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CONICS FIVE-FOLD TANGENT TO A PLANE CURVE

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Abstract

We study the question of determining the number of conics tangent to a general plane curve at five unassigned points. This is related to the number of rational curves in a system of curves on the $K3-$surface obtained as a double cover of the plane ramified along a sextic.

Resumo

Analisamos a questão de determinar o número de cônicas tangentes a uma curva plana genérica em cinco pontos não prefixados. Isto é relacionado com o número de curvas racionais em um sistema de curvas na superfície $K3$ obtida como recobrimento duplo do plano ramificado em uma séxtica.

To the memory of Prof. Claude Itzykson

1. Introduction

This article is motivated by a question posed by C. Itzykson regarding the number of conics five-fold tangent to a plane sextic. He asked it (priv. comm. on Aug. 30, 1994) in connection with the problem of enumerating rational curves on certain K3 surfaces. We present here only the general set-up, describing appropriate maps and parameter spaces. Actual enumerative calculations must be deferred until a few difficulties explained in the sequel are solved.

The subject of finding numerical invariants of families of curves of low genus has received many recent contributions in the context of Gromov-Witten theory,

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cf. Fulton-Pandharipande [3] for a survey. The case envisaged here is not covered by Yau-Zaslow [9] beautiful formula. This is also *not* a special case of the well-known formula of De Jonquières [2], [7] due to the presence of non–reduced conics. The device enabling us to get rid of these "bad" members in the family is the classical concept of complete conics (see [5] and the references therein).

Let C denote a general plane curve of degree d. Let $\mathbb{K}$ denote the variety of complete conics (cf. §2 below for a review). We set (cf. 3.1 for precision),

$$\mathbb{K}_C = \{((\kappa, \kappa'), P) \in \mathbb{K} \times C \mid P \in \kappa \cap C \text{ and } t_P C \in \kappa'\}.$$

Here $t_P C$ is the tangent line to C at P. We wish to employ multiple point formulas (cf. Kleiman [4]) to a suitable modification of the map

$$\begin{array}{ccc} \mathbb{K}_C & \xrightarrow{p} & \mathbb{K} \\ ((\kappa, \kappa'), P) & \mapsto & (\kappa, \kappa'). \end{array}$$

We show that for each $i = 1 \ldots 5$, the following holds (cf. 3.2).

1. The multiple point set $m_i(p) = \{y \in \mathbb{K}_C \mid \text{length}\, p^{-1}p(y) \geq i\}$ is of the right codimension $i - 1$ and

2. the image $p(m_5(p))$ in $\mathbb{K}$ consists of nondegenerate conics only.

The modification is required in order to resolve singularities of the source of the map p. Indeed, $\mathbb{K}_C$ turns out to be singular (not even l.c.i.) along the graph of the embedding $\iota : C \hookrightarrow \mathbb{K}$ defined by $P \mapsto ((t_P C)^2, (\check{P})^2)$, where $(\check{P})^2$ denotes twice the line dual to a point P. It can be shown that the blowup $\widehat{\mathbb{K}}_C$ of $\mathbb{K}_C$ along the graph of ι is smooth. But now the composite map $\widehat{\mathbb{K}}_C \to \mathbb{K}_C \to \mathbb{K}$ is no longer finite.

To get around this difficulty, we let $\widehat{\mathbb{K}}$ denote the blowup of $\mathbb{K}$ along the image of the embedding ι. We get an induced map $\widehat{p} : \widehat{\mathbb{K}}_C \longrightarrow \widehat{\mathbb{K}}$ which can be shown to be finite. Moreover, its restriction to the exceptional divisor of $\widehat{\mathbb{K}}_C$ is an embedding. Thus, roughly speaking, the original map p and its modification $\widehat{p}$ display the same multiple point loci, cf. [8]. In *loc. cit.* we hope to explain how to compute the relative tangent bundle of $\widehat{p} : \widehat{\mathbb{K}}_C \to \widehat{\mathbb{K}}$, a main ingredient for using multiple-point formulae. For this end, we will show in [8] that $\widehat{\mathbb{K}}_C$ is obtained by blowingup twice the conormal variety of C (embedded in pd5 by the Veronese map) along explicit, smooth centers.

A few words of caution are due. Recall that when one applies the double point formula to the Gauss map of a plane curve, one gets the number of bitangent lines *plus* the inflexional ones. Of course, in this case, one knows how to account for inflexions separately (e.g., employing ramification points formulae). Similarly, multiple point theory in the present setting gives us the *total* number of irreducible conics five-fold tangent to C. This includes stationary tangent conics of several types, one for each partition 5=4+1=3+2... (e.g., sextactic[1] conics).

Now, in order to apply the present setup to the counting of rational curves on the double cover of $\mathbb{P}^2$ ramified along a plane sextic C, we need to find the contributions of coalescing 5-fold points separately from the total of 5-fold points of $\widehat{p}$. Indeed, a sextactic point contributes less to genus reduction than 5 distinct simple tangencies to the curve C do. The same observation applies to other partitions of 5. Unfortunately, these coalescing points occur in wrong codimension for stationary multiple point theory: the known formulas (cf. [1]) are not applicable. Some of the types of stationary tangent conics may be computed by a De Jonquières like approach. That is the case when the number of distinct points of contact is at most 2. So far we haven't been able to calculate the remaining cases, e.g., irreducible conics cutting a divisor on C of the form $4P + 2Q + 2R + \cdots$. Thus, straightforward application of multiple point formula would give us, at the present stage, only an upper bound for the sought for number of rational curves in a general K3 surface with a polarization of genus 2.

A similar approach might work for the question of counting rational curves of arithmetic genus 9 on a general quartic surface in 3-space (cf. B. Segre [6]). See Fig. 2. We hope to report on this elsewhere.

[1] As Coolidge has observed, "it is hard to keep away from sex these days", [2], p. 280.

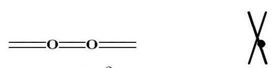
We work throughout over the field of complex numbers.

2. Complete conics

Let $\mathbb{K} \subset \check{\mathbb{P}}^5 \times \mathbb{P}^5$ denote the closure of the graph of the rational map

$$\check{\mathbb{P}}^5 \ni \kappa \longmapsto \kappa' \in \mathbb{P}^5$$

that assigns to an irreducible conic κ its dual conic κ'. The variety $\mathbb{K}$ is called the variety of complete conics. The map $p_1 : \mathbb{K} \to \check{\mathbb{P}}^5$ induced by projection is the blowup of the Veronese surface $V \cong \check{\mathbb{P}}^2 \subset \check{\mathbb{P}}^5$ of double lines. The exceptional divisor $p_1^{-1}V$ is the set of pairs (κ, κ') such that κ is a double line, κ' is a line pair in the dual plane and the pair of points (called *foci*) dual to the line pair κ' lies on κ.



$\kappa \subset \mathbb{P}^2$ $\qquad$ $\kappa' \subset \check{\mathbb{P}}^2$

A similar dual description holds for the second projection $p_2 : \mathbb{K} \to \mathbb{P}^5$.

We fix in the sequel affine coordinates x, y in $\mathbb{P}^2$ and $a_1, \dots, a_5$ in $\check{\mathbb{P}}^5$. The $a_i's$ are thought of as coefficients of the conic

$$y^2 + a_1xy + a_2x^2 + a_3y + a_4x + a_5.$$

The rational map $\kappa \mapsto \kappa'$ is expressed by

$$\begin{cases} b_0 = a_4^2 - 4a_5a_2, & b_1 = 4a_5a_1 - 2a_4a_3, \\ b_2 = 4a_3a_2 - 2a_4a_1, & b_3 = a_3^2 - 4a_5, \\ b_4 = 4a_4 - 2a_1a_3, & b_5 = a_1^2 - 4a_2. \end{cases} \tag{1}$$

Here the b_i's denote homogeneous coordinates in $\mathbb{P}^5$ corresponding to the coefficients of the conic

$$\kappa' = b_0\gamma^2 + b_1\alpha\gamma + b_2\beta\gamma + b_3\alpha^2 + b_4\alpha\beta + b_5\beta^2$$

in the dual plane with homogeneous coordinates α,β,γ.

Recall that the polynomials in the right hand side of (1) generate the ideal of the blowup center V. In the affine patch chosen above, the following three generators in fact suffice:

$$z_2 := a_2 - a_1^2/4,\ z_4 := a_4 - a_1a_3/2,\ z_5 := a_5 - a_3^2/4.$$

The pullback of this patch to $\mathbb{K}$ is covered by three affine pieces, one for each choice of z_2, z_4 or z_5 as principal generator for the ideal of the exceptional divisor. We fix the coordinate chart for $\mathbb{K}$ so that z_2 is a local equation of the exceptional divisor. This is the affine space $\mathbb{A}^5$ with coordinate functions a_1, a_2, a_3, u_4, u_5 such that the restriction of the blowup $\mathbb{K} \to \check{\mathbb{P}}^5$ is given by the homomorphism

$$\begin{array}{c} k[a_1, a_2, a_3, a_4, a_5] = k[a_1, z_2, a_3, z_4, z_5] \quad \hookrightarrow \quad k[a_1, z_2, a_3, u_4, u_5] \\ z_4 = u_4 z_2,\ z_5 = u_5 z_2. \end{array} \tag{2}$$

Solving the equations $b_4 = 4u_4(a_2 - a_1^2/4)$, $b_3 = -4u_5(a_2 - a_1^2/4)$ for a_4, a_5 (with b_4, b_3 as in (1)), we find

$$\begin{cases} a_4 = u_4a_2 + a_1a_3/2 - u_4a_1^2/4, \\ a_5 = u_5a_2 + a_3^2/4 - u_5a_1^2/4. \end{cases}$$

Setting $a_2 = z_2 + a_1^2/4$, the expression for each b_i in (1) becomes a multiple of z_2. Thus, after cancelling out z_2, we see that the rational map $\check{\mathbb{P}}^5 \cdots \to \mathbb{P}^5$ extends to a morphism $p_2 : \mathbb{K} \to \mathbb{P}^5$ given in the chosen affine patch of $\mathbb{K}$ by the assignment

$$\begin{cases} b_0 = z_2u_4^2 + a_1a_3u_4 - a_3^2 - (a_1^2 + 4z_2)u_5, & b_1 = 4a_1u_5 - 2a_3u_4, \\ b_2 = 4a_3 - 2a_1u_4, & b_3 = -4u_5, \\ b_4 = 4u_4, & b_5 = -4 \end{cases} \tag{3}$$

3. The completed conormal variety

Let C be a smooth plane curve of degree at least 3. We set

$$\mathbf{Co}_C := \{(P, \kappa) \in C \times \check{\mathbb{P}}^5 \,|\, \kappa \cdot C \geq 2P\}.$$

One checks easily that $\mathbf{Co}_C$ is a $\mathbb{P}^3$-bundle$/C$. It is usually referred to as the *conormal variety* of C for the Veronese embedding $C \subset \mathbb{P}^5$.

Let $q : \mathbf{Co}_C \to \check{\mathbb{P}}^5$ be induced by projection. Set $\check{C} := q(\mathbf{Co}_C)$. Thus $\check{C}$ is the dual variety of $C \subset \mathbb{P}^5$. The induced map $\mathbf{Co}_C \to \check{C}$ is finite, birational. For $\kappa \in \check{C}$ the fiber $q^{-1}\kappa$ is supported in the set of $P \in C$ such that $\kappa \cdot C \geq 2P$. If κ is a double line then clearly $q^{-1}\kappa$ is equal to $C \cap \kappa$ as sets. In particular, notice q admits ∞^2 many $r-$fold points for each $r = 1 \ldots \deg C$. Thus $q : \mathbf{Co}_C \to \check{\mathbb{P}}^5$ is *not* appropriately generic in the sense of [4]. Hence the formulas for $r-$fold loci do not apply to the map q.

Our goal is to replace the map q by a map obtained by blowing up its source and target. The new map will be shown to be appropriately generic.

The intuition supporting the idea of the construction is best explained by the picture below. The top complete conic is threefold tangent whereas the middle one is transversal and the bottom one is bitangent! Moreover, no degenerate complete conic contributes to the five-fold point locus, cf. 3.2.

Fig. 1

Set

$$W = \{(P, \ell) \in C \times \check{\mathbb{P}}^2 \,|\, \ell \cdot C \geq P\}.$$

Clearly W is a $\mathbb{P}^1$-bundle over C. It embeds in $C \times \check{\mathbb{P}}^5$ by squaring the linear form ℓ. Also note that C embeds in W as the graph of the Gauss map, with image

$$\{(P,\ell) \in C \times \check{\mathbb{P}}^2 \,|\, \ell \cdot C \geq 2P\}.$$

We get the commutative diagram,

$$\begin{array}{ccccc} C \times \check{\mathbb{P}}^5 & \supset & \mathbf{Co}_C & & \\ \cup & & \cup & & \\ C \times V & \supset & W & \supset & C. \end{array} \tag{4}$$

It is fair to say that the whole fun (or difficulty if you wish) in the sequel stems from the fact that the square in the previous diagram is not cartesian. Indeed, it turns out that the scheme intersection

$$W\!\!\!\!\!{-} := (C \times V) \cap \mathbf{Co}_C \tag{5}$$

presents an embedded component along the image of C in W by the Gauss map described above. This is revealed by the local-analytic calculations performed in the sequel.

3.1. We call the blowup $\mathbb{K}_C$ of $\mathbf{Co}_C$ along $W\!\!\!\!\!{-}$ the *completed conormal variety of* C.

Let x, y (resp. $a_1, \ldots, a_5$) be affine coordinates in $\mathbb{P}^2$ (resp. $\check{\mathbb{P}}^5$). Assume that the line $y = 0$ is tangent to C at the origin. Let $f(x) := c_2x^2 + \cdots$ be a power series so that $y = f(x)$ holds in the completion

$$\widehat{\mathcal{O}}_{C,0} \cong \mathbb{C}[\![x,y]\!]/(y - f(x)) \tag{6}$$

of the local ring of C at 0. Put

$$\left\{ \begin{array}{ll} z_1 := a_1/2 + f'(x), & z_2 := a_2 - a_1^2/4, \\ z_3 := a_3/2 + a_1x/2 + f(x), & z_4 := a_4 - a_1a_3/2, \\ z_5 := a_5 - a_3^2/4 & \end{array} \right. \tag{7}$$

We have the following local-analytic generators for the ideals of the embeddings displayed in the diagram (4):

$$C \times V \subset C \times \check{\mathbb{P}}^5 : \quad z_2, z_4, z_5 \tag{8}$$

$$\mathbf{Co}_C \subset C \times \check{\mathbb{P}}^5 : \left\{ \begin{array}{l} z_3^2 + z_2 x^2 + z_4 x + z_5, \\ 2 z_3 z_1 + 2 z_2 x + z_4 \end{array} \right. \tag{9}$$

Indeed, let α,β be affine coordinates in the dual plane $\check{\mathbb{P}}^2$. Put

$$\left\{ \begin{array}{l} \lambda := y + \alpha x + \beta \\ q := y^2 + a_1 xy + a_2 x^2 + a_3 y + a_4 x + a_5 \end{array} \right.$$

Matching the coefficients of λ^2 and q, we see the map $\check{\mathbb{P}}^2 \xrightarrow{\sim} V \hookrightarrow \check{\mathbb{P}}^5$ is given by

$$(\alpha, \beta) \mapsto (a_1 = 2\alpha, a_2 = \alpha^2, a_3 = 2\beta, a_4 = 2\alpha\beta, a_5 = \beta^2). \tag{10}$$

Eliminating α,β yields local equations for $V \subset \check{\mathbb{P}}^5$, whence (8). The second relation follows by substituting $y = f(x)$ in λ, q. For this, write $q = \lambda^2 + q - \lambda^2$. We get $q(x, f(x)) = z_3^2 + z_2 x^2 + z_4 x + z_5$. Differentiating with respect to x yields the second generator in (9).

It can be easily shown that the completed conormal variety $\mathbb{K}_C$ is equal to the closure in $C \times \mathbb{K}$ of the correspondence

$$\{(P, \kappa) \in C \times \check{\mathbb{P}}^5 \,|\, \kappa \text{ is an irreducible conic tangent to } C \text{ at } P\}.$$

The precise meaning of Fig. 1 above may be stated thus.

1. Let κ be a double line and let $(P, (\kappa, \kappa')) \in C \times \mathbb{K}$. We have

$$(P, (\kappa, \kappa')) \in \mathbb{K}_C \text{ if and only if } P \in C \cap \kappa \text{ and } t_P C \in \kappa'.$$

2. Suppose the support of a double line κ is transversal to C at P. Then

$$(P, (\kappa, \kappa')) \in \mathbb{K}_C \text{ if and only if } P \text{ is one of the foci of } (\kappa, \kappa'). \tag{11}$$

In fact, we compute below the possible dimensions for the artinian local k-algebra $\mathcal{A}$ of the fiber of $\mathbb{K}_C$ over (κ, κ') at $(0, (\kappa, \kappa'))$. The answers depend on whether

- κ_{red} is transversal to C at 0,
- the point 0 is inflexional and

- a focus of κ' is on 0.

For this end, we describe local-analytic equations of $\mathbb{K}_C$ in $C \times \check{\mathbb{P}}^5$.

The uniformizing parameter x (see 6) enables us to replace C by $\mathbb{A}^1$ (or a small disc) in the étale (or classical-analytic) topology. Accordingly, we replace $C \times \check{\mathbb{P}}^5$ by the affine space (or poly-disk) $\mathbb{A}^1 \times \mathbb{A}^5$ with coordinate functions $x, a_1, a_2, a_3, a_4, a_5$. In view of (7) we may as well employ the coordinate functions $x, z_1, z_2, z_3, z_4, z_5$.

Put

$$\begin{cases} v_5 := -x^2 - u_4 x - u_5, \\ v_4 := -2x - u_4. \end{cases} \tag{12}$$

Recalling (9), we see that the ideal of the *total* transform of $\mathbf{Co}_C$ in $C \times \mathbb{K}$, restricted to this $\mathbb{A}^6$, is generated by $\xi := z_3^2 - z_2 v_5$ and $\eta := 2z_3 z_1 - z_2 v_4$. Hence it also contains $2z_1\xi - z_3\eta = z_2(z_3 v_4 - 2z_1 v_5)$. Since z_2 is a local equation of the exceptional divisor, it follows that the last expression in parenthesis vanishes on the *strict* transform $\mathbb{K}_C$ of $\mathbf{Co}_C$ in $C \times \mathbb{K}$. We proceed to show that the ideal of $\mathbb{K}_C$ is in fact locally generated by the three quadrics,

$$z_3^2 - z_2 v_5,\ 2z_3 z_1 - z_2 v_4,\ z_3 v_4 - 2z_1 v_5. \tag{13}$$

The reader will at once identify (13) as the homogeneous equations in $\mathbb{P}^4$ (with homogeneous coordinates z_1, z_2, z_3, v_4, v_5) of a ruled cubic surface isomorphic to $\mathbb{P}^2$ blown up at a point, embedded by the system of conics through the point. Hence these quadrics define an integral subscheme of $\mathbb{A}^6$ which coincides with $\mathrm{Bl}_{\mathbb{W}}(\mathbf{Co}_C) = \mathbb{K}_C$ on the complement of the exceptional divisor. Therefore, the present affine patch of $\mathbb{K}_C$ is analytically isomorphic to a product $\mathbb{A}^1 \times V'$ of the affine line (with x coordinate) by the affine cone $V' \subset \mathbb{A}^5$ of the ruled surface.

The three local equations (13) will enable us to compute the fiber of $\mathbb{K}_C \to \mathbb{K}$ over a complete conic $(\kappa, \kappa') \in \mathbb{K}$. Assume κ is a double line, say

$$\kappa = (y + \alpha_0 x)^2.$$

Notation as in (10), we have

$$a_1(\kappa) = 2\alpha_0,\ a_2(\kappa) = \alpha_0^2,\ z_2(\kappa) = a_3(\kappa) = a_4(\kappa) = a_5(\kappa) = 0,$$

and from (3), we may write

$$\kappa' = -4\beta^2 - 4\bar{u}_5\alpha^2 + 4\bar{u}_4\alpha\beta + 8\bar{u}_5\alpha_0\alpha\gamma - 4\bar{u}_4\alpha_0\beta\gamma - 4\bar{u}_5\alpha_0^2\gamma^2, \tag{14}$$

for some $\bar{u}_4, \bar{u}_5 \in \mathbb{C}$. The local ring $\mathcal{A}$ of the fiber of $\mathbb{K}_C$ over (κ, κ') at $(0, (\kappa, \kappa'))$ is the quotient of

$$\mathbb{C}\big[a_1, a_2, a_3, u_4, u_5\big] = \mathbb{C}\big[z_1, z_2, z_3, u_4, u_5\big]$$

localized at the maximal ideal

$$\begin{gathered}\langle x,\ a_1 - 2\alpha_0,\ a_2 - \alpha_0^2,\ a_3,\ u_4 - \bar{u}_4,\ u_5 - \bar{u}_5\rangle = \\ \langle x,\ z_1 - f'(x) - \alpha_0,\ z_2,\ z_3 - \alpha_0 x - f(x),\ u_4 - \bar{u}_4,\ u_5 - \bar{u}_5\rangle\end{gathered}$$

by the ideal

$$\begin{gathered}\Big\langle a_1 - 2\alpha_0,\ a_2 - \alpha_0^2,\ a_3,\ u_4 - \bar{u}_4,\ u_5 - \bar{u}_5,\ z_3^2 - z_2 v_5,\ 2z_3 z_1 - z_2 v_4,\ z_3 v_4 - 2z_1 v_5\Big\rangle = \\ \Big\langle z_1 - f'(x) - \alpha_0,\ z_2,\ z_3 - \alpha_0 x - f(x),\ u_4 - \bar{u}_4,\ u_5 - \bar{u}_5,\ z_3^2,\ z_3 z_1,\ z_3 v_4 - 2z_1 v_5\Big\rangle.\end{gathered}$$

By (12), we have

$$\mathcal{A} \cong \mathbb{C}/\mathcal{J} \tag{15}$$

where $\mathcal{J}$ denotes the ideal

$$\begin{aligned}\mathcal{J} &= \Big\langle (\alpha_0 x + f(x))^2,\ (\alpha_0 x + f(x))(f'(x) + \alpha_0), \\ &\qquad (\alpha_0 x + f(x))(2x + \bar{u}_4) + 2(f'(x) + \alpha_0)(x^2 + \bar{u}_4 x + \bar{u}_5)\Big\rangle \\ &= \Big\langle (\alpha_0 x + f(x))^2,\ \alpha_0^2 x + \alpha_0 f(x) + \alpha_0 x f'(x) + f(x) f'(x), \\ &\qquad 2\alpha_0\bar{u}_5 + \alpha_0\bar{u}_4 x + 2\bar{u}_5 f'(x) - \bar{u}_4 f(x) + 2\bar{u}_4 x f'(x) - 2f(x)x + 2f'(x)x^2\Big\rangle.\end{aligned}$$

Clearly the quotient ring $\mathcal{A}$ is non-zero if and only if $\alpha_0\bar{u}_5 = 0$. Now $\alpha_0 = 0$ means that the reduced line of κ is tangent to C at 0; in this case the (point dual to the) line $t_P C$ is in κ'.

In order to grasp the condition $\bar{u}_5 = 0$, recall that the point (x_0, y_0) on the line $y + \alpha_0 x = 0$ corresponds by duality to the line $x_0\alpha + \beta - \alpha_0 x_0 = 0$ in the α, β plane. Thus the line pair in the α, β-plane corresponding to a pair of points $(x_0, -\alpha_0 x_0), (x_1, -\alpha_0 x_1)$ is given by the conic

$$\beta^2 + x_0 x_1\alpha^2 + (x_0 + x_1)\alpha\beta + \alpha_0^2 x_0 x_1 - 2\alpha_0 x_0 x_1\alpha - \alpha_0(x_0 + x_1)\beta.$$

Comparing with (14) (with $\gamma = 1$), we see that $\bar{u}_5 = 0$ holds if and only if the line pair κ' is dual to a pair of points on the line $y + \alpha_0 x = 0$, one of which coincides with the origin. Thus $t_P C \in \kappa'$ holds, thereby establishing (11).

We list below the possible lengths of the artinian local $\mathbb{C}$-algebra $\mathcal{A}$ of the fiber of $\mathbb{K}_C$ over (κ, κ') at $(0, (\kappa, \kappa'))$ for κ a double line. We assume for simplicity the intersection index of the tangent line $t_0 C$ with C at 0 is ≤ 3, since we will need the estimates only for generic C. The calculations follow easily from (15).

<table>
<tr><th>algebraic condition</th><th>geometric meaning</th><th>length $\mathcal{A}$</th></tr>
<tr><td>$\alpha_0 \bar{u}_5 \neq 0$</td><td>$\kappa_{\mathrm{red}}$ transversal, no focus at 0</td><td>0</td></tr>
<tr><td>$\alpha_0 \neq 0, \bar{u}_5 = 0$</td><td>$\kappa_{\mathrm{red}}$ transversal, some focus at 0</td><td>1</td></tr>
<tr><td rowspan="2">$\alpha_0 = 0, \bar{u}_5 \neq 0$</td><td rowspan="2">$\kappa_{\mathrm{red}}$ tangent, no focus at 0</td><td>1 (not inflexion)</td></tr>
<tr><td>2 (inflexion)</td></tr>
<tr><td rowspan="2">$\alpha_0 = \bar{u}_5 = 0 \neq \bar{u}_4$</td><td rowspan="2">$\kappa_{\mathrm{red}}$ tangent, one focus at 0</td><td>2 (not inflexion)</td></tr>
<tr><td>(inflexion)</td></tr>
<tr><td rowspan="2">$\alpha_0 = \bar{u}_5 = \bar{u}_4 = 0$</td><td rowspan="2">$\kappa_{\mathrm{red}}$ tangent, both foci at 0</td><td>3 (not inflexion)</td></tr>
<tr><td>4 (inflexion).</td></tr>
</table>

(16)

Proposition 3.2. *Keep the notation of 3.1. Let C be a generic plane curve. Let $p : \mathbb{K}_C \subset C \times \mathbb{K} \to \mathbb{K}$ be induced by projection. Then, for $2 \leq i \leq 5$,*

$$m_i(p) := \left\{ (P, (\kappa, \kappa')) \in \mathbb{K}_C \mid \mathrm{length}\, p^{-1}((\kappa, \kappa')) \geq i \right\}$$

is of codimension $i - 1$ in $\mathbb{K}_C$ and for each $(\kappa, \kappa') \in p(m_5(p))$ we have that κ is non-degenerate.

Proof. The hypothesis that C is generic allows us to assume that

- the inflexion and bitangent lines of C are all simple, i.e., for each $P, Q \in C$ and line $\ell \in \check{\mathbb{P}}^2$, if $\ell \cdot C \geq 2P + 2Q$ (resp. $\ell \cdot C \geq 3P$) then $P \neq Q$ and the divisor $\ell \cdot C - 2P - 2Q$ (resp. $\ell \cdot C - 3P$) is reduced and
- for each partition $1 \leq \pi_1 + \cdots + \pi_r \leq 5$, the set of reduced conics cutting C in a divisor of the form $\Sigma(1 + \pi_i)P_i + R$ with R reduced, is of the right dimension $5 - \Sigma\pi_i$.

By the previous discussion of the possible lengths of fibers of p, we see that if κ is a double line and $(\kappa, \kappa') \in \mathbb{K}_C$ then $p^{-1}((\kappa, \kappa'))$ is of length at most 4 (hitting that bound only when $\kappa_{\rm red}$ is inflexional). Therefore, $m_5(p)$ lies in $\mathbb{K}_C \setminus \mathbb{K}_{C|\mathbb{W}} = \mathbf{Co}_C \setminus \mathbb{W}$.

For each partition

$\pi = (\pi_1, \ldots, \pi_t) \in \{(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)\}$,

put

$$\Sigma_\pi = \Big\{(P, \kappa) \in \mathbf{Co}_C \mid \kappa \text{ is reduced and } \kappa \cdot C = (1+\pi_1)P_1 + \cdots \\ +(1+\pi_t)P_t + R \quad \text{with } R \text{ reduced}\Big\}.$$

Thus $m_5(p)$ is the union of the Σ_π. Each Σ_π is finite for general C by an easy count of constants. Since $m_5(p)$ is easily seen to be nonempty, it follows that each $m_i(p)$ must be of the right (minimal) dimension $5-i$ for $i = 2 \ldots 5$. Alternatively, the dimension estimates also follow easily by a count of constants in view of (16).

□

4. Final comments

In order to apply the present setup to count rational curves on the double cover of $\mathbb{P}^2$ ramified along C, we still face the following tasks.

1. To gather enough information to compute the relative tangent bundle of the map $\widehat{p} : \widehat{\mathbb{K}_C} \longrightarrow \widehat{\mathbb{K}}$ referred to at the introduction.

2. To account for the contributions of coalescing 5-fold points separately from the total of 5-fold points of $\widehat{p}$.

The first point will be the object of a forthcoming article [8].

The second task is, as of this writing, only "half-solved" at best. Indeed, we may use De Jonquières formulae to count Σ_π for $\pi \in \{(5), (4,1), (3,2)\}$. However, we haven't been able yet to detect the weights with which these enter the total number furnished by multiple point theory. The question requires a better understanding of the local structure of the Fitting ideal defining the scheme structure of the 5-point locus around a point of coalescence.

The picture below might convince the reader acquainted with notion of complete quadrics why a similar approach should work for the study of quadrics multi-tangent to a surface in $\mathbb{P}^3$. Thus, a plane-pair with two marks ($\bullet$ signs) is expected to be at most 8-fold tangent, this case occurring if and only if each plane is tritangent (say at the $\circ$'s) *and* the marks are contained in the intersection ($\star$'s) of the distinguished line with the surface.

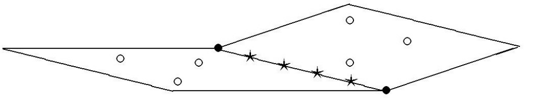


Fig. 2

Similarly, a double plane with a marked complete conic should be at most 8-fold tangent, this being so precisely when the supporting plane is tritangent *and* the marked conic is 5-fold tangent to the plane section.

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