Matemática Contemporânea, Vol. 14, 115–127
http://doi.org/10.21711/231766361998/rmc1411

ON PROJECTIVE MANIFOLDS WITH DEGENERATE SECANT VARIETIES

Masahiro Ohno

0. Introduction

Let X be an n-dimensional nondegenerate (i.e., not contained in a hyperplane) projective manifold in $\mathbf{P}^N$ over an algebraically closed field k of characteristic 0. Let Sec$X$ denote the secant variety of X in $\mathbf{P}^N$. We have always $\dim \mathrm{Sec}X \leq \min\{2n+1, N\}$. If $\dim \mathrm{Sec}X < \min\{2n+1, N\}$, we say that Sec$X$ is degenerate. The Linear Normality Theorem [Z, Chap.2, Corollary 2.17] implies that if Sec$X$ is degenerate then $\dim \mathrm{Sec}X \geq (3n+2)/2$. If equality holds, X is called a Severi variety. Severi varieties were completely classified by F. L. Zak [Z, Chap.4, Th. 4.7]. Zak also generalized the class of Severi varieties to a class of manifolds, named Scorza varieties, and classified Scorza varieties [Z, Chap.6]. In this paper, we propose a new class of projective manifolds with degenerate secant varieties, which is wider than the class of Scorza varieties, and investigate some properties of this class of manifolds.

Suppose that Sec$X$ is degenerate. Let $\varepsilon = 2\dim \mathrm{Sec}X - 3n - 2$. Let $\mathrm{Sm}(\mathrm{Sec}X)$ denote the smooth locus of Sec$X$, and $\gamma : \mathrm{Sm}(\mathrm{Sec}X) \to G(\dim \mathrm{Sec}X, \mathbf{P}^N)$ the Gauss map $u \mapsto T_u\mathrm{Sec}X$ of $\mathrm{Sm}(\mathrm{Sec}X)$. Then we have the following proposition.

Proposition 0.1. $\dim \mathrm{Im}(\gamma) = 2(\dim \mathrm{Sec}X - n - 1 - c)$ *for some integer* c $(0 \leq c \leq \varepsilon)$.

Note that if $X \subset \mathbf{P}^N$ is a Scorza variety then the integer c in Proposition 0.1 is zero ([Z, Chap.6, (1.4.11)]) but the converse is not true. Note also that, to the best of my knowledge, all examples of $c > 0$ are constructed from those of $c = 0$. In this paper we classify low dimensional projective manifolds with degenerate secant varieties satisfying $c = 0$.

The main result of this paper is the following.

Theorem 0.2. *Suppose that* $\mathrm{Sec}X$ *is degenerate and of dimension* $2n$ *and that* $\dim \mathrm{Im}(\gamma) = 2(n-1)$.
If $n = 4$, *then* $(X, \mathcal{O}_X(1))$ *is one of the following.*
1) $(\mathbf{P}_{\mathbf{P}^l}(\mathcal{E}), H(\mathcal{E}))$, *where* $\mathcal{E} = \mathcal{O}(1)^{\oplus(4-l)} \oplus \mathcal{O}(2)$ $(l = 2, 3, 4)$;
2) $(\mathbf{P}_{\mathbf{P}^2}(\mathcal{E}), H(\mathcal{E}))$, *where* $\mathcal{E} = \mathcal{O}(1) \oplus T_{\mathbf{P}^2}$.
If $n = 5$, *then* $(X, \mathcal{O}_X(1))$ *is one of the following.*
1) $(\mathbf{P}_{\mathbf{P}^l}(\mathcal{E}), H(\mathcal{E}))$, *where* $\mathcal{E} = \mathcal{O}(1)^{\oplus(5-l)} \oplus \mathcal{O}(2)$ $(l = 2, 3, 4, 5)$;
2) $(\mathbf{P}_{\mathbf{P}^l}(\mathcal{E}), H(\mathcal{E}))$, *where* $\mathcal{E} = \mathcal{O}(1)^{\oplus(6-2l)} \oplus T_{\mathbf{P}^l}$ $(l = 2, 3)$;
3) $X \subset \mathbf{P}^N$ *is a linear section of* $G(1, \mathbf{P}^5) \subset \mathbf{P}^{14}$ *a section cut out by codimension* 3 *linear subspace of* $\mathbf{P}^{14}$;
4) $(\Sigma_{10}, \mathcal{O}(1))$, *where* Σ_{10} *is the adjoint manifold of the simple algebraic group of exceptional type* G_2 *and* $\mathcal{O}(1)$ *is the fundamental line bundle on it. (In other words* Σ_{10} *is the* 5*-dimensional Mukai manifold of genus* 10 *([Mu]).)*

If $\mathrm{Sec}X$ is degenerate and of dimension $2n$, then $n \geq 2$ by the Linear Normality Theorem. If $n = 3$, then T. Fujita ([F. Th. (2.1)]) showed that $(X, \mathcal{O}_X(1))$ is one of the following: $(\mathbf{P}_{\mathbf{P}^l}(\mathcal{O}(1)^{\oplus 3-l} \oplus \mathcal{O}(2)), H(\mathcal{O}(1)^{\oplus 3-\ell} \oplus \mathcal{O}(2)))$ where $(l = 2, 3)$, or $(\mathbf{P}(T_{\mathbf{P}^2}), H(T_{\mathbf{P}^2}))$.

Notation and conventions

We work over an algebraically closed field k of characteristic 0. We follow the notation and terminology of [H]. We use the word *manifold* to mean a smooth variety. For a manifold X, we denote by $\kappa(X)$ the Kodaira dimension of X. We use the word *line* to mean a smooth rational curve of degree 1. Given two distinct points x, y on $\mathbf{P}^N$, let $x * y$ denote the line joining them. For subsets X, Y of $\mathbf{P}^N$, let $X * Y$ be the closure of the union of all lines $x * y$ joining two distinct points $x \in X$ and $y \in Y$. For a vector bundle E of rank $e+1$ on a variety X, we define the i-th Segre class $s_i(E)$ of E by the formula $s_i(E) \cap \alpha = p_*(c_1(\mathcal{O}_{\mathbf{P}(E^\vee)}(1)^{e+i} \cap p^*\alpha)$ where α is a k-dimensional cycle modulo rational equivalence and $p : \mathbf{P}(E) \to X$ is the projection. We also define the total Segre class $s(E)$ to be $1 + s_1(E) + s_2(E) + \cdots$. The total Chern class $c(E) = 1 + c_1(E) + c_2(E) + \cdots$ is defined by the formula $c(E)s(E) = 1$. These definitions of $s_i(E)$ and $c_i(E)$ are the same as those of [Fl]. By abuse of notation, we simply write $s_n(E)$ for $\deg s_n(E)$ when $n = \dim X$. We denote also by $H(E)$ the tautological line bundle $\mathcal{O}_{\mathbf{P}(E)}(1)$ on $\mathbf{P}(E)$. For a linear system Λ, BsΛ denotes the base locus of Λ. Let $[r]$ denote the greatest integer not greater than r for a real number r.

1. Preliminaries and Proof of Proposition 0.1

Let X be an n-dimensional nondegenerate closed submanifold in $\mathbf{P}^N$. Let B be the blowing-up of $X \times X$ along the diagonal Δ, and let $S_0 = \{(x, y, u) \in (X \times X \setminus \Delta) \times \mathbf{P}^N | x, y,$ and u are collinear$\}$. Since $\mathbf{P}(\Omega_X)$ is the exceptional divisor of B, we can identify $X \times X \setminus \Delta$ with $B \setminus \mathbf{P}(\Omega_X)$. Thus S_0 can be identified with a closed submanifold of $(B \setminus \mathbf{P}(\Omega_X)) \times \mathbf{P}^N$. We define S to be the closure of S_0 in $B \times \mathbf{P}^N$. We call S the complete secant bundle of X. Let $p : S \hookrightarrow B \times \mathbf{P}^N \to B$ be the first projection and $\sigma : S \hookrightarrow B \times \mathbf{P}^N \to \mathbf{P}^N$ the second projection. Then $\mathrm{Sec}X = \mathrm{Im}(\sigma)$. For a point $u \in \mathrm{Sec}X$, let $\Sigma_u = \sigma(p^{-1}(p(\sigma^{-1}(u))))$, $Q_u = \Sigma_u \cap X$, and $\theta_u = \{x \in X | u \in T_xX\}$. We call Σ_u the secant cone, Q_u the secant locus, and θ_u the tangent locus, with respect to u. Let $\mathrm{Sm}(\mathrm{Sec}X)$ denote the smooth locus of $\mathrm{Sec}X$. Let $\gamma : \mathrm{Sm}(\mathrm{Sec}X) \to G(\dim \mathrm{Sec}X, \mathbf{P}^N)$ be the Gauss map of $\mathrm{Sm}(\mathrm{Sec}X)$. For a point $u \in \mathrm{Sm}(\mathrm{Sec}X)$, C_u denotes the closure of $\gamma^{-1}(\gamma(u))$ in $\mathrm{Sec}X$. We call C_u the contact locus of $T_u\mathrm{Sec}X$ with $\mathrm{Sec}X$. We fix and will use these notations in the following sections.

We first observe that for a general point $u \in \mathrm{Sec}X$ $\dim Q_u = 2n + 1 - \dim \mathrm{Sec}X$, $\Sigma_u = u * Q_u$, and C_u is a linear subspace in $\mathbf{P}^N$ (see, for example, [Z, Chap. 1, Th. 2.3 c)]). We also have $\Sigma_u \subseteq C_u$ for any general point $u \in \mathrm{Sm}(\mathrm{Sec}X)$ (see, for example, [Oh, Cor. 1.2]). Therefore $2n+2-\dim \mathrm{Sec}X = \dim \Sigma_u \leq \dim C_u$ and hence $\dim \mathrm{Im}(\gamma) \leq 2\dim \mathrm{Sec}X - 2n - 2$.

Now we give a proof of Proposition 0.1. Let $D'_u = \overline{\cup_{v \in C_u:\ \mathrm{general}} Q_v}$. Then $C_u = \mathrm{Sec}D'_u$ and $1 + 2\dim D'_u = \dim Q_u + \dim C_u$ for any general point $u \in \mathrm{Sec}X$ (see [Oh, Lemma 1.4]). Let $X_u = \{x \in X | T_x X \subseteq T_u \mathrm{Sec}X\}$ for a point $u \in \mathrm{Sm}(\mathrm{Sec}X)$. Then $D'_u \subseteq X_u$ for any general point $u \in \mathrm{Sec}X$ by [Oh, Corollary 1.3]. Let $T(X_u, X) = \cup_{x \in X_u} T_x X$. Then $T(X_u, X) \subseteq T_u \mathrm{Sec}X$. Note that $T_u \mathrm{Sec}X \neq \mathbf{P}^N$ since $\mathrm{Sec}X$ is degenerate. Note also that $X \subseteq X_u * X$ and X is not contained in $T_u \mathrm{Sec}X$ since X is nondegenerate in $\mathbf{P}^N$ and $T_u \mathrm{Sec}X \neq \mathbf{P}^N$. Therefore $\dim X_u * X = \dim X_u + n + 1$ by [Z, Chap.1, Theorem 1.4]. Since $\dim X_u * X \leq \dim \mathrm{Sec}X$, we have $\dim D'_u \leq \dim \mathrm{Sec}X - n - 1$. Therefore we have $\dim C_u = 2n + 2 - \dim \mathrm{Sec}X + 2c$ for some integer c $(0 \leq c \leq \varepsilon)$. This completes the proof.

Suppose that $\mathrm{Sec}X$ is degenerate in the following. Then we have the following proposition.

Proposition 1.1. *Assume that* $\dim \mathrm{Im}(\gamma) = 2(\dim \mathrm{Sec}X - n - 1)$. *Then the secant cone* Σ_u *is a linear subspace of* $\mathbf{P}^N$ *of dimension* $2n + 2 - \dim \mathrm{Sec}X$ *for any general point* $u \in \mathrm{Sec}X$. *Moreover the secant locus* Q_u *is a smooth hyperquadric in* Σ_u, *and the tangent locus* θ_u *is a smooth hyperplane section of* Q_u *for any general point* $u \in \mathrm{Sec}X$. *In particular* X *is rationally connected,* $\kappa(X) = -\infty$, *and* $h^i(\mathcal{O}_X) = 0$ *for all* $i > 0$.

Proof. The first statement follows immediately from the fact that $\Sigma_u \subseteq C_u$. For a proof of the second statement, note first that a linear subspace Σ_u contains Q_u as a hypersurface. Second note that X is not a hypersurface in $\mathbf{P}^N$ because $\mathrm{Sec}X$ is degenerate, so that the trisecant lemma [F, (1.6)] shows that Q_u is a hyperquadric in Σ_u. For the rest of the second statement, refer to the proof of Th. 3 in [F-R, p.964, l.15 – p.967, l.7], and make obvious adjustments. For the definition of rational connectedness, see [Ko-Mi-Mo, (2.2)]. Since general two points can be joined by a positive dimensional quadric Q_u, X is rationally connected. The rest of the assertion follows immediately from [M-M, Th. 1] and [Ko-Mi-Mo, (2.5.2)].

$\square$

By generalizing [F, Lemma (2.3)], where $n = 3$, to arbitrary dimension n, we have the following proposition.

Proposition 1.2. *Assume that* $\dim \mathrm{Im}(\gamma) = 2(\dim \mathrm{Sec}X - n - 1)$. *If* $\dim \mathrm{Sec}X = 2n$, *then* $K_X.Q_u = -n-1$ *for a general point* $u \in \mathrm{Sec}X$.

2. Proof of Theorem 0.2

In this section we give a proof of Theorem 0.2. First of all, we state a couple of Lemmas.

Lemma 2.1. *Let* $X \subseteq \mathbf{P}^N$ *be an* n*-dimensional projective manifold. Then* $\dim \mathrm{Sec}X \leq 2n$ *if and only if*

$$(\deg X)^2 - \sum_{j=0}^{n} \binom{2n+1}{j} c_1(\mathcal{O}_X(1))^j s_{n-j}(T_X) \cap [X] = 0.$$

For a proof, see, for example, [F, (1.5) and (1.7)].

Lemma 2.2. *Let* $X \subseteq \mathbf{P}^N$ *be an* n*-dimensional projective manifold,* $L = \mathcal{O}_X(1)$, *and assume that* $(X, L) \cong (\mathbf{P}_Y(\mathcal{E}), H(\mathcal{E}))$ *for some locally free sheaf* $\mathcal{E}$ *of rank* $n - m + 1$ *on an* m*-dimensional projective manifold* Y. *Then*

$$(L^n)^2 - \sum_{j=0}^{n} \binom{2n+1}{j} c_1(L)^j s_{n-j}(T_X) \cap [X]$$
$$= (s_m(\check{\mathcal{E}}))^2 - \sum_{j=0}^{n}\sum_{p=0}^{n-j}\sum_{l=0}^{p} \binom{2n+1}{j}\binom{-n+m-l-1}{p-l} s_{j+p-l-n+m}(\check{\mathcal{E}}) s_l(\check{\mathcal{E}}) s_{n-j-p}(T_Y)$$
$$= \begin{cases} c_1(\mathcal{E})^2 - (2n+1)c_1(\mathcal{E}) - n(n+1)(g(Y)-1) & \text{if } m = 1 \\ \\ (L^2)^2 - (n^2+n+1)L^2 - (1/6)(2n+1)(n+1)nc_1(\mathcal{E})c_1(K_Y) \\ \quad - \binom{n+2}{4}(c_1(K_Y)^2 - c_2(T_Y)) - \binom{n+1}{2}c_1(\mathcal{E})^2 & \text{if } m = 2. \end{cases}$$

If $Y = \mathbf{P}^1$, *then* $\dim \mathrm{Sec}X \leq 2n$ *if and only if* $\mathcal{E} \cong \mathcal{O}(1)^{\oplus n}$ *or* $\mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2)$, *and under this equivalent condition we obtain* $\mathrm{Sec}X = \mathbf{P}(H^0(L))$.

Proof. We obtain these results by calculation and by Lemma 2.1.

□

We get the following lemma by calculation.

Lemma 2.3. *Let C be a smooth complete curve of genus g and $\mathcal{E}$ a vector bundle of rank n on C. Let X be a smooth irreducible effective Cartier divisor of $\mathbf{P}(\mathcal{E})$ such that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(X) \cong H(\mathcal{E})^{\otimes 2} \otimes \pi^* M$ for some line bundle M of degree m on C, where $\pi : \mathbf{P}(\mathcal{E}) \to C$ is the projection. Let $L = H(\mathcal{E}) \otimes \mathcal{O}_X$ and $d = L^n$. Then*

$$d^2 - \sum_{j=0}^{n} \binom{2n+1}{j} c_1(L)^j s_{n-j}(T_X) \cap [X] = d^2 - 4nd - m - 4n^2(g-1).$$

In the rest of this section, let X be an n-dimensional nondegenerate projective manifold in $\mathbf{P}^N$ with degenerate secant variety $\mathrm{Sec}X$ of dimension $2n$, and let $L = \mathcal{O}_X(1)$.

Lemma 2.4. *If* $\dim \mathrm{Im}(\gamma) = 2(n-1)$, *then we have* $\mathrm{Bs}|K_X + (n-1)L| = \emptyset$ *for all* $n \geq 3$.

Proof. If $\mathrm{Bs}|K_X + (n-1)L| \neq \emptyset$, then $(X, L) \cong (\mathbf{P}_C(\mathcal{E}), H(\mathcal{E}))$ for some vector bundle $\mathcal{E}$ of rank n on a smooth curve C by [S-V, (0.1)] since $\mathrm{Sec}X \neq \mathbf{P}^N$ and $n \geq 3$. Because X is rationally connected by Proposition 1.1, so is C, and hence $C = \mathbf{P}^1$. Therefore $\mathrm{Sec}X = \mathbf{P}(H^0(L))$ by Lemma 2.2, which contradicts the hypothesis that $\mathrm{Sec}X \neq \mathbf{P}^N$.

□

In the following, we always assume that $n \geq 4$. Let $\phi : X \to \mathbf{P}(H^0(K_X + (n-1)L))$ be the adjunction map, and let $\phi = s \circ r$ $(r : X \to Y, s : Y \to \mathbf{P}(H^0(K_X + (n-1)L)))$ be the Stein factorization of ϕ.

Theorem 2.5. *If* $\dim \mathrm{Im}(\gamma) = 2(n-1)$, *then there are the following possibilities.*

(1) *Y is a smooth rational projective surface, s is a closed immersion induced by $|K_Y + c_1(\mathcal{E})|$, $(K_Y + c_1(\mathcal{E}))^2 \le (n-3)^2$, $(X, L) \cong (\mathbf{P}_Y(\mathcal{E}), H(\mathcal{E}))$ for some vector bundle of rank $n-1$ on Y, and*

$$\begin{aligned}(L^2)^2 - (n^2+n+1)L^2 - (1/6)(2n+1)(n+1)nK_Y c_1(\mathcal{E}) \\ - \binom{n+2}{4}(K_Y^2 - c_2(T_Y)) - \binom{n+1}{2}c_1(\mathcal{E})^2 &= 0.\end{aligned}$$

Furthermore $(K_Y + c_1(\mathcal{E}))^2 \ge 5$ unless $(Y, \mathcal{E}) \cong (\mathbf{P}^2, \mathcal{O}(1)^{\oplus l} \oplus \mathcal{O}(2))$ or $(\mathbf{P}^2, \mathcal{O}(1)^{\oplus l-1} \oplus T_{\mathbf{P}^2})$ where $l = 2$ or 3;

(2) *Y is an n-dimensional rationally connected smooth projective variety, and r is the blowing-up of Y at a finite point set, and $L = r^*M - \Sigma E_i$ (E_i: exceptional divisors) for some ample line bundle M on Y, and $K_Y + (n-1)M$ is very ample. Moreover $(K_Y + (n-2)M)|_{r(Q_u)} \le n-5$ for a general point $u \in \mathrm{Sec}X$, and $K_Y + (n-2)M$ is nef if $n \ge 5$, and $(Y, M) \cong (\mathbf{P}^4, \mathcal{O}(2))$ if $n = 4$.*

Proof. First note that Y is rationally connected because so is X by Proposition 1.1. Since $(K_X + (n-1)L)|_{Q_u} = n-3 \ge 1$ by Proposition 1.2, we have $\dim \phi(X) \ge 1$. Assume that $\dim Y = 1$. Then r is a quadric fibration over Y by [S-V, (0.2)] and a contraction morphism of an extremal ray by [B-S-W, Th. (3.2.6)]. Therefore we can show, by the same argument as that in [Fb, p.100, l.10–l.27], that there exist a locally free sheaf $\mathcal{E}$ of rank $n+1$ on Y and a line bundle M on Y such that X is a Cartier divisor of $\mathbf{P}(\mathcal{E})$, that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(X) \cong H(\mathcal{E})^{\otimes 2} \otimes \pi^*M$, and that $L \cong H(\mathcal{E}) \otimes \mathcal{O}_X$, where $\pi : \mathbf{P}(\mathcal{E}) \to Y$ is the projection and $r = \pi|_X$. Since Y is rationally connected, Y is a smooth rational curve. Let $d = L^n$, $e = \deg c_1(\mathcal{E})$, and $m = \deg M$. Then we have $n - 3 = (K_X + (n-1)L)|_{Q_u} = (K_{\mathbf{P}(\mathcal{E})} + (n+1)H(\mathcal{E}) + \pi^*M)|_{Q_u} = \pi^*(\mathcal{O}_{\mathbf{P}^1}(e+m-2))|_{Q_u}$, and hence $e + m \le n-1$. On the other hand, $(L^n)^2 - \sum_{j=0}^{n}\binom{2n+1}{j}c_1(L)^j s_{n-j}(T_X) \cap [X] = (d-2n)^2 - m$ by Lemma 2.3, and therefore $\dim \mathrm{Sec}X = 2n$ implies that $(d-2n)^2 = m$. Let m' be a nonnegative integer such that $m = m'^2$. Then we have $e = n - (m'(m' \mp 1)/2)$ because

$d = 2e + m$. It follows from $e + m \leq n - 1$ that $m'^2 \pm m' + 2 \leq 0$, which is however a contradiction. Hence $\dim Y \geq 2$.

If $\dim Y = 2$, then Y is a smooth projective surface and $(X, L) \cong (\mathbf{P}_Y(\mathcal{E}), H(\mathcal{E}))$ for some vector bundle of rank $n-1$ on Y by [S-V, (0.2)]. Furthermore $K_Y + c_1(\mathcal{E})$ is very ample by [L-M, Th. B and Th. C] because $H(\mathcal{E})$ is very ample, so that s is a closed immersion. Note also that Y is rational since $\dim Y = 2$. For general three points $x, y, z \in X$, there exist two points $u, v \in \mathrm{Sec}X$ such that $x, y \in Q_u$ and $y, z \in Q_v$ and u, v are in general position. Since Q_u and Q_v are algebraically equivalent, so is $r_*(Q_u)$ and $r_*(Q_v)$. Since $r(y) \in r(Q_u) \cap r(Q_v)$ and $r(Q_u) \neq r(Q_v)$, we get $r(Q_u)^2 \geq 1$. We also have $(K_Y + c_1(\mathcal{E}))^2 r(Q_u)^2 \leq ((K_Y + c_1(\mathcal{E}))|_{r(Q_u)})^2 \leq (n-3)^2$ by the Hodge index theorem. Therefore $(K_Y + c_1(\mathcal{E}))^2 \leq (n-3)^2$. If $(K_Y + c_1(\mathcal{E}))^2 = 1$, then $Y = \mathbf{P}^2$ and $c_1(\mathcal{E}) \cong \mathcal{O}(4)$. If the rank of $\mathcal{E}$ is 3, then $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$ or $\mathcal{O}(1) \oplus T_{\mathbf{P}^2}$ by [E1]. In this case $\dim \mathrm{Sec}X = 8$ and $h^0(L) \geq 11$. The condition that $\dim C_u = 2$ is also satisfied. If the rank of $\mathcal{E}$ is 4, then $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 4}$. Hence $(X, L) \cong (\mathbf{P}^2 \times \mathbf{P}^3, \mathcal{O}(1) \otimes \mathcal{O}(1))$, which, however, does not satisfy the condition that $\dim \mathrm{Sec}X = 10$. If $(K_Y + c_1(\mathcal{E}))^2 \geq 2$, then $\mathrm{rk}\mathcal{E} \geq 4$ since $(n-3)^2 \geq 2$. If $(K_Y + c_1(\mathcal{E}))^2 = 2$, then $Y = \mathbf{P}^1 \times \mathbf{P}^1$ and $c_1(\mathcal{E}) \cong \mathcal{O}(3) \otimes \mathcal{O}(3)$. This contradicts the ampleness of $\mathcal{E}$. If $(K_Y + c_1(\mathcal{E}))^2 = 3$, then Y is either a cubic surface in $\mathbf{P}^3$ or $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(2))$ by [Fb, (17.2)]. If Y is cubic, then $c_1(\mathcal{E})|_l = 2$ for every l, one of the 27 lines on Y, which is a contradiction. For the scroll we have $c_1(\mathcal{E})|_f = 3$ where f is any fiber of the scroll, and this also contradicts the ampleness of $\mathcal{E}$. Suppose that $(K_Y + c_1(\mathcal{E}))^2 = 4$. Then Y is either a del Pezzo surface of degree 4, a scroll $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(3))$, a scroll $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(2))$, or a Veronese surface $\mathbf{P}^2 \subset \mathbf{P}^5$ by [Fb, (17.3)] since $\kappa(Y) = -\infty$. If Y is a del Pezzo surface, then $c_1(\mathcal{E})|_l = 2$ for any exceptional divisor l of Y, which is a contradiction. For the scrolls we have $c_1(\mathcal{E})|_f = 3$ where f is any fiber of the projection $Y \to \mathbf{P}^1$, and this is also a contradiction. If Y is a Veronese surface, we obtain $c_1(\mathcal{E}) \cong \mathcal{O}(5)$. If $\mathcal{E}$ is an ample vector bundle of rank 4, we have $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)$ or $\mathcal{O}(1)^{\oplus 2} \oplus T_{\mathbf{P}^2}$ by [E2, Th. 5.1]. For both bundles, we have $\dim \mathrm{Sec}X = 10$ and $h^0(L) \geq 14$. The condition that $\dim C_u = 2$ is also satisfied.

If the rank of $\mathcal{E}$ is 5, then $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 5}$. Hence $(X, L) \cong (\mathbf{P}^2 \times \mathbf{P}^4, \mathcal{O}(1) \otimes \mathcal{O}(1))$, which however does not satisfy the condition that $\dim \mathrm{Sec} X = 12$. The rest of the assertion in the case $\dim Y = 2$ follows from Lemma 2.2.

If $\dim Y > 2$, then $\dim Y = n$ by [S-V, (0.2)], and Y is smooth, r is the blowing-up of Y at a finite point set, and $L = r^*M - \Sigma E_i$ (E_i: exceptional divisors) for some ample line bundle M on Y by [S-V, (0.3)]. Moreover $K_Y+(n-1)M$ is very ample by [S-V, Th. (2.1)]. Since $n-3 = (K_X+(n-1)L)|_{Q_u} = (K_Y+(n-1)M)|_{r(Q_u)}$ and $M|_{r(Q_u)} \geq L|_{Q_u} = 2$, we obtain $(K_Y+(n-2)M)|_{r(Q_u)} \leq n-5$. If $n \geq 5$, then we know that $K_Y+(n-2)M$ is nef by [Fb, (11.8)], taking account of the fact that $K_Y + (n-1)M$ is ample. If $n = 4$, then $(K_Y + 2M)|_{r(Q_u)} \leq -1$ and hence $K_Y + 2M$ is not nef. Therefore $(Y, M) \cong (\mathbf{P}^4, \mathcal{O}(2))$ by [Fb, (11.8)] because $K_Y + 3M$ is ample.

□

Now we give a proof of Theorem 0.2.

Proof of Theorem 0.2. Suppose first that $n = 4$. Then by Theorem 2.5, $\dim Y = 2$ or 4. If $\dim Y = 2$, then (X, L) is isomorphic to $(\mathbf{P}_{\mathbf{P}^2}(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E} = \mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)$ or $\mathcal{O}(1) \oplus T_{\mathbf{P}^2}$ by Theorem 2.5 (1). If $\dim Y = 4$, then $(Y, M) \cong (\mathbf{P}^4, \mathcal{O}(2))$ by Theorem 2.5 (2). If r is an isomorphism, then $(X, L) \cong (\mathbf{P}^4, \mathcal{O}(2))$. If r is not an isomorphism, then $(X, L) \cong (\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1) \oplus \mathcal{O}(2))), H(\mathcal{O}(1) \oplus \mathcal{O}(2)))$. These polarized manifolds satisfy the assumptions that $\dim \mathrm{Sec} X = 10$ and that $\dim C_u = 2$.

Suppose in the following that $n = 5$. Then by Theorem 2.5, $\dim Y = 2$ or 5. If $\dim Y = 2$, then $(X, \mathcal{O}_X(1)) \cong (\mathbf{P}_{\mathbf{P}^2}(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E} = \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)$ or $\mathcal{O}(1)^{\oplus 2} \oplus T_{\mathbf{P}^2}$ by Theorem 2.5 (1). These two polarized manifolds satisfy the hypotheses that $\dim \mathrm{Sec} X = 10$ and that $\dim C_u = 2$.

Let us consider the case (2) of Theorem 2.5. Now we have $K_Y + 3M$ is nef and therefore $(K_Y + 3M)|_{r(Q_u)} = 0$. Since general two points can be joined by $r(Q_u)$, this implies that $K_Y + 3M = 0$ by [K-M-M, Th. 3-1-1 and Th. 3-2-1]. If M is not the fundamental line bundle, then $(Y, M) \cong (\mathbf{P}^5, \mathcal{O}(2))$. If r is an isomorphism, then $(X, L) \cong (\mathbf{P}^5, \mathcal{O}(2))$. If r is not an isomorphism, then

$(X, L) \cong (\mathbf{P}_{\mathbf{P}^4}(\mathcal{O}(1) \oplus \mathcal{O}(2)), H(\mathcal{O}(1) \oplus \mathcal{O}(2)))$. These two polarized manifolds satisfy the hypotheses that $\dim \mathrm{Sec}X = 10$ and that $\dim C_u = 2$. Assume that M is the fundamental line bundle of Y. Then Y is a Fano manifold of coindex 3 and M is very ample because $K_Y + 4M$ is very ample, so that (Y, M) satisfies the hypothesis (ES) of [M]. If $B_2(Y) \geq 2$, then (Y, M) is either $(\mathbf{P}^2 \times Q^3, \mathcal{O}(1) \otimes \mathcal{O}(1))$, $(\mathbf{P}(T_{\mathbf{P}^3}), H(T_{\mathbf{P}^3}))$, or $(\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)), H(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)))$ by [M, Th. 7]. Thus for every point $y \in Y$ there exists a line passing through y, which implies that r is an isomorphism by the ampleness of L. Since the secant variety of the manifold $(\mathbf{P}^2 \times Q^3, \mathcal{O}(1) \otimes \mathcal{O}(1))$ is 11-dimensional by [Z, Chap.3, Th. 1.6], (X, L) is either $(\mathbf{P}(T_{\mathbf{P}^3}), H(T_{\mathbf{P}^3}))$ or $(\mathbf{P}_{\mathbf{P}^3}(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)), H(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)))$. These polarized manifolds satisfy the hypothesis that $\dim C_u = 2$ and the condition that $\dim \mathrm{Sec}X = 10$.

Next let us consider the case that $B_2(Y) = 1$. Note that $g(Y, M) + 4 = h^0(M) \geq h^0(L) \geq \dim \mathrm{Sec}X + 2 = 12$.

Suppose that r is an isomorphism. Then we get $g(X, L) \geq 8$. Thus $X \subset \mathbf{P}^N$ is either a complete intersection of $G(1, \mathbf{P}^5) \subset \mathbf{P}^{14}$ and a codimension 3 linear subspace of $\mathbf{P}^{14}$ or the G_2 adjoint manifold $\Sigma_{10} \subset \mathbf{P}^{13}$ by [Mu, Th. 2], because $\mathrm{Sec}\Sigma_9 = \mathbf{P}^{13}$ by [K] and therefore the dimension of the secant variety of a general hyperplane section of Σ_9 is eleven, and all smooth hyperplane section of Σ_9 are isomorphic. A smooth complete intersection of $G(1, \mathbf{P}^5) \subset \mathbf{P}^{14}$ with a codimension 3 linear subspace of $\mathbf{P}^{14}$ satisfies the condition that $\dim \mathrm{Sec}X = 10$. The G_2 adjoint manifold $\Sigma_{10} \subset \mathbf{P}^{13}$ satisfies the assumptions that $\dim \mathrm{Sec}X = 10$ and that $\dim C_u = 2$ by [K-O-Y].

Suppose that r is not an isomorphism. Then $h^0(M) \geq h^0(L) + 1$ so that $g(Y, M) \geq 9$. Therefore Y is either a hyperplane section of $\Sigma_9 \subset \mathbf{P}^{13}$, or $\Sigma_{10} \subset \mathbf{P}^{13}$. For each point $y \in \Sigma_9$, Σ_9 contains a rational curve C passing through y such that $-K|_C \leq 7$ by [Ko, Chap.V, Th. 1.6.1]. Since the index of the Fano manifold Σ_9 is four, we have $-K|_C = 4$, and hence C is a line in Σ_9. Let $f : \mathbf{P}^1 \to C \hookrightarrow \Sigma_9$ be the normalization of C and let $f(0) = y$. Denote by ι the restriction of f to $\{0\}$. Then $\dim_{[f]} \mathrm{Hom}(\mathbf{P}^1, \Sigma_9; \iota) \geq -K|_C = 4$. On the other hand, we have $\dim \mathrm{Aut}(\mathbf{P}^1; 0) = 2$. Thus Σ_9 contains a closed cone of dimension ≥ 3 with vertex y. A hyperplane section Y of Σ_9 therefore contains a line passing through y for each point $y \in Y$. This contradicts the ampleness of L. For the G_2 adjoint variety $\Sigma_{10} \subset \mathbf{P}^{13}$ it follows from [Ko, Chap.V, Th. 1.15] that there exists a line C (i.e., $M|_C = 1$) on Σ_{10}. Hence for every point $y \in \Sigma_{10}$ there exists a line passing through y on Σ_{10}, which contradicts the ampleness of L.

□

Acknowledgements. The author expresses gratitude to Professors Takao Fujita and Hajime Kaji for their kind advice and encouragement, and to the referee for his useful suggestions. The author is partially supported by Waseda University Grant for Special Research Projects, 96A–136.

References

[B-S-W] Beltrametti, M. C., Sommese, A. J. and Wiśniewski, J. A., *Results on varieties with many lines and their applications to adjunction theory,* in Lecture Notes in Math. 1507 (eds. K. Hulek et al.), Springer-Verlag, (1992), 16–38.

[E1] Elencwajg, G., *Les fibrés uniformes de rang 3 sur* $\mathbf{P}_2(\mathbf{C})$ *sont homogènes,* Math. Ann. 231, (1978), 217–227.

[E2] Elencwajg, G., *Fibrés uniformes de rang élevé sur* $\mathbf{P}_2$, Ann. Inst. Fourier (Grenoble) 31 4, (1981), 89–114.

[F] Fujita, T., *Projective Threefolds with Small Secant Varieties,* Sci. Papers Coll. Gen. Educ., Univ. Tokyo 32 no.1, (1982), 33–46.

[Fb] Fujita, T., *Classification Theory of Polarized Varieties,* London Math. Soc. Lecture Note Ser., 155, (1990), Cambridge University Press.

[F-R] Fujita, T. and Roberts, J., *Varieties with small secant varieties: the extremal case,* Amer. J. Math. 103, (1981), 953–976.

[Fl] Fulton, W., *Intersection Theory,* Ergeb. Math. Grenzgeb. (3), Folge Band 2, (1984) Springer-Verlag, .

[H] Hartshorne, R., *Algebraic Geometry,* Graduate Texts in Math., 52, (1983), Springer-Verlag.

[K] Kaji, H., *On the homogeneous projective varieties with degenerate secants*, (1995), preprint.

[K-O-Y] Kaji, H., Ohno, M. and Yasukura, O., *Adjoint varieties and their secant varieties*, To appear in Indag. Math. (N.S.).

[K-M-M] Kawamata, Y., Matsuda, K. and Matsuki, K., *Introduction to the Mimimal Model Problem,* in Adv. Stud. Pure Math. 10 (ed. T. Oda), Kinokuniya, (1987), 283-360.

[Ko] Kollár, J., *Rational Curves on Algebraic Varieties,* Ergeb. Math. Grenzgeb. (3), Folge Band 32, (1996), Springer-Verlag.

[Ko-Mi-Mo] Kollár, J., Miyaoka, Y. and Mori, S., *Rationally connected varieties*, J. Algebraic Geom. 1, (1992), 429–448.

[L-M] Lanteri, A. and Maeda, H., *Adjoint bundles of ample and spanned vector bundles on algebraic surfaces*, J. Reine Angew. Math. 433, (1992), 181–199.

[M-M] Miyaoka, Y. and Mori, S., *A numerical criterion for uniruledness,* Ann. of Math. (2) 124, (1986), 65–69.

[Mu] Mukai, S., *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3,* Proc. Nat. Acad. Sci. U.S.A. 86, (1989), 3000–3002.

[Oh] Ohno, M., *On odd dimensional projective manifolds with smallest secant varieties*, To appear in Math. Z.

[S-V] Sommese, A. J. and Van de Ven, A., *On the Adjunction Mapping,* Math. Ann. 278, (1987), 593–603.

[Z] Zak, F. L., *Tangents and Secants of Algebraic Varieties,* Transl. Math. Monographs, vol. 127, (1993), American Mathematical Society.

Department of Mathematics
School of Science and Engineering
Waseda University
3-4-1, Okubo Shinjuku-ku Tokyo, 169
JAPAN
E-mail: mohno@mn.waseda.ac.jp