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TOPOLOGY AND EXTENSION OF SCHWARZSCHILD

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Abstract

Through the characterization of a spherically symmetric space-time as a local submanifold of a six dimensional pseudo-Euclidean spaces, with different signatures, we investigate the existence of a topological differences in the submanifold. In particular the Schwarzschild space-time and its Kruskal space-time extension are examined.

Resumo

Estudamos possíveis diferenças topológicas entre subvariedades esfericamente simétricas, imersas em espaços com mesma dimensão e diferentes assinaturas. Em particular, aplicamos aos espaços-tempo de Schwarzschild e Kruskal, ambos imersos em seis dimensões.

1. Introduction

The following problem is considered: *Determine topology of physical space outside of an approximately spherical body with mass M.* The physical space is modeled through a 4-dimensional space-time, solution of Einstein equations, whose geometry is described with good approximation by Schwarzschild's solution [3], representing the empty space-time with spherical symmetry outside of a body with spherical mass. Using spherical coordinates (t, r, θ, ϕ) this solution is given by

$$ds^2 = (1 - 2m/r)dt^2 - (1 - 2m/r)^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \qquad (1)$$

where $M = c^2 m G^{-1}$, c is the speed of light and G is the gravitational constant. Notice that in these coordinates regions $r = 0$ and $r = 2m$ are singular and need to be removed. When we remove the surface $r = 2m$, the manifold becomes separated in two disconnected components, one for $2m < r < \infty$ and the other for $0 < r < 2m$. Since we are dealing with the existence of the metric associated to a physical space, we require a connected space. Therefore, in analogy with [6] we define the following regions:

a)The exterior Schwarzschild space-time (V_4, g):

$V_4 = P_I^2 \times S^2$; $P_I^2 = \{(t,r) \in I\!R^2 |\, r > 2m\}$

b) The Schwarzschild black hole (B_4, g):

$B_4 = P_{II}^2 \times S^2$, $P_{II}^2 = \{(t,r) \in I\!R^2 |\, 0 < r < 2m\}$

In both cases, S^2 is the sphere of radius r and the metric g is given from (1). We know that (B_4, g) and (V_4, g) may be extendible for $r = 2m$. The extension of (V_4, g) was calculated by Kruskal [5] but it was suggested by C. Fronsdal one year before [2]. In the following section we use the isometric immersion formalism to establish the extension of $(E, g) = ([P_I^2 \cup P_{II}^2] \times S^2, g)$, denoted by $(E', g') = (Q^2 \times S^2, g')$, where Q^2 is the Kruskal plane [6]. Furthermore, we will prove that the topology of (E, g) is different from the topology of (E', g').

2. The Extension of Schwarzschild

Consider two known isometric immersions of space-time (E, g) into a pseudo Euclidean manifold of six dimensions, with different signatures:

- The Kasner immersion [4]

$$ds^2 = dY_1^2 + dY_2^2 - dY_3^2 - dY_4^2 - dY_5^2 - dY_6^2.$$

-The Fronsdal immersion [5]

$$ds^2 = dY'^2_1 - dY'^2_2 - dY'^2_3 - dY'^2_4 - dY'^2_5 - dY'^2_6,$$

Respectively given by (using $2m = 1$)

$$\begin{cases} Y_1 = (1-1/r)^{1/2}\cos t \\ Y_2 = (1-1/r)^{1/2}\sin t \\ Y_3 = f(r), \ \ (df/dr)^2 = \frac{1+4r^3}{4r^3(r-1)} \\ Y_4 = r\sin\theta\sin\phi \\ Y_5 = r\sin\theta\cos\phi \\ Y_6 = r\cos\theta \end{cases} \quad \text{and} \quad \begin{cases} Y_1' = 2(1-1/r)^{1/2}\sinh(t/2) \\ Y_2' = 2(1-1/r)^{1/2}\cosh(t/2) \\ Y_3' = g(r), \ \ (dg/dr)^2 = \frac{(r^2+r+1)}{r^3} \\ Y_4' = r\sin\theta\sin\phi \\ Y_5' = r\sin\theta\cos\phi \\ Y_6' = r\cos\theta \end{cases} \tag{2}$$

Notice that Y'_3 is defined for $r > 0$, while Y_3 is defined only for $r > 1$, suggesting the extension of (E, g). In order to determine the metric g' (extension of g), define the new coordinates u and v by:

- For $r > 2m$,

$$v = \frac{1}{4m}(\frac{r}{2m})^{1/2} exp(\frac{r}{4m}) Y'_1 \ \text{and} \ u = \frac{1}{4m}(\frac{r}{2m})^{1/2} exp(\frac{r}{4m}) Y'_2. \tag{3}$$

- For $0 < r < 2m$,

$$v = \frac{i}{4m}(\frac{r}{2m})^{1/2} exp(\frac{r}{4m}) Y'_1 \ \text{and} \ u = \frac{i}{4m}(\frac{r}{2m})^{1/2} exp(\frac{r}{4m}) Y'_2, \tag{4}$$

where

$$u^2 - v^2 = (\frac{r}{2m} - 1) exp(\frac{r}{2m}) \iff Y'^2_2 - Y'^2_1 = 16m^2(1 - \frac{2m}{r}). \tag{5}$$

Now $r = r(Y'_1, Y'_2)$ is implicitly defined by equation (5), while $t = t(Y'_1, Y'_2)$ is implicitly defined by

$$Y'_1/Y'_2 = tgh(\frac{t}{4m}). \tag{6}$$

Finally, the metric g' in the new coordinates is

$$ds^2 = (32m^3/r) exp(-r/2m)(dv^2 - du^2) - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{7}$$

which coincides exactly with the metric given by (1), when we replace u and v by the usual Schwarzschild coordinates. Curiously this metric is exactly the same metric encountered by Kruskal [5]. The u and v coordinates, Q^2 and all characteristics of Kruskal metric are given by $(E' = Q^2 \times S^2, g')$, without a singularity at $r = 2m$. We know that (E, g) is disconnected because it is composed by two connected components. When we calculated the extension (E', g') through the Fronsdal immersion we see that it is connected.

3. Schwarzschild's Topology

Let $(U_\alpha, \varphi_\alpha)$ be a coordinate system on a point $p \in M^n$. Then $\varphi_\alpha^{-1}(U_\alpha)$ is a coordinate neighborhood on p, where $U_\alpha \subset \mathbb{R}^n$. Generally speaking the topology of a manifold M^n is defined naturally through its open sets. If $A \subset M^n$, then A is an open set of M^n if $\varphi_\alpha(A \cap \varphi_\alpha^{-1}(U_\alpha))$ is an open set of $\mathbb{R}^n$, $\forall \alpha$. In other words, the atlas of M^n determines its topology [1].

The following theorem shows that the topology of (E, g) is different from that of (E', g').

Theorem 1. *The topology of a gravitational field outside of a body with spherical symmetry is given by* $\mathbb{R}^2 \times S^2$.

Proof: By construction, $E = [P_I^2 \cup P_{II}^2] \times S^2$ and $E' = Q^2 \times S^2$. The topology of E is the Cartesian product topology of $[P_I^2 \cup P_{II}^2]$ by S^2, while that the topology of E' is the Cartesian product topology of Q^2 by S^2. The topology of $S^2 \subset \mathbb{R}^3$ is the usual topology induced by the topological space $(\tau_3, \mathbb{R}^3)$. On the other hand, the topologies of $[P_I^2 \cup P_{II}^2] \subset \mathbb{R}^2$ and of $Q^2 \subset \mathbb{R}^2$, respectively τ_p and τ_q, will be induced from $(\tau_2, \mathbb{R}^2)$. Since Q^2 is an extension of $[P_I^2 \cup P_{II}^2]$, we may define one isometric embedding,

$$\psi : \; [P_I^2 \cup P_{II}^2] \longrightarrow Q^2.$$

Therefore, for an open set $A \subset \mathbb{R}^2$ given by

$$A = \{(t, r) \in \mathbb{R}^2 |\; t^2 + (r - 2m)^2 < m^2 \text{ and } r > 0\}.$$

we have $A \cap [P_I^2 \cup P_{II}^2] = A - \{(t, r) \in \mathbb{R}^2 |\; r = 2m\}$. This is an open set of the topological space $[P_I^2 \cup P_{II}^2]$, composed of two connected components. Observe that open sets form a topological basis for the semi-plane $t - r$, $r > 0$. However, we have that $\psi(A \cap [P_I^2 \cup P_{II}^2])$ is given for an open set composed by four connected components. As the lines L_1 and L_2 defined for $r = 2m$ from equation (5) are on Q^2 we have that

$$\{[\psi(A \cap [P_I^2 \cup P_{II}^2]) \cup L_1 \cup L_2]\} \cap D = B,$$

where D is an open disk on $I\!R^2$ with center in the origin of Q^2. The result B is a plane disk, in the new coordinates $r = r(Y'_1, Y'_2)$ and $t = t(Y'_1, Y'_2)$. In this manner the topology of Q^2 is given by open sets of $I\!R^2$. Finally we have that the topology of (E, g), $(I\!R^2 - \{(t, r) \in I\!R^2 |\ r = 2m\}) \times S^2$, is different from the topology $I\!R^2 \times S^2$ of (E', g').

$\square$

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