $p = q$ $(= 3, 5, 7)$. Let r the largest prime divisor of $|G|$ and $R \in Syl_r(G)$ such that $R \leq H$. If $\mathbf{O}_p(H) \neq 1$, we obtain that p divides $|N_G(R)|$ a contradiction.

(III) Assume $G \cong \mathbf{PSp}(4,3)$ $(|G| = 2^6 \cdot 3^4 \cdot 5)$
Since G has one conjugate class of involutions, the only case to check is $p = 3$. Let $P \in Syl_5(G)$ such that $P \leq H$. Since 3 is not a divisor of $|\mathbf{C}_G(P)|$, we obtain the contradiction $\mathbf{O}_3(H) = 1$.

(IV) Assume $G \cong \mathbf{PSL}(4,2) \cong A_8$ $(|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7)$
In this case, the only possibility is $p = 2$. Let $R \in Syl_7(G)$ such that $R \leq H$ and $L \in Syl_5(G)$. Since $|\mathbf{C}_G(R)| = 21$ by Sylow's theorem we obtain that $|\mathbf{O}_2(H)| = 8$. Since H is solvable by Hall's theorem (see [3] and

[4]) we may assume that $L \leq K$. Since $|\mathbf{C}_G(L)| = 7$ so $|\mathbf{O}_2(K)| \geq 8$. Hence $\mathbf{O}_2(H)\mathbf{O}_2(K) = \mathbf{O}_2(K)\mathbf{O}_2(H)$ in contradiction to proposition 2.6.

(V) Assume $G \cong \mathbf{PSU}(3,8)$ $(|G| = 2^9 \cdot 3^4 \cdot 7 \cdot 19)$
Let $S \in Syl_2(G)$. Then S is disjoint from its conjugates, so by proposition 2.4 (c) we have that the only possibility is $p = 3$. Let $P \in Syl_{19}(G)$, such that $P \leq H$. Since 3 is not a divisor of $|\mathbf{C}_G(P)|$ we have the contradiction, $\mathbf{O}_3(H) = 1$.

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