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SOME HOMOTOPY GROUPS OF THE DOUBLE SUSPENSION OF THE REAL PROJECTIVE SPACE $\mathbf{R}P^6$

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Introduction

Before we state our result, let us fix some notations. Let X be a based finite CW-complex and

$$\Sigma^n X = X \wedge S^n$$

be the n-fold reduced suspension of X. Let ι_X be the homotopy class of the identity map of X. The order of the element $\Sigma\iota_X = \iota_{\Sigma X}$ in $[\Sigma X, \Sigma X]$ is called the suspension order ([13]) or the characteristic ([2]) of X.

We denote by $\mathbf{R}P^n$ the real n-dimensional projective space. The purpose of the present note is to prove the following.

Theorem 1. *The suspension order of* $\Sigma^2\mathbf{R}P^6$ *is* 8.

As a direct consequence of this theorem, we have

Theorem 2. *The suspension order of* $\Sigma^2\mathbf{R}P^{2n}$ *is* $2^{\varphi(2n)}$, *where* $\varphi(m)$ *stands for the number of integers* k *satisfying* $1 \leq k \leq m$ *and* $k \equiv 0, 1, 2$ *or* $4 \bmod 8$.

We set

$$M^n = \Sigma^{n-2}\mathbf{R}P^2 \ (n \geq 2).$$

This is a Moore space of type $(\mathbf{Z}_2, n-1)$. To prove Theorem 1, we need to determine the following unstable homotopy groups

$$\pi_i(\Sigma^2\mathbf{R}P^6) \text{ and } [M^i, \Sigma^2\mathbf{R}P^6] \ (i \leq 8).$$

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Our method is the composition methods developed by Toda([12]). We shall also use the homotopy excision theorem ([5]) about the metastable relative homotopy groups $\pi_k(X, A)$, where A is a finite CW-complex of dimension less than $n-1$ and $X = A \cup e^n$.

The results and notations of [12] will be used freely. It is assumed that we are especially familiar with the results and relations in $\pi_{n+k}(S^n)$ $(k \leq 6)$. We will also deal with the Whitehead products ([3], [1]) freely.

Let $\iota_n = \iota_{S^n}, \eta_2$ be the Hopf map, $\eta_n = \Sigma^{n-2}\eta_2$ and $\eta_n^2 = \eta_n \circ \eta_{n+1}$. Let $\nu' \in \pi_6(S^3)$ be a generator of the 2-primary component $\mathbf{Z}_4$ of $\pi_6(S^3)$, $\nu_4 \in \pi_7(S^4)$ be the Hopf map and $\nu_n = \Sigma^{n-4}\nu_4$ $(n \geq 4)$.

1. Recalling some fundamental results

We set $\mathrm{P}^n = \mathbf{R}\mathrm{P}^n$ for simplicity. We set $\mathrm{P}_k^n = \mathrm{P}^n/\mathrm{P}^{k-1}$ $(k \leq n)$. Let $\gamma_n : S^n \longrightarrow \mathrm{P}^n$ be the projection. We denote by $i_{n-1} : S^{n-1} \hookrightarrow M^n$ an inclusion and by $p_n : M^n \to S^n$ a collapsing map. Let $X = A \cup CB$ be a mapping cone of a mapping from B to A, where $B = S^{n-1}$ or M^{n-1}. For an element $\alpha \in \pi_{k-1}(B)$, we denote by $\hat{\alpha} \in \pi_k(X, A)$ an element satisfying $p'_*\hat{\alpha} = \Sigma\alpha$, where $p' : (X, A) \to (\Sigma B, *)$ is a collapsing map. We recall

$$\Sigma\gamma_2 = 2i_2\eta_2;\ \Sigma^2\gamma_2 = 0,$$

and so we have

$$\Sigma^2\mathrm{P}^3 = M^4 \vee S^5.$$

Let

$$k_1 : M^4 \hookrightarrow M^4 \vee S^5 \text{ and } k_2 : S^5 \hookrightarrow M^4 \vee S^5$$

be the inclusions, respectively. Then, by the cell structure of $\Sigma^2\mathrm{P}^4$, we have

$$\Sigma^2\gamma_3 = \pm k_1\tilde{\eta}_3 + 2k_2.$$

Here $\tilde{\eta}_3 \in \pi_5(M^4)$ is a coextension of η_3 satisfying $p_{4*}\tilde{\eta}_3 = \eta_4$. We recall ([10]) that

$$\pi_4(M^3) = \mathbf{Z}_4\{\alpha\},\ \text{where } \Sigma\alpha = \tilde{\eta}_3.$$

We denote by $p'_n : \mathrm{P}^{2n+2} \longrightarrow \mathrm{P}^{2n+2}_{2n+1} = M^{2n+2}$ a map given by pinching P^{2n} to a single point. We set

$$\tau_n = \Sigma\gamma_{2n}.$$

We know that τ_n is of order 2. We define an extension

$$\bar{\tau}_n \in [M^{2n+2}, \Sigma \mathrm{P}^{2n}]$$

of τ_n as the third map of the cofiber sequence

$$\mathrm{P}^{2n} \xrightarrow{i} P^{2n+2} \xrightarrow{p'_n} M^{2n+2} \longrightarrow \Sigma \mathrm{P}^{2n}.$$

By abuse of notation, we often use p to represent a collapsing map and use i to represent an inclusion map. We define an element $\beta \in \pi_5(\Sigma^2\mathrm{P}^4)$ as the composite ik_2 for the inclusion $i : \Sigma^2\mathrm{P}^3 \hookrightarrow \Sigma^2\mathrm{P}^4$. From the definition of β, we have $p'_1\beta = i_5 \in \pi_5(M^6)$ and $\Sigma\beta$ is taken as a coextension $\tilde{\imath}_5$ of i_5. Let $\bar{\eta}_3 \in [M^5, S^3]$ be an extension of η_3, set $\bar{\eta}_n = \Sigma^{n-3}\bar{\eta}_3 \in [M^{n+2}, S^n]$ and $\tilde{\eta}_n = \Sigma^{n-3}\tilde{\eta}_3 \in \pi_{n+2}(M^{n+1})$ $(n \geq 3)$. We recall ([7]) that $\pi_6(\Sigma^3\mathrm{P}^4) = \mathbf{Z}_8\{\tilde{\imath}_5\}$ and the following.

Lemma 1.1. $\Sigma\bar{\tau}_1 = \tilde{\eta}_3 p \in [M^5, M^4]$ *and* $\Sigma^2\bar{\tau}_2 = \tilde{\imath}_5\bar{\eta}_6 \in [M^8, \Sigma^3\mathrm{P}^4]$.

We have the following.

Lemma 1.2. $\pi_5(\Sigma^2\mathrm{P}^4) = \mathbf{Z}_8\{\beta\}$, *where* $2\beta = \pm i'\tilde{\eta}_3$ *and* $H(\beta) = i \in \pi_5(\Sigma(\Sigma\mathrm{P}^4 \wedge \Sigma\mathrm{P}^4))$.

Proof. We consider the exact sequence

$$\pi_6(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) \xrightarrow{\partial} \pi_5(\Sigma^2\mathrm{P}^3) \xrightarrow{i_*} \pi_5(\Sigma^2\mathrm{P}^4) \longrightarrow 0.$$

We have $\pi_6(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) = \mathbf{Z}\{\omega\}$, $\pi_5(\Sigma^2\mathrm{P}^3) \cong \pi_5(M^4) \oplus \pi_5(S^5)$, where ω is the characteristic map of the 6-cell of $\Sigma^2\mathrm{P}^4$. We have $\partial\omega = \Sigma^2\gamma_3 = \pm k_1\tilde{\eta}_3 + 2k_2$. From the definition of β, we have the result apart from the last assertion.

In the EHP sequence

$$\pi_4(\Sigma\mathrm{P}^4) \xrightarrow{\Sigma} \pi_5(\Sigma^2\mathrm{P}^4) \xrightarrow{H} \pi_5(\Sigma(\Sigma\mathrm{P}^4 \wedge \Sigma\mathrm{P}^4)) \xrightarrow{\Delta} \pi_4(\Sigma\mathrm{P}^4),$$

we have $\pi_5(\Sigma(\Sigma P^4 \wedge \Sigma P^4)) \cong \pi_5(\Sigma(M^3 \wedge M^3)) \cong \pi_5(M^6) \cong \mathbf{Z}_2$ and $\Delta i = i''\Delta\iota_5 = 2i\eta_2 = i''\Sigma\gamma_2 = 0$, where $i'' : M^3 \hookrightarrow \Sigma P^4$ is an inclusion satisfying $\Sigma i'' = i'$. So H is an epimorphism and $H(\beta) = i$. This completes the proof.

□

We denote by ι'_n the identity map of M^n. We recall that $M^2 \wedge M^2 = M^3 \cup_{2\iota'_3} CM^3$ and that $2\iota'_3 = i_2\eta_2 p_3 \in [M^3, M^3]$.

We have the following([10]) ; $\pi_3(M^2 \wedge M^2) = \mathbf{Z}_8\{\tilde{i}'_2\}$ and $\pi_{2n}(\Sigma(M^n \wedge M^n)) = \mathbf{Z}_4\{\Sigma^{2n-3}\tilde{i}'_2\}$ for $n \geq 3$, where $\tilde{i}'_2$ is a coextension of i_2 and it satisfies $2\tilde{i}'_2 = i\eta_2$. We also recall ([10]) that $\pi_6(M^4) = \mathbf{Z}_4\{\delta\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3\eta_5\}$, where $\delta \in \pi_6(M^4)$ is the attaching map in the Stiefel manifold $V_{5,2} = M^4 \cup_\delta e^7$. We have the relations $2\delta = i_3\nu' = \Delta(\Sigma^5\tilde{i}'_2)$, $\Sigma\delta = 2i_4\nu_4$ and $H(\delta) = \Sigma^3\tilde{i}'_2$. We show

Lemma 1.3. $\pi_6(\Sigma^2 P^4) = \mathbf{Z}_4\{i'\delta\} \oplus \mathbf{Z}_2\{\beta\eta_5\}$ *and* $\beta\eta_5 = \Sigma^2\gamma_4 + 2i'\delta \in \pi_6(\Sigma^2 P^4)$ *so that* $\pi_6(\Sigma^2 P^5) = \mathbf{Z}_4\{i'\delta\}$.

Proof. In the exact sequence

$$\pi_7(\Sigma^2 P^4, \Sigma^2 P^3) \xrightarrow{\partial} \pi_6(\Sigma^2 P^3) \xrightarrow{i_*} \pi_6(\Sigma^2 P^4) \longrightarrow 0,$$

we have

$$\pi_7(\Sigma^2 P^4, \Sigma^2 P^3) = \mathbf{Z}_2\{\hat{\eta}_5\} \text{ and } \pi_6(\Sigma^2 P^3) \cong \pi_6(M^4) \oplus \pi_6(S^5).$$

Since $\partial\hat{\eta}_5 = \Sigma^2\gamma_3 \circ \eta_5 = k_1\tilde{\eta}_3\eta_5$, we have the first half.

We consider the EHP sequence

$$\pi_7(\Sigma^2 P^4) \xrightarrow{\Sigma} \pi_8(\Sigma^3 P^4) \xrightarrow{H} \pi_8(\Sigma(\Sigma^2 P^4 \wedge \Sigma^2 P^4)) \xrightarrow{\Delta} \pi_6(\Sigma^2 P^4) \longrightarrow \cdots.$$

Since

$$\pi_8(\Sigma(\Sigma^2 P^4 \wedge \Sigma^2 P^4)) = \pi_8(\Sigma(\Sigma^2 P^3 \wedge \Sigma^2 P^3)) \cong \pi_8(\Sigma(M^4 \wedge M^4)) = \mathbf{Z}_4\{\Sigma^5\tilde{i}'_2\},$$

we have an isomorphism $\Sigma(i' \wedge i')_* : \pi_8(\Sigma(M^4 \wedge M^4)) \to \pi_8(\Sigma(\Sigma^2 P^4 \wedge \Sigma^2 P^4))$. Lemmas 1.1 implies that $\Sigma^3\gamma_4 = (\Sigma\beta)\eta_6 \in \pi_7(\Sigma^3 P^4)$ so that $\Sigma^2\gamma_4 \equiv \beta\eta_5 \bmod \Delta(\Sigma(i' \wedge i')\Sigma^5\tilde{i}'_2) = 2i'\delta$. Since $H(\beta\eta_5) = i\eta_5$ and $2H(\delta) = H(i_3\nu') = i\eta_5$, we have the relation. The structure of $\pi_6(\Sigma^3 P^5)$ follows, completing the proof. □

Let $i'_n : M^{2n-1} \longrightarrow M^n \wedge M^n$ be the inclusion and set $i''_n = \Sigma i'_n$. By inspecting the proof of Theorem 2.4 of Chapter 12 of [14], by [1] and by Remark in §1 of [10], we have the following.

Lemma 1.4. $\Delta(i''_{n+1}) = [\iota'_{n+1}, \iota'_{n+1}]i''_n = [\iota'_{n+1}, i_n] \in [M^{2n}, M^{n+1}]$ *for* $n \geq 3$.

Next we show

Lemma 1.5.

i) $[M^5, M^4] = \mathbf{Z}_2\{i_3\bar{\eta}_3\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3 p_5\}$
and $\Sigma : [M^5, M^4] \to [M^6, M^5]$ *is an isomorphism.*

ii) $[M^4, M^3] = \mathbf{Z}_2\{\bar{\tau}_1\} \oplus \mathbf{Z}_2\{\alpha p_4\}$ *and* $\Sigma\bar{\tau}_1 = \Sigma(\alpha p_4) = \tilde{\eta}_3 p_5$.

iii) $[M^7, M^5] = \mathbf{Z}_2\{i_4\eta_4\bar{\eta}_5\} \oplus \mathbf{Z}_2\{\tilde{\eta}_4\eta_6 p_7\} \oplus \mathbf{Z}_2\{i_4\nu_4 p_7\}$;
$[M^6, M^4] = \mathbf{Z}_2\{\delta p_6\} \oplus \mathbf{Z}_2\{i_3\eta_3\bar{\eta}_4\} \oplus \mathbf{Z}_2\{\tilde{\eta}_3\eta_5 p_6\}$;
$[M^8, M^5] = \mathbf{Z}_4\{\tilde{\eta}_4\bar{\eta}_6\} \oplus \mathbf{Z}_2\{[\iota'_5, i_4]\} \oplus \mathbf{Z}_2\{i_4\nu_4\eta_7 p_8\}$
and $[M^7, M^4] = \mathbf{Z}_4\{\tilde{\eta}_3\bar{\eta}_5\} \oplus \mathbf{Z}_2\{\delta\eta_6 p_7\} \oplus \mathbf{Z}_2\{\overline{i_3\nu'}\}$, *where* $\overline{i_3\nu'}$ *is an extension of* $i_3\nu' \in \pi_6(M^4)$.

iv) $\Delta(i''_4) = [\iota'_4, i_3] = \delta p_6 \in [M^6, M^4]$; $H[\iota'_5, i_4] = (\Sigma^5\tilde{i}'_2)p_8 \in [M^8, \Sigma(M^4 \wedge M^4)]$
and $\Sigma\overline{i_3\nu'} = i_4\nu_4\eta_7 p_8 \in [M^8, M^5]$.

Proof. By making use of the cofibration $i_4 : S^4 \hookrightarrow M^5$, an EHP sequence and the fact $[\iota_3, \iota_3] = 0$, we have i). By making use of the cofibration starting with $2\iota_3$ and by Lemma 4.1 of [10], we have ii).

By making use of cofibrations starting with $2\iota_n$ for $n = 5$, 6 and 7 and by Lemma 2.2 of [7], we have iii). We note that $2\overline{i_3\nu'} = i_3\nu'\eta_6 p_7 = 2\delta\eta_6 p_7 = 0$.

In the EHP sequence

$$[M^7, M^4] \xrightarrow{\Sigma} [M^8, M^5] \xrightarrow{H} [M^8, \Sigma(M^4 \wedge M^4)] \xrightarrow{\Delta} [M^6, M^4] \xrightarrow{\Sigma} [M^7, M^5],$$

we have ([10]) $[M^8, \Sigma(M^4 \wedge M^4)] = \mathbf{Z}_2\{i''_4\} \oplus \mathbf{Z}_2\{(\Sigma^5\tilde{i}'_2)p_8\}$. Since $\Sigma(\delta p_6) = 2i_4\nu_4 p_7 = 0, \Delta((\Sigma^5\tilde{i}'_2)p_8) = 2\delta p_6 = 0$ and $H(i_4\nu_4\eta_7 p_8) = i\eta_7 p_8 = 2(\Sigma^5\tilde{i}'_2)p_8 = 0$, we have iv) by iii) and Lemma 1.4. This completes the proof.

□

2. Some homotopy groups of $\Sigma^2 P^6$

By making use of the cofibration $i_5 : S^5 \hookrightarrow M^6$ and by Lemmas 1.2 and 1.3, we have the following.

Lemma 2.1. *We have the following isomorphisms;*

$$[M^6, \Sigma^2 P^4] = \mathbf{Z}_2\{i\eta_3\bar{\eta}_4\} \oplus \mathbf{Z}_2\{i'\delta p_6\} \oplus \mathbf{Z}_2\{\beta\eta_5 p_6\};$$

$$[M^6, \Sigma^2 P^6] = \mathbf{Z}_2\{i\eta_3\bar{\eta}_4\} \oplus \mathbf{Z}_2\{i'\delta p_6\}.$$

We show

Lemma 2.2.

i) $\pi_7(\Sigma^2 P^4) = \mathbf{Z}_2\{\beta\eta_5^2\} \oplus \mathbf{Z}_2\{[i,\beta]\}$ *and* $[i,\beta]$ *has an extension* $[i',\beta] \in [M^8, \Sigma^2 P^4]$, *where* $i' : M^4 \hookrightarrow \Sigma^2 P^4$ *is the inclusion.*

ii) $\pi_7(\Sigma^2 P^6) = \mathbf{Z}_2\{i'[i,\beta]\}$.

iii) $[M^7, \Sigma^2 P^6] = \mathbf{Z}_2\{i'\overline{\imath_3\nu'}\} \oplus \mathbf{Z}_2\{i'[i,\beta]p_7\}$.

Proof. Since $\Sigma^2 P^3 = M^4 \vee S^5$, we have a split exact sequence

$$0 \longrightarrow \pi_8(M^4 \times S^5, \Sigma^2 P^3) \xrightarrow{\partial} \pi_7(\Sigma^2 P^3) \xrightarrow{i_*} \pi_7(M^4 \times S^5) \longrightarrow 0.$$

We know $M^4 \times S^5 = (M^4 \vee S^5) \bigcup_{[k_1,k_2]} CM^8$, $\pi_8(M^4 \times S^5, \Sigma^2 P^3) = \mathbf{Z}_2\{\hat{\imath}_7\}$ and $\partial\hat{\imath}_7 = [k_1, k_2] \circ i_7$.

We consider the exact sequence

$$\pi_8(\Sigma^2 P^4, \Sigma^2 P^3) \xrightarrow{\partial} \pi_7(\Sigma^2 P^3) \xrightarrow{i_*} \pi_7(\Sigma^2 P^4) \xrightarrow{j_*} \pi_7(\Sigma^2 P^4, \Sigma^2 P^3) \xrightarrow{\partial} \pi_6(\Sigma^2 P^3).$$

By the homotopy excision theorem, we have

$$\pi_8(\Sigma^2 P^4, \Sigma^2 P^3) = \mathbf{Z}_2\{\widehat{\eta_5^2}\} \oplus \mathbf{Z}_2\{[\omega, i]\} \text{ for } i = k_1 i_3$$

and $\pi_7(\Sigma^2 P^4, \Sigma^2 P^3) = \mathbf{Z}_2\{\hat{\eta}_5\}$. By Lemma 1.5.iv), we have

$$\partial\hat{\eta}_5 = \Sigma^2\gamma_3 \circ \eta_5 = k_1\tilde{\eta}_3\eta_5, \; \partial\widehat{\eta_5^2} = \Sigma^2\gamma_3 \circ \eta_5^2 = k_1\tilde{\eta}_3\eta_5^2$$

and

$$\begin{aligned}\partial[\omega, i] = [\Sigma^2\gamma_3, i] &= [2k_2, i] + [k_1\tilde{\eta}_3, i] \\ &= k_1[\tilde{\eta}_3, i_3] \\ &= k_1[\iota_4', i_3]\tilde{\eta}_5 \\ &= k_1\delta\eta_6.\end{aligned}$$

So i_* is an epimorphism and $i[k_1, k_2]i_7 = [i', \beta]i_7 = [i, \beta]$. This leads us to i).

In the homotopy exact sequence

$$\pi_8(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^4) \xrightarrow{\partial} \pi_7(\Sigma^2\mathrm{P}^4) \xrightarrow{i_*} \pi_7(\Sigma^2\mathrm{P}^6) \xrightarrow{j_*} \pi_7(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^4) \xrightarrow{\partial} \pi_6(\Sigma^2\mathrm{P}^4),$$

we have

$$\pi_8(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^4) = \mathbf{Z}_2\{\hat{\imath}_6\hat{\eta}_6\} \text{ and } \pi_7(\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^4) = \mathbf{Z}_2\{\hat{\imath}_6\}.$$

By Lemma 1.3, $\partial\hat{\imath}_6 = \Sigma^2\gamma_4$ and $\partial(\hat{\imath}_6\hat{\eta}_6) = \Sigma^2\gamma_4 \circ \eta_6 = \beta\eta_5^2$. So we have ii).

By ii) and Lemma 1.3, $[M^7, \Sigma^2\mathrm{P}^6]$ is generated by $i'\overline{\imath_3\nu'}$ and $i''[i, \beta]p_r$. This leads us to iii) and completes the proof.

□

Remark. The result ii) of Lemma 2.2 was pointed out by Milgram ([6]).

Next we shall determine $\pi_8(\Sigma^2\mathrm{P}^n)$. Since $\eta_3\nu_4 = \nu'\eta_6$, $\tilde{\eta}_3\nu_5 \in \pi_8(M^4)$ is a coextension of $\nu'\eta_6$.

Lemma 2.3. *$\tilde{\eta}_3\nu_5$ is of order 2 and $\tilde{\eta}_4\nu_6 = i_4\nu_4\eta_7^2$.*

Proof. We have $2\tilde{\eta}_3\nu_5 = i_3\eta_3^2\nu_5 = i_3(\eta_3\Sigma\nu')\eta_7 = 0$. So we have the first half. From the definition of the Toda bracket, we have

$$\begin{aligned}\overline{\imath_3\nu'}\tilde{\eta}_6 &\in \{i_3\nu', 2\iota_6, \eta_6\} \\ &\subset \{i_3, 2\nu', \eta_6\} \\ &\supset \{i_3, 2\iota_3, \nu'\eta_6\} \\ &\ni \tilde{\eta}_3\nu_5.\end{aligned}$$

So we have $\tilde{\eta}_3\nu_5 \equiv \overline{i_3\nu'}\tilde{\eta}_6 \bmod i_{3*}\pi_8(S^3) + \pi_7(M^4) \circ \eta_7 = \{\delta\eta_6^2\}$. We have $\Sigma(\delta\eta_6^2) = 2i_4\nu_4\eta_7^2 = 0$. So, by Lemma 1.5.iv), we have $\tilde{\eta}_4\nu_6 = \Sigma(\overline{i_3\nu'}\tilde{\eta}_6) = i_4\nu_4\eta_7^2$. This completes the proof.

□

Lemma 2.4. $\pi_8(M^4) = \mathbf{Z}_2\{\tilde{\eta}_3\nu_5\} \oplus \mathbf{Z}_2\{\delta\eta_6^2\} \oplus \mathbf{Z}_2\{[\delta, i_3]\}$.

Proof, By the exact sequence

$$\pi_7(V_{5,2}, M^4) \xrightarrow{\partial} \pi_6(M^4) \xrightarrow{i_*} \pi_6(V_{5,2}) \longrightarrow 0,$$

we have $\pi_6(V_{5,2}) = \mathbf{Z}_2\{i\tilde{\eta}_3\eta_5\}$.

We consider the homotopy exact sequence for the S^3-bundle $V_{5,2}$ over S^4:

$$\pi_{10}(S^4) \xrightarrow{\Delta} \pi_9(S^3) \xrightarrow{i'_*} \pi_9(V_{5,2}) \xrightarrow{p'_*} \pi_9(S^4) \xrightarrow{\Delta} \pi_8(S^3) \xrightarrow{i'_*} \pi_8(V_{5,2}) \xrightarrow{p'_*} \pi_8(S^4)$$
$$\xrightarrow{\Delta} \pi_7(S^3) \xrightarrow{i'_*} \pi_7(V_{5,2}) \xrightarrow{p'_*} \pi_7(S^4) \xrightarrow{\Delta} \pi_6(S^3) \xrightarrow{i'_*} \pi_6(V_{5,2}) \xrightarrow{p'_*} \pi_6(S^4).$$

Since the last p'_* is an isomorphism and $\Delta(\Sigma\nu') = 2\nu'$, we have $\Delta(\nu_4) = \pm\nu'$. Hence, by chasing the diagram, we have the following result which overlaps with the results of [11]:

$$\pi_7(V_{5,2}) \cong \mathbf{Z} \oplus \mathbf{Z}_2, \pi_8(V_{5,2}) = \mathbf{Z}_2\{i\tilde{\eta}_3\nu_5\} \text{ and } \pi_9(V_{5,2}) = \mathbf{Z}_2\{i\tilde{\eta}_3\nu_5\eta_8\}.$$

We consider the exact sequence

$$\pi_9(M^4) \xrightarrow{i_*} \pi_9(V_{5,2}) \xrightarrow{j_*} \pi_9(V_{5,2}, M^4) \xrightarrow{\partial} \pi_8(M^4)$$
$$\xrightarrow{i_*} \pi_8(V_{5,2}) \xrightarrow{j_*} \pi_8(V_{5,2}, M^4) \xrightarrow{\partial} \pi_7(M^4).$$

There exists an element $\tilde{\eta}_3\nu_5\eta_8 \in \pi_9(M^4)$ of order 2, because $p_{4*}\tilde{\eta}_3\nu_5\eta_8 = \eta_4\nu_5\eta_8 = (\Sigma\nu')\eta_7^2 \neq 0$. So, by the above result and by Lemma 2.3, both i_* are split epimorphisms, respectively. Therefore we have a split exact sequence

$$0 \longrightarrow \pi_9(V_{5,2}, M^4) \xrightarrow{\partial} \pi_8(M^4) \xrightarrow{i_*} \pi_8(V_{5,2}) \longrightarrow 0.$$

By the homotopy excision theorem, we have $\pi_9(V_{5,2}, M^4) = \mathbf{Z}_2\{\widehat{\eta_6^2}\} \oplus \mathbf{Z}_2\{[\omega, i_3]\}$, where ω is the charactesitic map of the 7-cell of $V_{5,2}$. We have $\partial\widehat{\eta_6^2} = \delta\eta_6^2$ and $\partial[\omega, i_3] = [\delta, i_3]$. This completes the proof.

□

Finally we show the main result in this section.

Lemma 2.5.

i) $\pi_8(\Sigma^2\mathrm{P}^4) = \mathbf{Z}_4\{\beta\nu_5\} \oplus \mathbf{Z}_2\{i'[\delta, i_3]\} \oplus \mathbf{Z}_2\{[i,\beta]\eta_7\}$ *and* $2\beta\nu_5 = i'\tilde{\eta}_3\nu_5$.

ii) $[M^8, \Sigma^2\mathrm{P}^4] = \mathbf{Z}_4\{[i',\beta]\} \oplus \mathbf{Z}_2\{\beta\eta_5\bar{\eta}_6\} \oplus \mathbf{Z}_2\{i'[\delta, i_3]p_8\} \oplus \mathbf{Z}_2\{\beta\nu_5 p_8\}$ *and* $2[i',\beta] = [i,\beta]\eta_7 p_8$.

iii) $\pi_8(\Sigma^2\mathrm{P}^5) = \mathbf{Z}_4\{i''\beta\nu_5\} \oplus \mathbf{Z}_2\{i'[\delta, i_3]\}$, $i''[i,\beta]\eta_7 = 0$ *and* $\pi_8(\Sigma^2\mathrm{P}^6) \cong \pi_8(\Sigma^2\mathrm{P}^5)$.

iv) $[M^8, \Sigma^2\mathrm{P}^6] = \mathbf{Z}_2\{i''[i',\beta]\} \oplus \mathbf{Z}_2\{i'[\delta, i_3]p_8\} \oplus \mathbf{Z}_2\{i''\beta\nu_5 p_8\}$.

Proof. In the exact sequence

$$\pi_9(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) \xrightarrow{\partial} \pi_8(\Sigma^2\mathrm{P}^3) \xrightarrow{i_*} \pi_8(\Sigma^2\mathrm{P}^4) \xrightarrow{j_*} \cdots,$$

we have $\pi_8(\Sigma^2\mathrm{P}^3) \cong \pi_8(M^4) \oplus \pi_8(S^5) \oplus \pi_9(M^4 \times S^5, M^4 \vee S^5) \cong (\mathbf{Z}_2)^3 \oplus \mathbf{Z}_{24} \oplus \mathbf{Z}_2$ and i_* is an epimorphism by the proof of Lemma 2.2.i). We have

$$\pi_9(\Sigma^2\mathrm{P}^4, \Sigma^2\mathrm{P}^3) = \mathbf{Z}_{24}\{\hat{\nu}_5\} \oplus \mathbf{Z}_2\{[\omega, i]\hat{\eta}_7\}.$$

Since $\partial\hat{\nu}_5 = \Sigma^2\gamma_3 \circ \nu_5 = k_1\tilde{\eta}_3\nu_5 + 2k_2\nu_5$ and $\partial([\omega, i]\hat{\eta}_7) = k_1\delta\eta_6^2$, we have i).

In the exact sequence

$$0 \longleftarrow \pi_7(\Sigma^2\mathrm{P}^4) \xleftarrow{i_7^*} [M^8, \Sigma^2\mathrm{P}^4] \xleftarrow{p_8^*} \pi_8(\Sigma^2\mathrm{P}^4) \xleftarrow{2} \pi_8(\Sigma^2\mathrm{P}^4),$$

we find elements $[i',\beta]$ and $\beta\eta_5\bar{\eta}_6$ in $[M^8, \Sigma^2\mathrm{P}^4]$ by Lemma 2.2.i). We have $2[i',\beta] = [i',\beta]i_7\eta_7 p_8 = [i,\beta]\eta_7 p_8$. This determine the group extension of ii).

In the exact sequence

$$\pi_9(\Sigma^2\mathrm{P}^5, \Sigma^2\mathrm{P}^4) \xrightarrow{\partial} \pi_8(\Sigma^2\mathrm{P}^4) \xrightarrow{i_*} \pi_8(\Sigma^2\mathrm{P}^5) \xrightarrow{j_*} \pi_8(\Sigma^2\mathrm{P}^5, \Sigma^2\mathrm{P}^4) \xrightarrow{\partial} \pi_7(\Sigma^2\mathrm{P}^4),$$

we have $\pi_8(\Sigma^2\mathrm{P}^5, \Sigma^2\mathrm{P}^4) = \mathbf{Z}_2\{\hat{\eta}_6\}$ and $\partial\hat{\eta}_6 = \Sigma^2\gamma_4 \circ \eta_6 = \beta\eta_5^2$ by Lemma 1.3. So i_* is an epimorphism. We have

$$\pi_9(\Sigma^2\mathrm{P}^5, \Sigma^2\mathrm{P}^4) = \mathbf{Z}_2\{\widehat{\eta_6^2}\} \oplus \mathbf{Z}_2\{[\omega, i]\}.$$

Since $\partial\widehat{\eta_6^2} = \beta\eta_5^3 = 4\beta\nu_5 = 2i'\tilde{\eta}_3\nu_5 = 0$ and $\partial[\omega, i] = [\beta\eta_5, i] + [2i'\delta, i] = [i, \beta]\eta_7$, we have the result of iii) apart from the last assertion.

In the exact sequence

$$\pi_9(\Sigma^2 \mathrm{P}^6, \Sigma^2 \mathrm{P}^5) \xrightarrow{\partial} \pi_8(\Sigma^2 \mathrm{P}^5) \xrightarrow{i_*} \pi_8(\Sigma^2 \mathrm{P}^6) \longrightarrow 0,$$

we have $\pi_9(\Sigma^2 \mathrm{P}^6, \Sigma^2 \mathrm{P}^5) = \mathbf{Z}_2\{\hat{\eta}_7\}$.

Since $\Sigma^2\gamma_5 \in \{i'', \Sigma^2\gamma_4, 2\iota_6\}$, we have, by Lemma 1.3,

$$\begin{aligned} \partial\hat{\eta}_7 = \Sigma^2\gamma_5 \circ \eta_7 &\in \{i'', \Sigma^2\gamma_4, 2\iota_6\} \circ \eta_7 \\ &= -i''\{\Sigma^2\gamma_4, 2\iota_6, \eta_6\} \\ &\subset -i''\{i\nu', 2\iota_6, \eta_6\} - i''\{\beta\eta_5, 2\iota_6, \eta_6\}. \end{aligned}$$

By Lemma 2.2, we have $\{i\nu', 2\iota_6, \eta_6\} \subset \{i, 2\nu', \eta_6\} \supset i'\{i_3, 2\iota_3, \nu'\eta_6\}$. So we have $i'\tilde{\eta}_3\nu_5 \in \{i\nu', 2\iota_6, \eta_6\}$ mod $i_*\pi_8(S^3) + \pi_7(\Sigma^2 \mathrm{P}^4) \circ \eta_7 = \{[i, \beta]\eta_7\}$ and $\{\beta\eta_5, 2\iota_6, \eta_6\} \ni 2\beta\nu_5$ mod $\pi_7(\Sigma^2 \mathrm{P}^4) \circ \eta_7 = \{[i, \beta]\eta_7\}$. Therefore we have $\partial\hat{\eta}_7 \equiv i''(i'\tilde{\eta}_3\nu_5 + 2\beta\nu_5) = 0$ mod $i''_*[i, \beta]\eta_7 = 0$. This leads us to iii).

In the exact sequence for $X = \Sigma^2 \mathrm{P}^6$:

$$\pi_7(X) \xleftarrow{2} \pi_7(X) \xleftarrow{i^*} [M^8, X] \xleftarrow{p^*} \pi_8(X) \xleftarrow{2} \pi_8(X),$$

we have $i''[i', \beta] \circ i = i''[i, \beta]$ by the proof of Lemma 2.2.i). We also have $2i''[i', \beta] = i''[i, \beta]\eta_7 p_8 = 0$ by ii) and iii). This completes the proof.

□

3. The suspension order of $\Sigma^2 \mathrm{P}^6$

Let $\varphi \in [M^5, M^4]$ be the attaching map in $\mathrm{P}_3^6 = M^4 \cup_\varphi CM^5$. We consider the cofiber sequence

$$M^4 \xrightarrow{i} \mathrm{P}_3^6 \xrightarrow{p'} M^6 \xrightarrow{\Sigma\varphi} M^5. \tag{3.1}$$

The squaring operation

$$\mathrm{Sq}^2 : \tilde{H}^i(\mathrm{P}_3^6; \mathbf{Z}_2) \to \tilde{H}^{i+2}(\mathrm{P}_3^6; \mathbf{Z}_2)$$

is nontrivial for $i = 3$ and trivial for $i = 4$. So, by Lemma 1.5.i), we can take $\varphi = i_3\bar{\eta}_3$.

We set $p'' = p_6 p' : \mathrm{P}^6_3 \longrightarrow S^6$. Since $[\mathrm{P}^6_3, M^6]$ is stable, we have $2p' = 2\iota_6 \circ p'$. So we have

$$2p' = i_5\eta_5 p'' \in [\mathrm{P}^6_3, M^6]. \tag{3.2}$$

Let $\bar{p} \in \{p, i\bar{\eta}_4, p'\} \subset [\Sigma\mathrm{P}^6_3, S^5]$ be an extension of $p = p_5 : M^5 \longrightarrow S^5$. Because $\iota_5 \in \{2\iota_5, p, i\}$, a bracket with indeterminancy $\{2\iota_5\}$, we have

$$2\bar{p} = \pm\bar{\eta}_5\Sigma p'. \tag{3.3}$$

We show

Lemma 3.1.

i) $[M^6, S^3] = \mathbf{Z}_2\{\eta_3\bar{\eta}_4\} \oplus \mathbf{Z}_2\{\nu'\Sigma p_6\}$;
$[M^7, S^3] = \mathbf{Z}_2\{\eta_3^2\bar{\eta}_5\} \oplus \mathbf{Z}_2\{\nu'\eta_6 p_7\}$;
$[M^8, S^3] = \mathbf{Z}_4\{\nu'\bar{\eta}_6\}$;
$[M^7, S^4] = \mathbf{Z}_2\{\eta_4\bar{\eta}_5\} \oplus \mathbf{Z}_2\{\nu_4 p_7\} \oplus \mathbf{Z}_2\{(\Sigma\nu')p_7\}$;
$[M^8, S^4] = \mathbf{Z}_2\{\eta_4^2\bar{\eta}_6\} \oplus \mathbf{Z}_2\{\nu_4\eta_7 p_8\} \oplus \mathbf{Z}_2\{(\Sigma\nu')\eta_7 p_8\}$.

ii) $[\Sigma^2\mathrm{P}^6_3, S^4] = \mathbf{Z}_2\{\eta_4^2\Sigma\bar{p}\} \oplus \mathbf{Z}_2\{\nu_4\eta_7\Sigma^2 p''\} \oplus \mathbf{Z}_2\{(\Sigma\nu')\eta_7\Sigma^2 p''\}$.

iii) $[\Sigma^2\mathrm{P}^6_3, S^3] = \mathbf{Z}_4\{\nu'\Sigma\bar{p}\}$.

Proof. i) is easily obtained. By use of the cofiber sequence (3.1), we have an exact sequence for $\gamma = \Sigma^2\varphi = i_5\bar{\eta}_5$:

$$[M^7, S^4] \xleftarrow{\gamma*} [M^6, S^4] \xleftarrow{i*} [\Sigma^2\mathrm{P}^6_3, S^4] \xleftarrow{\Sigma^2 p'*} [M^8, S^4] \xleftarrow{\Sigma\gamma*} [M^7, S^4].$$

By i), we have

$$\bar{\eta}_4\gamma = \eta_4\bar{\eta}_5,\ \eta_4\bar{\eta}_5\Sigma\gamma = \eta_4^2\bar{\eta}_6 \text{ and } \nu_4 p_7\Sigma\gamma = (\Sigma\nu')p_7\Sigma\gamma = 0.$$

This leads us to ii). By a parallel argument to ii) and by (3.3), we have iii). This completes the proof.

□

We set $\theta = i\eta_3\bar{\eta}_4 \in [M^6, \Sigma^2 P^4]$. By Lemma 1.2, we have

$$\theta \circ i\bar{\eta}_5 = i\eta_3^2\bar{\eta}_5 = 2i'\tilde{\eta}_3\bar{\eta}_5 = 4\beta\bar{\eta}_5 = 0 \in [M^7, \Sigma^2\mathrm{P}^4].$$

So we can define an extension $\bar{\theta} \in [\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^4]$ of θ. We have

$$\bar{\theta} \in \{\theta, i_5\bar{\eta}_5, \Sigma p'\} \subset [\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^4].$$

We show

Lemma 3.2.

i) $[\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^6]$ *is generated by* $i'''\bar{\theta}, i'\delta\Sigma\bar{p}, i''\beta\nu_5\Sigma^2 p''$, $i'[\delta, i_3]\Sigma^2 p''$ *and* $i''[i', \beta]\Sigma^2 p'$.

ii) $2\bar{\theta} \equiv 0 \bmod \{[i, \beta]\eta_7\Sigma^2 p'', 2i'\delta\Sigma\bar{p}\}$, $2i'''\bar{\theta} \equiv 0 \bmod 2i'\delta\Sigma\bar{p}$ *and* $\Sigma\bar{\theta}$ *is of order* 2.

Proof. By (3.1), we have an exact sequence

$$[M^7, \Sigma^2\mathrm{P}^6] \xleftarrow{\gamma^*} [M^6, \Sigma^2\mathrm{P}^6] \xleftarrow{i^*} [\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^6] \xleftarrow{\Sigma^2 p'^*} [M^8, \Sigma^2\mathrm{P}^6] \xleftarrow{\Sigma\gamma^*} [M^7, \Sigma^2\mathrm{P}^6].$$

By Lemma 2.1, there exist elements $\bar{\theta}$ with $\theta = i\eta_3\bar{\eta}_4$ and $i'\delta\Sigma\bar{p}$ in $[\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^4]$. So we settle the generators of $[\Sigma^2\mathrm{P}^6_3, \Sigma^2\mathrm{P}^6]$ by Lemma 2.5.iii). This leads us to i).

From the definition of $\bar{\theta}$ and by (3.2), we have

$$\begin{aligned} 2\bar{\theta} &\in \{i\eta_3\bar{\eta}_4, i_4\bar{\eta}_5, i_6\eta_6\Sigma p''\} \\ &\subset \{i\eta_3, \eta_4^2, \eta_6\Sigma p''\} \\ &\supset i'\{i_3\eta_3, \eta_4^2, \eta_6\Sigma p''\}. \end{aligned}$$

So we have

$$2\bar{\theta} \in i'\{i_3\eta_3, \eta_4^2, \eta_6\Sigma p''\} \bmod i\eta_3[\Sigma^2\mathrm{P}^6_3, S^4] + \pi_7(\Sigma^2\mathrm{P}^4) \circ \eta_7\Sigma^2 p''.$$

We also have

$$\{i_3\eta_3, \eta_4^2, \eta_6\Sigma p''\} \subset \{i_3, 2\nu', \eta_6\Sigma p''\} \supset \{i_3, 2\iota_3, \nu'\eta_6\Sigma p''\}.$$

So we have

$$\tilde{\eta}_3\nu_5\Sigma^2 p'' \in \{i_3\eta_3, \eta_4^2, \eta_6\Sigma p''\} \bmod i_{3*}[\Sigma^2 \mathrm{P}_3^6, S^3] + \pi_7(M^4)\circ\eta_7\Sigma^2 p''.$$

In $[\Sigma^2\mathrm{P}_3^6, \Sigma^2\mathrm{P}^4]$, we have $i'\tilde{\eta}_3\nu_5\Sigma^2 p'' = 2\beta\nu_5\Sigma^2 p'' = 0$. We have $\pi_7(M^4)\circ\eta_7\Sigma^2 p'' = \{\delta\eta_6^2\Sigma^2 p''\} = 0$, because $\delta\eta_6^2\Sigma^2 p'' = 2\delta\bar{\eta}_6\Sigma^2 p' = 4\delta\Sigma\bar{p} = 0$ by (3.2) and (3.3). By Lemma 3.1, we have $i_3[\Sigma^2\mathrm{P}_3^6, S^3] = \{i_3\nu'\Sigma\bar{p}\} = \{2\delta\Sigma\bar{p}\}$ and $i\eta_3[\Sigma^2\mathrm{P}_3^6, S^4] = 0$. By Lemmas 2.2.i) and 2.5.ii), we have $\pi_7(\Sigma^2\mathrm{P}^4)\circ\eta_7\Sigma^2 p'' = \{[i,\beta]\eta_7\Sigma^2 p''\}$. This completes the proof.

□

Remark. By use of the exact sequence induced from (3.1) and by Lemma 2.5.ii), we have

$$2\bar{\theta} \in \{[i',\beta], i'[\delta, i]p_8, \beta\eta_5\bar{\eta}_6 p_8, \beta\nu_5 p_8\}\circ\Sigma^2 p'.$$

By (3.3), we have $\beta\eta_5\bar{\eta}_6\Sigma^2 p'' = 0$. By Theorem 3.4 of [7], $\beta\nu_5\Sigma^2 p''$ survives in the stable range. So we have $2\Sigma\bar{\theta} = 0$.

Finally we prove Theorem 1. We consider the cofiber sequence

$$\mathrm{P}^2 \xrightarrow{i} \mathrm{P}^6 \xrightarrow{q} \mathrm{P}_3^6 \xrightarrow{\psi} M^3 \longrightarrow \cdots \tag{3.4}$$

and the exact sequence induced from (3.4):

$$[M^4, \Sigma^2\mathrm{P}^6] \xleftarrow{i^*} [\Sigma^2\mathrm{P}^6, \Sigma^2\mathrm{P}^6] \xleftarrow{\Sigma^2 q^*} [\Sigma^2\mathrm{P}_3^6, \Sigma^2\mathrm{P}^6] \xleftarrow{\Sigma^2\psi^*} [M^5, \Sigma^2\mathrm{P}^6].$$

By Lemmas 2.5 and 3.2, we have the relations for $X = \mathrm{P}^6$:

$$\begin{aligned} 4\Sigma^2\iota_X &\in \mathrm{Im}\,\Sigma^2 q^* \\ &= \{i'''\bar{\theta}, i'\delta\Sigma\bar{p}, i''\beta\nu_5\Sigma^2 p'', i'[\delta, i_3]\Sigma^2 p'', i''[i',\beta]\Sigma^2 p'\}\Sigma^2 q \end{aligned}$$

and

$$8\Sigma^2\iota_X \equiv 0 \bmod 2i'\delta\Sigma\bar{p}\Sigma^2 q.$$

So we have $8\Sigma^3\iota_X \equiv 0 \bmod 4i\nu_4\Sigma^2\bar{p}\Sigma^3 q = 0$. Hence we have

$$8\Sigma^3\iota_X = 0.$$

This completes the proof of Theorem 1.

The proof of Thorem 2 is immediately obtained from the method of the appendix of [8].

Conjecture. $i'''\bar{\theta}$ is of order 2 and the suspension order of ΣP^6 is 8.

Remark.

i) The author announced the result $\pi_{12}(M^5) \cong (\mathbf{Z}_2)^2$ ([10]). This result should be corrected as follows: $\pi_{12}(M^5) \cong (\mathbf{Z}_2)^3$([4]).

ii) The author obtained the result $\{2\iota_5, \eta_5, \nu_6\} = 0$ and $\{2\iota_5, \eta_5, \zeta_6\} = 0$ by Lemma 4.3 of [9]. By Lemma 2.3, we have $\tilde{\eta}_5\nu_7 = i\nu_5\eta_7^2$. So it should be corrected as follows: $\{2\iota_5, \eta_5, \nu_6\} \equiv \nu_5\eta_7^2 \bmod 0$ and $\{2\iota_5, \eta_5, \zeta_6\} \equiv \nu_5\eta_8\mu_9 \bmod 0$.

The author's mistake was induced from using the following wrong formula about the Toda bracket: $\{i, 2\iota_5, 0\}_2 \ni 0$. This formula should be corrected as follows:

$$\{i_5, 2\iota_5, 0\}_2 \subset \{i_5, 2\iota_5, 0\} \ni 0 \bmod i_5\nu_5\eta_7^2.$$

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