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AMPLE AND SPANNED VECTOR BUNDLES OF TOP CHERN NUMBER TWO ON PROJECTIVE VARIETIES

Atsushi Noma

1. Introduction

We work over an algebraically closed field k of characteristic zero. Let X be a normal projective variety of dimension n. Let E be an ample and spanned vector bundle of rank $r \geq n$ on X. Here E is said to be *ample* if the tautological line bundle $\mathcal{O}_{\mathbb{P}}(1)$ of the projective bundle $\mathbb{P} = \mathbb{P}_X(E)$ is ample. And E is said to be *spanned* if E is generated by its global sections.

The purpose here is to classify ample and spanned vector bundles E of top Chern number two when X is a smooth projective variety of arbitrary dimension and when X is a normal Gorenstein surface: for the first case, we give only an outline of a proof here and for the details we refer the reader to [11]; for the second case, we give a proof in detail.

There are several studies on the problem of classifying ample and spanned vector bundles of small top Chern number on normal projective varieties.

In case $c_n(E) = 1$, Lanteri-Sommese [9] have shown that $(X, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$ when X is a normal surface or a Gorenstein 3-fold with only isolated singularities. In the higher dimensional case, Wiśniewski [14] has shown that the same is true when X is a smooth projective variety. Fujita has pointed out that the same result for a variety X with only log terminal singularities also follows from Zhang's result [15] together with Lanteri-Sommese's argument in [9] (see Proposition 3.1). These results for $c_n(E) = 1$ have been proved in this way: if $c_n(E) = 1$, then the adjoint bundle $K_X \otimes \det E$ is not nef by

Lanteri-Sommese [9], namely there is a curve C with $(K_X \otimes \det E, C) < 0$; hence theorems on bundles with non-nef adjoint bundles in [14] and [15] imply $(X, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$.

In case $c_2(E) = 2$, the structure of (X, E) has been described explicitly when X is a smooth surface, along the idea of Ballico [1] ([10], see also [8]).

Our main results are the following:

Theorem 1.1. ([11, Theorem 1.1]). *Let X be a smooth projective variety of dimension $n \geq 2$, and E an ample and spanned vector bundle of rank r with $r \geq n$ and $c_n(E) = 2$ on X. Then $r = n$ and (X, E) is one of the following:*

(1) *There exists a finite morphism $f\ X \to \mathbb{P}^n$ of degree 2 over a projective space $\mathbb{P}^n$, and $E \cong f^*\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n}$.*

(2) $(X, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-1})$.

(3) $(X, E) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n})$, *where $\mathbb{Q}^n$ is a smooth quadric hypersurface in $\mathbb{P}^{n+1}$, and $\mathcal{O}_{\mathbb{Q}^n}(1)$ is the hyperplane line bundle.*

(4) *X is isomorphic to a projective space bundle $\mathbb{P}_C(\mathcal{F})$ over an elliptic curve C with the projection $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ and with the tautological line bundle $\mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$, and $E \cong \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$. Here $\mathcal{F}$ and $\mathcal{E}$ are indecomposable rank-n vector bundles of degree 1 on C.*

(5) *X is isomorphic to a projective space bundle $\mathbb{P}_C(\mathcal{F})$ over a hyperelliptic curve C of genus $g \geq 2$ with the projection $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ and with the tautological line bundle $\mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$, and $E \cong \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$. Here $\mathcal{F}$ and $\mathcal{E}$ are vector bundles of rank n on C such that the system $|\det \mathcal{F} \otimes \det \mathcal{E}|$ is the hyperelliptic pencil g^1_2.*

Conversely, every bundle in the all cases but (5) is ample and spanned with $c_n(E) = 2$.

Remark. When $n = 2$, the case (5) is void (see Theorem 1.2 below and [10, Theorem 6.1 and Proposition 5.4]). When $n \geq 3$, I do not know whether a bundle in the case (5) exists or not.

Theorem 1.2. *Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a normal Gorenstein projective surface X. If $c_2(E) = 2$, then $r = 2$ and (X, E) is one of the following :*

(1) *There exists a finite morphism $f : X \to \mathbb{P}^2$ of degree 2 whose branch divisor has no double components and $E \cong f^*\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$.*

(2) $(X, E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$.

(3) $(X, E) \cong (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$.

(4) $(X, E) \cong (\mathbb{Q}^2_0, \mathcal{O}_{\mathbb{Q}^2_0}(1)^{\oplus 2})$, *where $\mathbb{Q}^2_0$ is an (integral) quadric cone in $\mathbb{P}^3$.*

(5) *X is isomorphic to a geometrically ruled surface $\mathbb{P}_C(\mathcal{F})$ over an elliptic curve C with the projection $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ and with the tautological line bundle $\mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$, and $E \cong \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$. Here $\mathcal{F}$ and $\mathcal{E}$ are indecomposable rank-2 vector bundles of degree 1 on C.*

Conversely, every bundle above is ample and spanned with $c_2(E) = 2$.

To prove both theorems, the essential case is $\operatorname{rank} E = \dim X$. In this case, an invariant $\operatorname{sp}(E)$ of Ballico [1] divides our situation into two cases: $\operatorname{sp}(E) = n$ and $\operatorname{sp}(E) > n$. When $\operatorname{sp}(E) = n$, Ballico's theorem [1] and, for a smooth higher dimensional variety X, Zhang's theorem [15] together with Lanteri-Sommese's argument [9] imply the precise structure. When $\operatorname{sp}(E) > n$, a key step is to show that the adjoint bundle $K_X \otimes \det E$ fails to be ample. Then Fujita's theorem [4] for smooth X, and a normal surface version of Lanteri-Maeda's theorem for a normal Gorenstein projective surface X tell us the structure of (X, E). We pick up the bundles in our case and determine the precise structure.

Our exposition proceeds as follows. In §2, we recall the definition of Ballico's invariant $\operatorname{sp}(E)$ and some results on it. In §3, we outline a proof of Theorem

1.1. In §4, we study the adjoint bundle of an ample and spanned vector bundle on a normal Gorenstein projective surface. In §5, we give a proof of Theorem 1.2.

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2. Ballico's invariant sp(E)

In this section, we recall the definition of the invariant $\mathrm{sp}(E)$ of Ballico [1] (see also [10]).

Let X be a normal projective variety of dimension n, and E a spanned vector bundle of rank n with $c_n(E) > 0$ on X. By $|E|$ we denote $\mathbb{P}(H^0(X,E)^\vee) = (H^0(X,E) \setminus \{0\})/k^*$ and by $[t]$ the point of $|E|$ corresponding to $t \neq 0 \in H^0(X,E)$. We consider the family of zeros $F = \{([t],p) \in |E| \times X; t(p) = 0\}$, the projective space bundle $\mathbb{P}_X(\mathcal{F}_E)$ associated with the vector bundle $\mathcal{F}_E$ whose dual bundle is the kernel of the evaluation map $\mathrm{ev} : H^0(X,E) \otimes \mathcal{O}_X \to E$. Hence we get the following diagram:

$$\begin{array}{ccccc} |E| \times X & \supset & F & \overset{\Psi}{\to} & X \\ & & \phi \downarrow & & \\ & & |E|. & & \end{array} \tag{2.1}$$

Here $\Phi : F \to |E|$ and $\Psi : F \to X$ denote the first and second projections. Then Φ is a finite flat morphism of degree $c_n(E)$ over a dense and open subset

$$|E|_{\mathrm{reg}} := \{[t] \in |E|; \dim(t)_0 = 0 \text{ and } (t)_0 \text{ is on the Cohen-Macaulay locus of } X\}.$$

Thus we have a morphism $\tau : |E|_{\mathrm{reg}} \to \mathrm{Hilb}_X^{c_n(E)}$ by the universality of the Hilbert scheme $\mathrm{Hilb}_X^{c_n(E)}$ of $c_n(E)$ points in X. By considering the norm morphism $\nu : \mathrm{Hilb}_X^{c_n(E)} \to S^{c_n(E)}(X)$, we also have a morphism $\rho = \nu \circ \tau : |E|_{\mathrm{reg}} \to S^{c_n(E)}(X)$ to the $c_n(E)$th symmetric product of X.

Then we define $\mathrm{sp}(E)$ by

$$\mathrm{sp}(E) = \dim \tau(|E|_{\mathrm{reg}}).$$

Since ν is isomorphic over the open subset parameterizing distinct smooth $c_n(E)$ points of X, we have $\mathrm{sp}(E) = \dim \rho(|E|_{\mathrm{reg}})$. Hence this invariant is Ballico's in [1]. By definition, we have $n \leq \mathrm{sp}(E) \leq n \cdot c_n(E)$. By [10, Proposition 1.1], for every $[t] \in |E|_{\mathrm{reg}}$, the fibre $\tau^{-1}(\tau([t]))$ of τ over $\tau([t])$ is $|E \otimes \mathcal{I}_{(t)_0}| \cap |E|_{\mathrm{reg}}$, where $|E \otimes \mathcal{I}_{(t)_0}| := \mathbb{P}(H^0(X, E \otimes \mathcal{I}_{(t)_0})^\vee) \subseteq |E|$. Hence, for a general section $t \in H^0(X, E)$, it holds that

$$\mathrm{sp}(E) = \dim_k H^0(X, E) - \dim_k H^0(X, E \otimes \mathcal{I}_{(t)_0}). \tag{2.2}$$

If $\mathrm{sp}(E) = n$ and if E is ample, the finite morphism from X to the Grassmann of n-quotients associated with the evaluation map $H^0(X, E) \otimes \mathcal{O}_X \to E$ maps $(t)_0$ to one point. Thus we have the following proposition.

Proposition 2.3 (Ballico [1, Theorem 4.1], see also [11, Proposition 4.1]). *Let E be an ample and spanned vector bundle of rank n on a normal projective variety X of dimension n. Assume that $sp(E) = n$. Then there exist an ample and spanned vector bundle E' of rank n with $c_n(E') = 1$ on a normal projective variety X' of dimension n, and a finite morphism $f : X \to X'$ of degree $c_n(E)$ such that $E \cong f^*(E')$. Moreover if $n = 2$, then $X' \cong \mathbb{P}^2$ and $E' \cong \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$.*

3. Outline of proof of Theorem 1.1

Now we outline a proof of Theorem 1.1. First we assume that $r = n$. In view of Ballico's invariant $\mathrm{sp}(E)$, we divide the proof into two cases; $\mathrm{sp}(E) = n$ and $\mathrm{sp}(E) > n$.

If $\mathrm{sp}(E) = n$, by Proposition 2.3, X has a double cover $f : X \to X'$ over a normal projective variety X' of dimension n and E is the pull-back of an ample and spanned vector bundle E' with $c_n(E') = 1$ on X'. Since X and X' are normal and $\deg f = 2$, there exists the involution ι on X such that the quotient $X/\langle \iota \rangle$ is X', and hence, X' has only log terminal singularities. Thus by the following Proposition, which is a consequence of Zhang's Theorem [15, Theorem 1] along with Lanteri-Sommese's argument [9], we have $X' \cong \mathbb{P}^n$ and $E' \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n}$; this is the case (1). (The above argument determining the structure of (X', E') is due to Fujita).

Proposition 3.1 (See [11, Proposition 5.1]). *Let X be a projective variety of dimension n with only log terminal singularities and E an ample and spanned vector bundle of rank n with $c_n(E) = 1$ on X. Then $(X, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$.*

If $\mathrm{sp}(E) > n$, for a general point $p \in X$ and a general line $\ell \subset |E \otimes \mathcal{I}_{\{p\}}|$,

$$C := \overline{\{q \in X | \{p, q\} = (t)_0 \text{ for some } t \in \ell \cap |E|_{\mathrm{reg}}\}}$$

is a complete curve on X, where $\bar{\cdot}$ denotes the closure. Assume that

$$\ell \subset |E|_{\mathrm{reg}}. \tag{$*$}$$

Then we have $(K_X \otimes \det E, C) \leq 0$: Indeed, Φ is a double cover over ℓ, and hence $\Phi^{-1}(\ell) = \ell \times \{p\} \cup \tilde{\ell}$, where $\tilde{\ell} := \{(q, t) \in F | t \in \ell, (t)_0 = \{p, q\}\}$. Note that $\Phi(\tilde{\ell}) = C$. Thus for the canonical line bundle K_F of F, we have $(K_F, \tilde{\ell}) = (K_F, \ell \times \{p\})$. On the other hand, $(\mathcal{O}_F(1), \tilde{\ell}) = (\mathcal{O}_{|E|}(1), \ell) = (\mathcal{O}_F(1), \ell \times \{p\})$. Since $K_F \cong \Psi^*(K_X \otimes \det E) \otimes \mathcal{O}_F(-N + n - 1)$, we have $(\Psi^*(K_X \otimes \det E), \tilde{\ell}) = (\Psi^*(K_X \otimes \det E), \ell \times \{p\})$. Since the right hand side is zero, we have $(K_X \otimes \det E, C) = (\Psi^*(K_X \otimes \det E), \tilde{\ell}) = 0$ by the projection formula.

Unfortunately, the assumption $(*)$ above is not true in general (for example, the case (4) in Theorem 1.1): namely, Φ has a positive dimensional fibre on ℓ even if $p \in X$ and $\ell \subset |E \otimes \mathcal{I}_{\{p\}}|$ are general, hence we cannot apply the above argument directly. But we can modify it so that we have the following proposition (see [11, Proposition 7.1] for the details).

Proposition 3.2. *Let X be a normal projective variety of dimension n and E a spanned vector bundle of rank n with $c_n(E) = 2$ on X. Assume that $sp(E) > n$ and that X is Gorenstein. Then there exists a projective integral curve C on*

X through a general point of X such that $(K_X \otimes \det E, C) \leq 0$. In particular, $K_X \otimes \det E$ is not ample.

Therefore the adjoint bundle $K_X \otimes \det E$ is not ample if $\mathrm{sp}(E) > n$. Hence by Fujita's theorem [4], the following three cases fit our situation:

(a) $(X, E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n-1})$;

(b) $(X, E) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n})$;

(c) *There exist vector bundles $\mathcal{F}$ and $\mathcal{E}$ of rank n on a smooth projective curve C, such that X is isomorphic to a projective space bundle $\mathbb{P}_C(\mathcal{F})$ over C with the projection $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ and with the tautological line bundle $\mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$, and $E \cong \pi^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)$.*

The cases (a) and (b) correspond to (2) and (3) respectively. If the case is (c), we have (4) if $g(C) \leq 1$ and (5) if $g(C) \geq 2$, and conversely, we can check that the bundles but in (5) are ample and spanned with $c_n(E) = 2$. (For the details, see [11, §8]).

To complete our proof, we have to show that there is no bundle when $r > n$, by using a standard argument (see, [8], [11]): If $r > n$, by Serre's lemma, we have an exact sequence of vector bundles

$$0 \to \mathcal{O}_X^{\oplus r-n} \to E \to E' \to 0,$$

since E is spanned. Then E' is an ample and spanned vector bundle of rank n with $c_n(E) = 2$. By Theorem 1.1 for $r = n$, we easily check in each case that $H^1(X, E'^{\vee}) = 0$. Therefore the exact sequence above is split in each case. This contradicts the ampleness of E. This completes the proof of Theorem 1.1.

4. The adjoint bundle of an ample and spanned bundle on a normal surface

To prove Theorem 1.2, we need the following proposition.

Proposition 4.1 (Normal surface version of Theorem of Lanteri-Maeda [7]). *Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a normal projective surface X. Then the adjoint system $|K_X \otimes \det E|$ on the Gorenstein locus* $\mathrm{Gor}(X)$ *of X has no base points on* $\mathrm{Gor}(X)$ *unless* $(X, E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$.

Before proving Proposition 4.1, we need the following three lemmas.

Lemma 4.2. *Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a projective curve C. Then E contains a subline bundle of degree ≥ 1 and hence* $\deg(E) \geq r$.

Proof. See [7, Lemma 2].

□

Lemma 4.3. *Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a Gorenstein projective surface X. Assume that $K_X \otimes \det E \cong \mathcal{O}_X$. Then (X, E) is isomorphic to one of the following.*

(1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$.

(2) $(\mathbb{P}^2, T_{\mathbb{P}^2})$, *where $T_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.*

(3) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3})$.

(4) $(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$ *where $\mathbb{Q}^2$ is a smooth quadric surface in $\mathbb{P}^3$.*

(5) $(\mathbb{Q}^2_0, \mathcal{O}_{\mathbb{Q}^2_0}(1)^{\oplus 2})$ *where $\mathbb{Q}^2_0$ is an (integral) quadric cone in $\mathbb{P}^3$.*

Proof. By the assumption, X is a normal Del Pezzo surface (See Brenton [2], Hidaka-Watanabe [6]). Set $L = \det E$. Let $\pi : \widetilde{X} \to X$ be the minimal resolution. If $\widetilde{X}$ is non-rational, by [6, Theorem 2.2], $\widetilde{X}$ is a geometrically ruled surface $p : \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \to C$ over an elliptic curve C, where $\deg(\mathcal{L}) > 0$. And X is obtained by contracting the minimal section C_0, and $K_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(C_0) = \pi^* K_X$. Since $K_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-2C_0) \otimes p^*\mathcal{L}^{-1}$, we have $\pi^* L = \pi^* K_{\widetilde{X}}^{-1} = \mathcal{O}_{\widetilde{X}}(C_0) \otimes p^*\mathcal{L}$. Hence for $x \in C$, we have $\pi^* L \cdot p^{-1}(x) = 1$. By (4.2), this is a contradiction.

Thus $\widetilde{X}$ is rational. By [6, Theorem 3.4], X has only rational double points as singularities, and hence $K_{\widetilde{X}} = \pi^* K_X = \pi^* L^{-1}$. By (4.2), $\widetilde{X}$ has no (-1)-curve. By [6, Theorem 3.4] again, $X \cong \mathbb{P}^2$, $X \cong \mathbb{Q}^2$, or $X \cong \mathbb{Q}_0^2$.

When $X \cong \mathbb{P}^2$, we have $L \cong \mathcal{O}_{\mathbb{P}^2}(3)$, and $r = 2$ or 3, by (4.2). If $r = 2$, by a theorem of van de Ven [13], $(X, E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ or $(\mathbb{P}^2, T_{\mathbb{P}^2})$, which is the case (1) or (2). If $r = 3$, by uniformity, $(X, E) \cong (\mathbb{P}^2, \mathcal{O}(1)^{\oplus 3})$, which is the case (3).

When $X \cong \mathbb{Q}^2$, we have $L = K_X^{-1} = \mathcal{O}_{\mathbb{Q}^2}(2)$. For every fibre $f \cong \mathbb{P}^1$ of the projection $p : \mathbb{Q}^2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, we have $L|f \cong \mathcal{O}_{\mathbb{P}^1}(2)$ and hence $E|f \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ by ampleness. Thus $p_*(E \otimes \mathcal{O}_{\mathbb{Q}^2}(-1))$ is locally free of rank 2 on $\mathbb{P}^1$ and $E \otimes \mathcal{O}_{\mathbb{Q}^2}(-1) \cong p^*(p_*(E \otimes \mathcal{O}_{\mathbb{Q}^2}(-1)))$ by Grauert's theorem ([5, ch. III, Corollary 12.9]). Hence E is the direct sum of ample line bundles with $\det E \cong \mathcal{O}_{\mathbb{Q}^2}(2)$. Therefore we have $(X, E) \cong (\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2})$, which is the case (4).

When $X \cong \mathbb{Q}_0^2$, we have $\widetilde{X} \cong \mathbb{F}_2 := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ and $\pi : \widetilde{X} \to X$ is the contraction of the minimal section C_0. Let $p : \widetilde{X} = \mathbb{F}_2 \to \mathbb{P}^1$ be the projection. Since every fibre f of p is not mapped to a point by π, $\pi^* E|f$ is ample and spanned. Since $\pi^* L = \pi^* K_X^{-1} = K_{\widetilde{X}}^{-1} = \mathcal{O}(2C_0 + 4f)$, we have $\pi^* E|f \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Thus $\pi^* E \cong \mathcal{O}(C_0 + af) \oplus \mathcal{O}(C_0 + bf)$ with $a + b = 4$ by Grauert's theorem. Since $\pi^*(E)$ is spanned, $a = b = 2$. Since $\mathcal{O}_{\widetilde{X}}(C_0 + 2f) \cong \pi^* \mathcal{O}_{\mathbb{Q}_0^2}(1)$, we have $(X, E) \cong (\mathbb{Q}_0^2, \mathcal{O}_{\mathbb{Q}_0^2}(1)^{\oplus 2})$, which is the case (5).

□

Lemma 4.4. *Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a normal projective surface X. Then $c_1^2(E) - c_2(E) \geq 3$ holds.*

Proof. By considering an exact sequence of vector bundles $0 \to \mathcal{O}_X^{\oplus(r-2)} \to E \to E'' \to 0$ for $r \geq 3$, we may assume that $r = 2$. Set $L = \det E$. Let $t \in H^0(X, E)$ be a general section so that the Koszul sequence $0 \to \mathcal{O}_X \to E \to L \otimes \mathcal{I}_{(t)_0} \to 0$ is exact and $(t)_0$ is a set of smooth $c_2(E)$ points. And $\widetilde{X} := \mathbf{Proj}(\oplus_{n \geq 0} \mathcal{I}_{(t)_0}^n)$ is the blowing-up of $(t)_0$ with the projection $\pi : \widetilde{X} \to X$

and with the exceptional Cartier divisor A defined by the ideal $\pi^{-1}\mathcal{I}_{(t)_0}\mathcal{O}_{\widetilde{X}}$. Let $\iota : \widetilde{X} \hookrightarrow \mathbb{P}_X(E)$ be the inclusion corresponding to $E \to L \otimes \mathcal{I}_{(t)_0}$. Set $\mathcal{O}_{\widetilde{X}}(1) = \mathcal{O}_{\mathbb{P}}(1)|\widetilde{X}$. Then $\mathcal{O}_{\widetilde{X}}(1) = \pi^*L \otimes \mathcal{O}_{\widetilde{X}}(-A)$ and $\mathcal{O}_{\widetilde{X}}(1)^2 = c_1^2(E) - c_2(E) > 0$. We have to show that $\mathcal{O}_{\widetilde{X}}(1)^2 = 1$, 2 do not occur.

If $\mathcal{O}_{\widetilde{X}}(1)^2 = 1$, since $\mathcal{O}_{\widetilde{X}}(1)$ is ample and spanned, then the Δ-genus $\Delta(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) := \mathcal{O}_{\widetilde{X}}(1)^2 + 2 - h^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 0$ (see [3, (4.2)]), and hence $(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}^2(1))$ by [3, (5.10) and (5.15)]. This contradicts the minimality of $\mathbb{P}^2$.

If $\mathcal{O}_{\widetilde{X}}(1)^2 = 2$, then $\Delta(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 0$ or 1. If $\Delta(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 0$, $\widetilde{X}$ is isomorphic to an integral quadric surface in $\mathbb{P}^3$, which has no (-1)-curve supported on the nonsingular locus, contradiction. If $\Delta(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 1$, then $h^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 3$, so $\psi = \phi_{|\mathcal{O}_{\widetilde{X}}(1)|} : \widetilde{X} \to \mathbb{P}^2$ is a flat morphism of degree 2, since $\widetilde{X}$ is normal of $\dim \widetilde{X} = 2$. Hence $\widetilde{X}$ is of the form $\mathrm{Spec}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-g-1))$ for the sectional genus $g = g(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1))$ as a scheme over $\mathbb{P}^2$ (see [3, p.49, (6.11) and (6.12)]). In particular, $\widetilde{X}$ is Gorenstein with the canonical line bundle $K_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(g-2)$. Since π is an isomorphism around the singular points, X is also Gorenstein. So $\mathbb{P}_X(E)$ is Gorenstein with the canonical line bundle $K_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-2) \otimes \pi^*(K_X \otimes \det E)$. Since $\widetilde{X} \in |\mathcal{O}_{\mathbb{P}}(1)|$, by the adjunction formula, we also have $K_{\widetilde{X}} = \pi^*K_X \otimes \mathcal{O}_{\widetilde{X}}(A)$. Calculating the intersection number $K_{\widetilde{X}} \cdot A$ by using the two expression of $K_{\widetilde{X}}$, we have $-c_2(E) = (g-2)c_2(E)$ and hence $g = 1$. Thus $\pi^*L^{-1} \otimes \mathcal{O}_{\widetilde{X}}(A) = \mathcal{O}_{\widetilde{X}}(-1) = K_{\widetilde{X}} = \pi^*K_X \otimes \mathcal{O}_{\widetilde{X}}(A)$. Hence $K_X \otimes L \cong \mathcal{O}_X$. But, by (4.3), we have $\mathcal{O}_{\widetilde{X}}(1)^2 = c_1^2(E) - c_2(E) > 2$, contradiction. Thus we have the claim.

□

Proof of Proposition 4.1. Set $L = \det E$. As in (4.4), we may assume that $r = 2$. Note that $c_2(E) > 0$ since E is ample and spanned. If $c_2(E) = 1$, then by [9, Theorem 1], $(X, E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$. If $c_2(E) \geq 2$, then $c_1^2(E) \geq 5$ by (4.4). Suppose to the contrary that $|K_X \otimes \det E|$ has a base point in $\mathrm{Gor}(X)$. By Reider-type-Theorem of Sakai [12, Theorem 4], there exists a nonzero effective divisor B on X such that $B.L \leq 1$. This contradicts the ampleness and spannedness of E by (4.2).

$\square$

5. Proof of Theorem 1.2

Now we prove Theorem 1.2. If E is ample and spanned with $c_n(E) = 2$, and if $\operatorname{rank} E = 2$, as in the proof of Theorem 1.1, we divide the proof into two cases, $\operatorname{sp}(E) = 2$ and $\operatorname{sp}(E) > 2$.

When $\operatorname{sp}(E) = 2$, by (2.3), there exists a finite morphism $f : X \to \mathbb{P}^2$ of degree 2 and $E \cong f^*(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$, i.e., this is the case (1). Note that the condition on the branch divisor is a necessary and sufficient condition for a double cover X to be integral and normal.

Next we assume that $\operatorname{sp}(E) > 2$. By (4.1), $\varphi = \phi_{|K_X \otimes \det E|} : X \to \mathbb{P}(H^0(K_X \otimes \det E))$ is a morphism defined everywhere on X. By (3.2), we have $\dim \varphi(X) = 0, 1$. If $\varphi(X)$ is a point, then $K_X \otimes \det E \cong \mathcal{O}_X$. By (4.3), the case (2), (3) or (4) occurs. If $\varphi(X) = C'$ is an integral curve, we take Stein factorization $\pi : X \to C$ and $C \to C'$ of $\varphi : X \to C'$. Then C is a nonsingular projective curve, and π is a flat morphism of connected fibres, and $K_X \otimes \det E$ is the pull-back of a line bundle on C. Since a general fibre f is smooth and irreducible and since

$$\deg K_f = (K_X + f) \cdot f = K_X \cdot f = -\det E \cdot f \leq -2,$$

we have $f \cong \mathbb{P}^1$. By the theorem of Noether-Enriques, X is ruled. Since $\deg(K_X|\pi^{-1}(x)) = -2$, we have $\deg(\det E|\pi^{-1}(x)) = 2$ for every $x \in C$. By (4.2), every fibre $\pi^{-1}(x)$ is irreducible and reduced. Since π is flat, we have $\pi^{-1}(x) \cong \mathbb{P}^1$. Therefore X is a (smooth) geometrically ruled surface over C. Thus by [10, Theorem 6.1] (see also [10, Proposition 5.4]), we have (5).

As in Theorem 1.1, there is no bundle with $r > 2$. Conversely every bundle in the cases is ample and spanned with $c_2(E) = 2$ as in [10] (see also [11]). This completes the proof of Theorem 1.2.

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Department of Mathematics
Faculty of Education and Human Sciences
Yokohama National University
Tokiwadai, Hodogaya, Yokohama 240-8501
Japan
E-mail: noma@ms.ed.ynu.ac.jp