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CRITICAL SETS OF PROPER WHITNEY FUNCTIONS IN THE PLANE

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Abstract

A characterization of critical sets and their images for suitable proper Whitney functions in the plane is given. More precisely, necessary and sufficient conditions are provided for the existence of proper Whitney extensions F of $f : C \to I\!R^2$ such that C is the critical set of F. Although this result relies strongly on previous theorems of Blank and Troyer, it is presented in a self-contained fashion.

Resumo

Descreve-se uma caracterização do conjunto crítico e sua imagem para funções próprias de Whitney do plano no plano. Mais precisamente, dada uma função f de um conjunto C para oplano, são obtidas condições necessárias e suficientes para a existência de uma extensão própria F doplano no plano, tendo C como conjunto crítico. O resultado é apresentado de forma auto-contida, apesar de depender de teoremas de Blank e Troyer.

Introduction

The purpose of this paper is to provide a characterization of critical sets and their images for suitable proper Whitney functions from the plane to the plane. Knowledge of the critical set of such a function F is essentially sufficient for the understanding of the global behaviour of F and is most useful for the numerical inversion of F (i.e., for the computation of all the solutions of the equation $F(x) = y$) by continuation methods. In [MST2], numerical inversion and a geometric description of F are provided for generic proper Whitney functions

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with bounded critical set. Interest in this last problem arose from the study of Rankine-Hugoniot equations for hyperbolic conservation laws as considered in [MT]. Indeed, the interest in the problem considered in this paper derived from the approach used in [MST2]. The need to find the critical set of a function F naturally leads to the following purely topological question:

- In which conditions a given set C of smooth curves and a smooth function $f : C \to \mathbb{R}^2$ can be the critical set of a Whitney function $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F|_C = f$?

In other words,

- When is it possible to extend $f : C \to \mathbb{R}^2$ to a Whitney function $F : \mathbb{R}^2 \to \mathbb{R}^2$ whose critical set is C?

In this paper, a complete answer to this question is provided for C being a finite union of smooth curves, $f(C)$ having a finite number of cusps and intersection points, all of them being generic, and the additional requirement that the extension F be proper.

This problem can be reduced to two virtually independent issues: the existence of a local extension of f to a tubular neighbourhood of C and the existence of extensions to regular regions (the connected components of the complement of a neighbourhood of C) of proper immersions defined on their boundaries. In Section 1, the class of functions considered is introduced and the existence of the desired local extension is proved.

The problem of existence of extensions to a disk of immersions defined on the boundary was solved by Blank ([B], [P]). Troyer generalized Blank's result for disks with finitely many holes. A slightly modified version of Blank's theorem together with Troyer's result are presented in section 2. Proofs are included because the version of Blank's result presented in this paper admits a somewhat simpler proof and Troyer's work is not readily available; furthermore, Troyer's theorem is proved by induction, Blank's theorem being the initial step. The existential part of both theorems has an essentially constructive proof.

In section 3, with heavy use of Blank's methods, these results are extended to unbounded regions in the plane with finitely many boundary curves, assuming finitely many intersection points in the image of the boundary; it is now essential that immersions be proper. Finally, in section 4, the main theorem is stated and proved. Formulae for the topological degree and number of pre-images of a regular point are also given. As with Blank's theorem, the existential part of our results is essentially constructive.

1. Preliminaries and local problems

Let C be the finite union of disjoint curves in $\mathbb{R}^2$ and let $f : C \to \mathbb{R}^2$ be a smooth proper function. Since C is to be the critical set of a Whitney function, curves in C are assumed to be images of smooth proper embeddings of S^1 or $\mathbb{R}$ in $\mathbb{R}^2$.

As mentioned in the introduction, to prove the existence of F, we first extend f to a proper Whitney function $\tilde{f}$, defined in a thin closed tubular neighbourhood $\overline{U}$ of C whose critical set is C. We then extend $\tilde{f}|_{\partial U}$ to obtain a function $\tilde{F}$ which is a topological immersion outside C and is a Whitney function outside ∂U. To get the desired Whitney function F it is enough to regularize $\tilde{F}$ at ∂U, a standard procedure which we shall not discuss in this paper.

The existence of F is thus equivalent to:

(a) the existence of an extension $\tilde{f}$ of f as above,

(b) the existence of an immersion of a region in the plane with a prescribed behaviour at its boundary.

Item (a) was studied by Francis and Troyer in [FT] but, for completeness, we also present it in this section (Proposition 1.1). Item (b) is the subject of Sections 2 and 3.

Recall that a Whitney function is a smooth function whose critical points are all *folds* or *cusps*. A fold is a critical point such that, after smooth local changes of coordinates in the domain and image, the function is of the form

$$F(x,y) = (x, y^2),$$

the critical point being taken to the origin. For a cusp, after a change of coordinates, the function is of the form

$$F(x,y) = (x, \pm y^3 - xy),$$

where again the critical point is taken to the origin. If changes of coordinates preserve orientation, the sign in the formula above is well defined.

A critical curve for a Whitney function F admits a natural orientation, which will be called the *sense of folding*, leaving nearby points with positive $\det DF$ to the left of the oriented curve. If we take a smooth curve following parallel to the critical curve, slightly to the left (resp., right) the image of the curve will form loops precisely at the cusps for which the sign in the above normal form is $+$ (resp., $-$). We therefore say the cusp is *effective to the left* if the said sign is $+$; otherwise, the cusp is said to be *effective to the right*.

If we parametrize a critical curve γ of a Whitney function by a smooth regular function g, then $\alpha = F \circ g$ has critical points precisely at the cusps and at all cusps in γ the sign of $\alpha'' \wedge \alpha'''$ is the same. More generally, for a parametrized curve β, the generic critical point (in which $\beta' = 0$) satisfies $\beta'' \wedge \beta''' \neq 0$ and its image is called a *cusp of the curve*. Thus, in our context, images of cusps of a Whitney function F are cusps of the curves in $F(C)$. If g follows the sense of folding, $\alpha'' \wedge \alpha''' < 0$ and all cusps of $F(C)$ point to the right of the curve $F(\gamma)$ with the induced orientation.

We now characterize critical curves for Whitney functions with orientation given by the sense of folding, assuming a finite number of cusps. Let γ be an oriented proper curve embedded in $\mathbb{R}^2$ (i.e., the image of a proper embedding of S^1 or $\mathbb{R}$). A smooth proper function $f : \gamma \to \mathbb{R}^2$ is called *extension-compatible* if there exists only a finite number of points on which the derivative of f is zero and at each such point we have $\alpha'' \wedge \alpha''' < 0$ where α is the composition with f of an orientation-preserving regular parametrization of the curve. In other words, the critical values of the image of γ under f are all cusps of $f(\gamma)$ and point to the right of $f(\gamma)$.

Proposition 1.1 ([FT]): *Let γ be an embedded smooth proper oriented curve in $\mathbb{R}^2$. If $f : \gamma \to \mathbb{R}^2$ is an extension-compatible function then there exist an open neighbourhood U of γ and a smooth Whitney function $\tilde{f} : U \to \mathbb{R}^2$ extending f such that γ is the critical set of $\tilde{f}$ and the sense of folding corresponds to the prescribed orientation on γ. Furthermore, $\tilde{f}$ can be chosen so that cusps are effective to whichever side we prescribe.*

Proof. We first prove a local version of the theorem, i.e., that given a point p_0 in γ there exists a neighbourhood U_0 of p_0 and a Whitney function $\tilde{f}_0 : U_0 \to \mathbb{R}^2$ such that $\tilde{f}_0|_{\gamma\cap U_0} = f|_{\gamma\cap U_0}$. Furthermore, if p_0 is a cusp, $\tilde{f}_0$ can be taken so that the cusp is effective to the left or right, as we choose.

Assume first that p_0 is a fold point, i.e., p_0 is not a critical point of f. There exists an orientation preserving diffeomorphism $\phi : U_0 \to U_0'$ taking p_0 to the origin and $\gamma \cap U_0$ to the x-axis. Similarly, for V_0 a sufficiently small neighbourhood of $f(p_0)$, there exists an orientation preserving diffeomorphism $\psi : V_0 \to V_0'$ such that, after reducing U_0 and U_0', $(\psi \circ f \circ \phi^{-1})(x, 0) = (x, 0)$. Let $G_1(x, y) = (x, y^2)$; we can take $\tilde{f}_0 = \psi^{-1} \circ G_1 \circ \phi$, which clearly satisfies the required conditions.

If p_0 is a cusp, ϕ is taken as above and, for a sufficiently small V_0, there is an orientation preserving diffeomorphism $\psi_1 : V_0 \to V_0'$ such that the function $\xi(t) = (h_1(t), h_2(t)) = (\psi_1 \circ f \circ \phi^{-1})(t, 0)$ satisfies $\xi(0) = (0, 0)$, $\xi''(0) = (6, 0)$ and $\xi'''(0) = (0, -12)$ (notice that $\xi'(0)$ is automatically 0): this is possible because we require $\alpha'' \wedge \alpha''' < 0$ and ξ must satisfy the same property (ψ_1 can be taken as an orientation preserving affine transformation). Let

$$\tilde{f}_1(u, v) = (h_1(u) \pm v, h_2(u) \pm \frac{h_2'(u)}{h_1'(u)} v),$$

signs varying together. It is clear that $\tilde{f}_1 : U_0' \to V_0'$ coincides with $\psi_1 \circ f \circ \phi^{-1}$ on the x-axis and it is now easy to check that the origin is a cusp for $\tilde{f}_1$ ([W]), that the sense of folding corresponds to the usual orientation for the x-axis and that the cusp is effective to the right if we take the $+$ sign and to the left if we take the $-$ sign. Finally, take $\tilde{f} = \psi_1^{-1} \circ \tilde{f}_1 \circ \phi$.

Now, using a partition of unity, glue functions $\tilde{f}$ for a covering of γ. Since they all coincide in γ, $\tilde{f}$ thus defined is indeed an extension of f. It is furthermore easy to verify that such partitions of unity preserve folds (cusps are never touched).

□

In this paper, we are interested in considering only certain very well behaved Whitney functions, which we call *nice*: smooth Whitney functions $F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the properties below.

- The function F is proper.
- The critical set C is the union of finitely many curves (necessarily disjoint).
- There are only finitely many cusps.
- There are finitely many intersection points in $F(C)$, i.e., points in $F(C)$ with more than one pre-image in C.
- All intersection points are double, transversal and are not cusps, that is, all intersection points have precisely two pre-images in C, both of them being fold points, so that tangent vectors to C at these two points are taken by DF to linearly independent vectors.

Thus, the precise question we answer in this paper is: when is it possible to extend $f : C \to \mathbb{R}^2$ to a nice function $F : \mathbb{R}^2 \to \mathbb{R}^2$ whose critical set is C?

Recall that a critical curve of a nice function has a natural orientation, the sense of folding. Thus, given a nice function F, with the curves in the critical set C of F oriented by sense of folding, the definition of a nice function imposes the following conditions on C and $f = F|_C$:

(a) The set C is the disjoint union of a finite number of embedded proper curves.

(b) Each connected component of $\mathbb{R}^2 - C$ admits an orientation such that the induced orientation on its boundary coincides with the sense of folding.

(c) The function f restricted to any of the curves in C is extension-compatible.

(d) There exist only finitely many intersection points (i.e., points in $f(C)$ with more than one pre-image), all of them being double (i.e., with two pre-images) and transversal (tangent vectors to C at the two pre-images are taken by f' to linearly independent vectors).

These properties then lead us to consider only sets C of oriented curves and smooth proper functions $f : C \to \mathbb{R}^2$, which we call *adequate*, satisfying conditions (a) to (d), above. In particular, a proper immersion $f : C \to \mathbb{R}^2$ satisfying (a), (b) and (d) (hold automatically) will be called an *adequate immersion*. So, except for the requirement that the number of intersections be finite (which is automatically satisfied if C is compact), adequate immersions correspond to what is known as normal immersions. Notice that curves in C have to be appropriately oriented.

Proposition 1.1 tells us that it is always possible to extend an adequate f to a Whitney function $\tilde{f}$ defined on a neighbourhood U of C such that the critical set of $\tilde{f}$ is C; we may even choose to which side cusps are effective. Notice that U can be taken to be a *tubular* neighbourhood of C, with one connected component per curve and boundary made of smooth curves; furthermore, given $\tilde{f}$, U can be made smaller if necessary so that $\tilde{f}|_{\partial U}$ is an adequate immersion.

2. The Theorems of Blank and Troyer

In this section, we present the results of Blank and Troyer on the existence of extensions of boundary immersions for bounded regions in the plane. First of all we recall some properties of the turning of locally simple closed curves which will be necessary to state and prove the theorems.

Let $g : S^1 \to I\!R^2$ be a locally injective continuous function and $\delta > 0$ be such that g is injective in any interval of size δ. The *turning* of g, $\tau(g)$, is defined as the degree of the function

$$\theta \mapsto \frac{g(\theta+\delta) - g(\theta)}{|g(\theta+\delta) - g(\theta)|}$$

from S^1 to S^1. Notice that $\tau(g)$ does not depend on the choice of δ. In particular, if g is regular, $\tau(g)$ is the degree of $\theta \mapsto g'(\theta)/|g'(\theta)|$. Clearly, if ρ is a homeomorphism of S^1, $\tau(g \circ \rho) = \pm\tau(g)$, the sign depending on whether ρ preserves orientation or not. If γ is an oriented simple closed curve and $f : \gamma \to I\!R^2$ is continuous and locally injective we define $\tau(f) = \tau(f \circ g)$ where g is any orientation preserving simple parametrization of γ by S^1; the previous property guarantees that τ is independent from g. We will make use of the following proposition, whose proof, being standard, is omitted.

Proposition 2.0. *Let A be a disk with k holes, with exterior and interior boundaries given by the smooth curves γ_0 and $\gamma_1, \ldots, \gamma_k$, respectively. If $\gamma_0, \gamma_1, \ldots, \gamma_k$ are oriented counterclockwise and $\phi : A \to I\!R^2$ is an immersion then*

$$\tau(\phi|_{\gamma_0}) = \operatorname{sgn}(\det(D\phi))(1-k) + \sum_{1 \le i \le k} \tau(\phi|_{\gamma_i}).$$

As a corollary, given an orientation for A, if the curves $\gamma_0, \gamma_1, \ldots, \gamma_k$ are oriented as the boundary of A and $\phi : A \to I\!R^2$ is an immersion then

$$\sum_{0 \le i \le k} \tau(\phi|_{\gamma_i}) = \operatorname{sgn}(\det(D\phi)) \operatorname{sgn}(A) \chi(A),$$

where $\operatorname{sgn}(A) = \pm 1$ indicates the orientation of A and $\chi(A)$ is the Euler characteristic of A.

Let C be the union of finitely many disjoint oriented simple closed curves and let $f : C \to I\!R^2$ be an adequate function. Define $\tau(f) = \sum_{\gamma \subseteq C} \tau(\gamma)$. The cycles of the *Seifert decomposition* of $f(C)$ are the simple closed oriented curves obtained by following the curves in $f(C)$, respecting orientation and turning at every intersection. It is also well known that $\tau(f) = m - n$ where m and n are the number of positively and negatively oriented cycles, respectively.

Let γ be a positively oriented simple closed curve in the plane and let $f : \gamma \to \mathbf{r}^2$ be a adequate immersion. Consider in the image $f(\gamma)$ the orientation induced by f from the orientation of γ. Blank's theorem gives us a necessary and sufficient condition for the existence of an orientation preserving immersion F, from the disk bounded by γ to the plane, extending f. We begin with a simple example. Consider $f(\gamma)$ as in Figure 2.1(a). How can we decide whether an orientation preserving immersion F exists, extending f to the disk bounded by γ? Such an F exists, and can be constructed by juxtaposing three homeomorphisms taking three sub-disks in the domain to three disks in the image as indicated in Figure 2.1(b), where the homeomorphisms coincide in the common part of the boundary.

Figure 2.1

Blank's theorem describes a construction of F by decomposing the domain into disks inside which F is a homeomorphism. For a closed subset X of $\mathbb{R}^2$, we call the connected components of $\mathbb{R}^2 - X$ the *tiles* for X. The following definitions are exemplified in Figure 2.1(c). A *ray* for f is a proper embedding $r : [0, +\infty) \to \mathbb{R}^2$ which is transversal to $f(\gamma)$ and never goes through an intersection point in $f(\gamma)$. The images of rays, oriented by the given parametrization, are also called rays. A *system of rays* for f is a finite family of disjoint rays with the following properties:

- the origin of each ray is in some bounded tile for $f(\gamma)$,
- each bounded tile for $f(\gamma)$ contains the origin of a unique ray.

Since $f(\gamma)$ and the rays are oriented, their intersections naturally have an orientation: when the curve crosses the ray from right to left, we call the orientation *positive*, otherwise *negative*. Each intersection also has a *height* associated to it: it is the number of other intersections on the same ray which are closer to the origin of the ray. Therefore, the first intersection of a ray with a curve always has height zero. The *Blank word* is obtained following $f(\gamma)$ once, respecting the orientation, and writing down, at each intersection, a letter corresponding to the ray, the sign of the intersection (as an exponent) and its height (as an lower index). Blank words are defined only up to cyclic permutation: any intersection can be taken as the beginning of the word. In the example of Figure 2.1, the Blank word is

$$Bw = a_0^+ b_2^+ c_1^+ d_1^+ e_0^+ f_1^+ b_1^+ c_0^+ f_0^+ g_0^+ b_0^- d_0^- e_1^+ f_2^+ g_1^+ .$$

Actually, Blank's original theorem does not involve the concept of height in its statement, and there are indeed generalizations of Blank's and Troyer's theorems which do not make use of the concept of heights ([CW]). Heights are, however, essential for Troyer's theorem and the generalizations we have in mind and the modified version of Blank's theorem with which we work also involves heights in its statement. We prefer to work with heights from the beginning for three reasons. First, heights have to be introduced at some point anyway. Second, Blank's theorem admits a simpler proof if heights are taken into account. Finally, and most importantly, if Blank's theorem (or any of its generalizations) is interpreted as providing a combinatorial criterion for the existence of extensions, the criterion becomes computationally simpler if heights are used.

A Blank word admits a *simplification* if there existis a pair of letters $\mathbf{z}^+$, $\mathbf{z}^-$, such that, after a cyclic permutation if necessary, there are no letters with negative exponent between $\mathbf{z}^+$ and $\mathbf{z}^-$. In this case, the simplification is obtained by deleting from the word $\mathbf{z}^+$, $\mathbf{z}^-$ and the letters between them; we say that $\mathbf{z}^-$ was *cancelled* with $\mathbf{z}^+$. A simplification is *positive* if the height of $\mathbf{z}^-$ is smaller than that of $\mathbf{z}^+$. A Blank word *groups* (or *admits a grouping*) if we can sequentially simplify it until we get to a word with no negative exponents. A grouping is *positive* when all simplifications are positive. In the previous example, a positive grouping is given by

$$Bw \to a_0^+ b_2^+ c_1^+ d_1^+ e_0^+ f_1^+ d_0^- e_1^+ f_2^+ g_1^+ \to a_0^+ b_2^+ c_1^+ e_1^+ f_2^+ g_1^+ .$$

We indicate simplifications connecting two equal letters with opposite signs by brackets, as in the example below:

$$Bw = a_0^+ b_2^+ c_1^+ \overbrace{d_1^+ e_0^+ f_1^+ \overbrace{b_1^+ c_0^+ f_0^+ g_0^+ b_0^-} d_0^-} e_1^+ f_2^+ g_1^+ .$$

Notice that, in a grouping, brackets are not allowed to intersect. Since the word is cyclic, for each simplification we can choose between the two brackets which cover complementary parts of the word: both are equivalent with respect to the possibility of grouping. Simplifications describe the previously mentioned decompositions of the domain into sub-disks, as we shall see. We introduce the notation D_α for the closed disk bounded by the simple closed curve α.

Theorem 2.1. *Let γ be a regular, positively oriented, simple closed curve, $f : \gamma \to \mathbf{r}^2$ an adequate immersion and consider a system of rays for f. Then there exists an orientation preserving immersion F extending f to D_γ if and only if*

(a) *the turning of f is 1,*

(b) *the Blank word groups positively.*

As mentioned, this is a modified version of Blank's Theorem. In the original one ([B], [P]), the condition in item (b) is that the Blank word groups. In fact, Blank proves that a word groups if and only if it groups positively.

To prove Theorem 2.1, we need a few technical lemmas.

Lemma 2.2. *Let ζ be a closed oriented curve which is the image of an adequate function. If ζ is not simple, then it has more than one Seifert cycle.*

Proof. Consider the boundary α of the unbounded connected component of the complement of ζ: if α has a consistent orientation, it forms a Seifert cycle which can only be the whole curve if ζ is simple. On the other hand, segments of α with opposite orientations belong to different Seifert cycles: the orientation of a segment of α determines the orientation of its Seifert cycle. Indeed, the orientation of this arc tells us on which side lies the unbounded connected component of the complement of the cycle.

□

Lemma 2.3. *Consider γ, f and a system of rays as in the theorem. If the Blank word has no negative exponents, the turning of f is strictly positive. Moreover, if the turning is equal to 1, $f(\gamma)$ is a simple closed curve.*

Proof. Consider the Seifert decomposition of $f(\gamma)$. If there exists a cycle with negative orientation, there exists a ray beginning inside the cycle which, when leaving the cycle, creates a negative intersection. There are, therefore, no cycles with negative orientation. The result now follows from the previous lemma.

□

Lemma 2.4. *Let γ, f and a system of rays be as in the theorem. If the Blank word groups positively then the turning of $f(\gamma)$ is strictly positive.*

Proof. Define the *depth* of an intersection as the number of other intersections on the same ray closer to infinity — thus, depth plus height equals the number of intersections on the ray minus one. To each f and system of rays for f, assign the polynomial in the variable ω with natural coefficients $\sum a_n \omega^n$, where a_n is the number of negative intersections with depth n. Define on this set of polynomials the following order: $P < Q$ if and only if the coefficient of highest degree of $Q - P$ is positive. It is easy to see that this set is well ordered: each non-empty subset contains a smallest polynomial (the reader may have noticed that these polynomials correspond to ordinals below ω^ω). Suppose by contradiction that there are counter-examples to the lemma, and consider one such with minimal associated polynomial.

There exists at least one simplification, since, by Lemma 2.3, there are letters with negative exponents. Consider therefore the first simplification (positive, by hypothesis), associating $f(p_-)$ and $f(p_+)$, intersections of the curve $f(\gamma)$ with a ray r. Connect points p_- and p_+ of γ by a simple, regular curve contained entirely in D_γ. Define new curves γ_1 and γ_2 bounding disks D_{γ_1} and D_{γ_2} as indicated in Figure 2.2. Let $f_1 : \gamma_1 \to \mathbf{r}^2$ and $f_2 : \gamma_2 \to \mathbf{r}^2$ be such that, on γ, they coincide with f, and on the new segment, they follow the ray. By summability, $\tau(f) = \tau(f_1) + \tau(f_2) - 1$.

Figure 2.2

The original problem then gives rise to two analogous problems on two curves. In order to deal with the problem involving f_2 and γ_2, we now describe a system of rays for f_2. Take the original rays, moving slightly to the left the ray over which the simplification was performed (follow the argument in Figure

2.3(a)): the two reasons for which the set of rays may not suit the new problem are the possible existence of two or more rays starting from the same tile for $f_2(\gamma_2)$ and the existence of tiles for $f_2(\gamma_2)$ from which no ray departs. The first difficulty is solved simply by omitting a few rays. The only tiles for which the second difficulty may apply are those immediately to the right of the segment of the original ray going from $f(p_-)$ to $f(p_+)$. In any case, we can construct rays for these tiles crossing the original ray and following it to infinity. Thus the word for f_2, γ_2 and the system of rays described above has positive intersections only: indeed, being a (first) simplification, there are no negative intersections in the segment of the original curve between $f(p_-)$ and $f(p_+)$. From the previous lemma, $\tau(f_2) \geq 1$.

As to f_1 and γ_1, take the rays of the original problem, moving slightly to the right the ray over which the simplification was performed (Figure 2.3(b)). Omit and add rays exactly as above. We claim that the word associated to this system of rays also groups positively. Indeed, for each negative intersection over an old ray, the simplification that cancelled this intersection in the word for f still takes care of it. For the new rays, mimic groupings along the ray they follow.

Figure 2.3

cancelled this intersection in the word for f still takes care of it. For the new rays, mimic groupings along the ray the follow.

Notice that the polynomial (ordinal) associated to f_1 is strictly less than that associated to f. In fact, going from f to f_1, we eliminate a negative intersection, and any new negative intersection possibly arising must have strictly smaller depth. Notice also that possible eliminations of rays only help and the depth of a given negative intersection may go down, but never up.

Since by hypothesis f is a minimal counter-example, $\tau(f_1) \geq 1$. Thus, we know that $\tau(f_1) \geq 1$ and $\tau(f_2) \geq 1$, and hence $\tau(f) = \tau(f_1) + \tau(f_2) - 1 \geq 1$, contradicting the hypothesis of f being a counter-example.

□

Proof of Theorem 2.1. We initially show necessity. Since F is an orientation preserving immersion, Proposition 2.0 with $k = 0$ tells us that $\tau(f) = 1$. For each intersection of $f(\gamma)$ with a system of rays we shall indicate its unique pre-image (by f) by the corresponding letter in the Blank word. The inverse images of rays by F (as for the example in Figure 2.4) satisfy the following properties:

- each connected component is a regular curve with boundary,
- these curves are disjoint and have a natural orientation induced by the orientation of rays,
- curves always start either at a negative intersection or at a pre-image of the origin of the ray,
- curves always end at a positive letter,
- every negative letter is the starting point of such a curve,
- curves which begin and end on γ associate a negative letter with a positive one of greater height.

Figure 2.4

The positive grouping is defined by matching letters indicating points connected by curves which begin and end on γ.

To prove the other implication, associate to each situation the same polynomial (ordinal) as in the proof of Lemma 2.4 and assume the existence of a minimal counter-example.

Again, by Lemma 2.3, there is at least one simplification. Given a simplification involving p_- and p_+, perform the construction in the proof of Lemma 2.4 in order to obtain f_1, f_2 and systems of rays as in the lemma. Since $\tau(f) = 1$, $\tau(f_1) \geq 1$, $\tau(f_2) \geq 1$ and $\tau(f) = \tau(f_1) + \tau(f_2) - 1$ we have $\tau(f_1) = \tau(f_2) = 1$. So, by Lemma 2.3, $f_2(\gamma_2)$ is the boundary of a disk since the corresponding Blank word has no negative exponent: f_2 therefore extends to an immersion F_2 on D_2. By the minimality hypothesis, since the polynomial (ordinal) for f_1 is smaller than that for f, f_1 also extends to an immersion F_1 on D_1. The juxtaposition of these immersions gives us a local homeomorphism $\check{F}$ extending f to D.

It is now possible by classical methods to render $\check{F}$ smooth by changing it in a neighborhood of the segment connecting p_- and p_+ in such a way as to obtain an immersion F as desired, which contradicts the hypothesis of f being a counter-example.

□

Although our proof is somewhat indirect, it shows us how to construct an immersion with the required properties. Indeed, the rays on which we perform groupings indicate how to cut domain and image into disks inside which the immersion clearly exists. The proof is necessary only to guarantee that this process works, i.e., that the curves so constructed are indeed boundaries of disks. Alternatively, the reader may think of the proof not as by contradiction but as by (transfinite) induction. In this case, we have a recursive procedure to construct the desired immersions.

We now prove the generalization of this result to a disk with k holes, due to Troyer ([T]). Let A be a disk with k holes in the plane, with positively oriented exterior boundary γ_0 and negatively oriented interior boundaries $\gamma_1, \ldots, \gamma_k$. Let $f : \partial A \to \mathbf{r}^2$ be an adequate immersion. Troyer's theorem provides a criterion for the existence of an orientation preserving immersion $F : A \to \mathbf{r}^2$ extending f, also involving rays and Blank words.

Systems of rays are defined exactly as before, for tiles for $f(\partial A)$. Consider in the curve $f(\gamma_i)$ the orientation induced by f and the orientation of γ_i. Intersections with rays are classified as positive or negative as before. Therefore, following $f(\gamma_i)$, we have a Blank word for each γ_i. A *concatenation* of two Blank words from a pair of intersections in the same ray $\mathbf{z}^-$ e $\mathbf{z}^+$, one in each word, is obtained by cyclically permuting the two words so as to leave $\mathbf{z}^-$ at the right extreme and $\mathbf{z}^+$ at the left extreme of their respective words, juxtaposing both words and eliminating the pair $\mathbf{z}^-\mathbf{z}^+$ produced at the juxtaposition. A concatenation is *positive* when the height of $\mathbf{z}^-$ is smaller than that of $\mathbf{z}^+$. A family of Blank words *groups positively* if there exist positive concatenations giving rise to a single word which in turn groups positively.

Theorem 2.5 (Troyer [T]): *Let A be a (positively oriented) disk with k holes and $f : \partial A \to \mathbb{R}^2$ be an adequate immersion. Consider a system of rays and the associated Blank words. Then there exists an orientation preserving immersion*

$F : A \to \mathbf{r}^2$ which extends the f_i to A if and only if

(a) *the turning of f is $1 - k$,*

(b) *the Blank words group positively.*

Notice that Troyer's theorem for $k = 0$ is Theorem 2.1. We first prove a technical lemma.

Lemma 2.6. *Let A be a (positively oriented) disk with k holes and $g : \partial A \to \mathbf{r}^2$ be an adequate immersion. Consider a system of rays and the associated Blank words. If the Blank words group positively, then $\tau(g) \geq 1 - k$.*

Figure 2.5

Proof. We proceed by induction on k: the case $k = 0$ is Lemma 2.4. Let $k \neq 0$. Assume the first concatenation to be made from $g(p_-)$ to $g(p_+)$ — clearly, p_- and p_+ are in different boundary components of A. Let $\tilde{A}$ be obtained from A by removing an open tubular neighbourhood of a simple arc contained in A joining p_- and p_+ as in Figure 2.5; $\tilde{A}$ is thus a disk with $k - 1$ holes. We construct an adequate immersion $\tilde{g} : \partial\tilde{A} \to I\!R^2$ and a system of rays for $\tilde{g}$ such that the associated Blank words group positively. Let $\tilde{g}$ be equal to g on $\partial A \cap \partial\tilde{A}$ and on the two remaining arcs define $\tilde{g}$ by closely following the ray along which concatenation was performed, again as in Figure 2.5. As in Lemma 2.4, we construct a system of rays for $\tilde{g}$ from the given system of rays for g by omitting superfluous rays and adding, when necessary, rays following the concatenation ray. The result now follows since, by construction, $\tau(\tilde{g}) = \tau(g) + 1$.

□

Proof of Theorem 2.5. First, assume the existence of F. Item (a) is the corollary to Proposition 2.0. As to item (b), take the inverse image by F of the rays. These pre-images satisfy all the properties listed in the proof of Blank's Theorem. Pre-images of rays going from the boundary to the boundary indicate concatenations or groupings (see Figure 2.6); of course, concatenations always come from pre-images joining distinct boundary components. It now suffices to see that there exist pre-images of rays connecting the various boundary curves in such a way that we can select k connected pre-images of rays to be the k necessary concatenations. Thus, it remains only to prove that the union of the boundaries with the pre-images of the rays is a connected set. If this set were disconnected, we would have $\ell \geq 2$ disjoint sets of (boundary) curves with $1 + k_j$ curves, $1 \leq j \leq \ell$, for which the corresponding sets of Blank words group positively, independently from one another, since a concatenation and grouping is obtained by associating intersections connected by pre-images of rays which begin and end in ∂A. By the previous lemma, the turning of F restricted to each such set of curves is greater than or equal to $1 - k_j$; notice that the lemma can be applied to such sets of curves even when they do not include the exterior boundary, as the position of curves in the domain is irrelevant. The turning of F restricted to ∂A would therefore be greater than or equal to $2\ell - 1 - k$ (remember that $\sum(1 + k_j) = 1 + k$), contradicting item (a), according to which this turning is $1 - k$.

Figure 2.6

We shall now prove the converse by induction on k, the case $k = 0$ being given by Theorem 2.1. By performing the same construction as in Lemma 2.6, we obtain a disk $\tilde{A}$ with $k-1$ holes and an adequate immersion $\tilde{f} : \partial\tilde{A} \to I\!\!R^2$ which coincides with f on the common parts of the domains. By induction, $\tilde{f}$ can be extended to an immersion $\tilde{F} : \tilde{A} \to I\!\!R^2$. Clearly, $\tilde{F}$ can be extended to an immersion $F : A \to I\!\!R^2$ (extending f) by taking the tubular neighbourhood in the domain to the tubular neighbourhood in the image (see Figure 2.5).

□

The above theorems deal with orientation preserving extensions but we shall also need an orientation reversing version of Troyer's Theorem. In order to simplify our exposition, we present both versions in a single theorem.

Let A be an oriented disk with k holes and give the induced orientation to its boundary, that is, A is to the left of each boundary curve. Given an adequate immersion $f : \partial A \to I\!\!R^2$ define a system of rays and Blank words exactly as for Troyer's theorem taking into account the induced orientations for $f(\partial A)$. We are interested in immersions F extending f such that $\det(DF) > 0$ iff A is positively oriented.

Theorem 2.7. *Let A be an oriented disk with k holes and $f : \partial A \to I\!\!R^2$ be* *an adequate immersion. Given a system of rays, consider the associated Blank words. Then there exists an immersion* $F : A \to I\!R^2$ *extending* f*, with the sign of* $\det(DF)$ *agreeing with the orientation of* A*, if and only if*

(a) *the turning of* f *is* $1 - k$*,*

(b) *the Blank words group positively.*

Proof. If A is positively oriented, this is exactly Troyer's theorem.

If A is negatively oriented, change the orientation of the domain by composing with $R(x, y) = (x, -y)$. Let $\overline{A} = R(A)$ and $\overline{f} = f \circ R$. In one direction, apply Troyer's theorem to $\overline{f} : \partial\overline{A} \to I\!R^2$ in order to get an extension $\overline{F}$ of $\overline{f}$ and let $F = \overline{F} \circ R$. For the other direction, let $\overline{F} = F \circ R$ and again apply Troyer's theorem. In both cases turnings and Blank words are unaltered.

□

3. Boundary immersions for unbounded regions

In this section we give the versions of Theorem 2.7 for unbounded regions in the plane.

Let A be an open connected oriented subset of the plane with boundary given by a finite number of smooth curves, which are either smooth embeddings of S^1 or smooth proper embeddings of R. We divide such regions A into three types: A is of type I if bounded, of type II if unbounded with bounded boundary, and of type III if unbounded with unbounded boundary. Regions of type I have been already considered in Theorem 2.7. We shall need versions of this theorem for regions of type II and III.

Let $\sigma_A = +1$ if A is positively oriented and $\sigma_A = -1$ otherwise. The bounded boundary components of A will be called $\gamma_i, i = 1, \ldots, k$ and the unbounded ones, $\beta_j, j = 1, \ldots, \ell$. The boundary curves are oriented compatibly with A: A is to their left iff A is positively oriented. In this section, we assume that $f : \partial A \to I\!R^2$ is an adequate immersion. We are concerned with the possible existence of an immersion F from A to $\mathbf{r}^2$ extending f such that the induced orientation in the image is positive. In other words, we are looking for an immersion F with $\operatorname{sgn}(\det(DF)) = \sigma_A$.

In our proofs, we often will make use of auxiliary closed curves. We define an *enveloping curve in the domain* to be a smooth simple oriented closed curve δ with the following properties:

- δ is oriented counterclockwise iff $\sigma_A > 0$.
- δ encloses the bounded components of ∂A.
- δ is transversal to ∂A.
- For each unbounded component β_j of ∂A (if any), β_j meets δ at exactly two points.

If A is of type II the two last conditions in the definition of δ are vacuosly satisfied. Similarly, an *enveloping curve in the image* is a smooth simple oriented closed curve ζ with the following properties:

- ζ is oriented counterclockwise.
- ζ encloses the image of the bounded components of ∂A.
- ζ surrounds the intersection points of the image.
- ζ is transversal to $f(\partial A)$.
- For each unbounded component β_j of ∂A (if any), $f(\beta_j)$ meets ζ at exactly two points.

As before, if A is of type II the three last conditions are vacuosly satisfied.

Let A be a region of type III and δ be an enveloping curve in the domain. The unbounded connected components of $A - \delta$ naturally correspond to the *ends* of A (see [S]). The set $\beta_j - \delta$, on the other hand, has two unbounded connected components: following the orientation induced by β_j, one component, α_j, goes from infinity to δ, while the other, ω_j, goes from δ to infinity. The symbols

α_j and ω_j shall be interpreted as names for the two ends of β_j. Tracing δ according to its orientation and keeping track of the ends of the β_j's, we build a cyclic word with letters α_j and ω_j, the *word for the domain*. This is a precise formulation of the rather geometric concept of *order of arrival at infinity* of the components of the boundary of A. It can easily be verified that this word is independent of the choice of the enveloping curve δ. We assume the curves β_j to be labeled so that the word for the domain is $\alpha_1\omega_1\alpha_2\ldots\omega_\ell$. In particular, the boundary of an unbounded component of $A-\delta$ is composed of ω_j, an arc of δ and α_{j+1} (where $\beta_{\ell+1}=\beta_1$). Similarly, using an enveloping curve ζ in the image and letters $f(\alpha_j)$ and $f(\omega_j)$, we define the order of arrival at infinity of the curves $f(\beta_j)$, or the *word for the image* $\mathcal{W}$.

Alternatively, consider compact subsets K of $\mathbf{r}^2$ and take the unbounded connected component of $\mathbf{r}^2 - K - \partial A$: if K is large enough, we will have 2ℓ such connected components and a well defined cyclic order among them. If $K_1 \subseteq K_2$, we can naturally map the connected components for K_2 to those for K_1: again, if these are large enough, this map is a bijection. Thus, eventually, these connected components can be interpreted as the gaps between neighboring letters in the word of order of arrival at infinity: for each connected component, both neighbors correspond to boundary curves. Thus, the set A has ℓ ends: we call them E_j, $j=1,\ldots,\ell$, where E_j is bounded by ω_j and α_{j+1} (indices are to be interpreted cyclically).

The cyclic word $\mathcal{W}$ induces a permutation π of $\{1,2,\ldots,2\ell-1\}$. Turn $\mathcal{W}$ into a linear word leaving $f(\omega_\ell)$ at the last position. The numbers $\pi(2j-1)$ and $\pi(2j)$ give the positions of $f(\alpha_j)$ and $f(\omega_j)$ in the linear word, respectively. We now define ρ_A to be the *number of runs* of π ([K]), i.e., the number of maximal intervals in $\{1,2,\ldots,2\ell-1\}$ where π is increasing. Equivalently, ρ_A is the number of times required to go through $\mathcal{W}$ in order to pass sequentially by the letters $f(\alpha_1), f(\omega_1),\ldots, f(\omega_\ell), f(\alpha_1)$. A more directly relevant interpretation is the following: let δ and ζ be enveloping curves in the domain and image, respectively. Consider orientation preserving immersions from δ to ζ taking α_j to $f(\alpha_j)$ and ω_j to $f(\omega_j)$, i.e., taking the points in $\beta_j \cap \delta$ to the corresponding points in $f(\beta_j) \cap \zeta$. The minimum degree (or turning) of such immersions is ρ_A.

In Figure 3.1, the word for the image is $f(\alpha_1)f(\alpha_2)f(\alpha_3)f(\omega_2)f(\omega_1)f(\omega_3)$, the permutation π is given by $\pi(1) = 1$, $\pi(2) = 5$, $\pi(3) = 2$, $\pi(4) = 4$ and $\pi(5) = 3$ and therefore $\rho_A = 3$.

Figure 3.1

The turning of an oriented closed curve was defined in the previous section: we now define the *turning* of an unbounded oriented curve with a finite number of self-intersections. Take two points p and q on the curve, one on each unbounded connected component of the curve minus its self-intersections. Connect p and q by a simple arc which does not intersect the portion pq of the curve between p and q and consider the oriented closed curve formed by the arc and pq, respecting in pq the original orientation of the curve. There are two possible values, differing by 2, for the turning of the closed curve thus constructed, depending on the choice of the simple arc. The turning of the unbounded curve is by definition the average of these two values. In particular, a curve with no self-intersections has turning number 0. We then define $\tau(\beta_j)$ as the turning of $f(\beta_j)$, where $f(\beta_j)$ has the orientation induced by f and the orientation of β_j. Similarly, $\tau(\gamma_i) = \tau(f|_{\gamma_i})$.

We first prove some preliminary results about immersions $F : A \to \mathbb{R}^2$.

Lemma 3.1. *Let A be a region of type II or III, $F : A \to \mathbf{r}^2$ a smooth* *proper immersion such that* $F|_{\partial A}$ *is an adequate immersion. Suppose that* A *is oriented so that* $\sigma_A = sgn(\det(DF))$. *Let* ζ *be an enveloping curve in the image of* $f = F|_{\partial A}$. *Then*

(i) *If* A *is of type II, the pre-image of* ζ *by* F *is a simple closed curve.*

(ii) *If* A *is of type III, then each connected component of the pre-image of* ζ *by* F *is a simple arc lying in* A *joining end* ω_j *to* α_{j+1}.

Proof. Since F is a proper immersion, the connected components of the (non-empty) pre-image of a simple, closed, regular curve are either simple closed curves or simple arcs whose endpoints belong to ∂A. If this closed curve is the enveloping curve ζ, any arc must go from some ω_j to $\alpha_{j'}$. Indeed, a neighbourhood in A of a point in β_j is taken to the left of $F(\beta_j)$ if F preserves orientation and to the right if F reverses orientation. Since ζ is an enveloping curve, there are only two intersections with each $F(\beta_j)$ and, by the remark above, if an arc starts at some ω_j it must end at some $\alpha_{j'}$.

Let A be of type II. From the previous paragraph, the connected components of the pre-image of ζ are simple, closed curves. In order to prove (i), then, it suffices to show that there is only one such component. We first show this for an auxiliary curve $\tilde{\zeta}$, defined as follows. By properness of F, there is an enveloping curve δ bounding a disk D_δ which contains the pre-image of $F(\partial A)$. Notice that, since A is of type II, δ has to be contained in A. Now, let $\tilde{\zeta}$ be an enveloping curve in the image surrounding $F(D_\delta)$ and ζ. The connected components of the pre-image of $\tilde{\zeta}$ are necessarily simple closed curves surrounding the disk D_δ, by construction. The existence of two such components would give rise to a critical point of F in the annulus between them, contradicting the fact that F is an immersion. Thus the pre-image of $\tilde{\zeta}$ is connected; we now proceed to prove that the pre-image of ζ is also connected. Consider the open annulus R between the curves ζ and $\tilde{\zeta}$ and set $S = F^{-1}(R)$. It suffices to show now that S is also an annulus. For this we first claim that S is connected. Indeed, being an immersion, F must take the boundary of S to the boundary of R, and the pre-image of a point in the boundary of R must be in the boundary of S. The outer boundary $\tilde{\zeta}$ of R has a connected pre-image, which can only bound one connected component of S. The set S can have no other connected components because the image of their boundary would have to be contained in ζ, a contradiction. Thus, we conclude that S is an open disk with holes, with outer boundary given by the pre-image of $\tilde{\zeta}$ and inner boundaries given by all the connected components of the pre-image of ζ. Since the restriction of F to S is a proper, local homeomorphism onto R, we have that S is a connected covering space of R, and hence must also be an annulus. This proves (i).

Let A be of type III. Construct in the domain an enveloping curve δ surrounding $F^{-1}(D_\zeta)$. Let $\tilde{\zeta}$ be an enveloping curve in the image surrounding $F(D_\delta)$. We first prove (ii) for the auxiliary curve $\tilde{\zeta}$. Remember that since δ is an enveloping curve in the domain, each connected component of $A - D_\delta$ is an end bounded by ω_j and α_{j+1}. Thus, from the fact that $\tilde{\zeta}$ surrounds $F(D_\delta)$ it is clear that any connected component of $F^{-1}(\tilde{\zeta})$ is contained in one of these ends. Arcs must therefore join ω_j to α_{j+1} and it remains to prove that there are no closed curves in the pre-image of $\tilde{\zeta}$. Indeed, the image of the disk bounded by such a closed curve would be $D_{\tilde{\zeta}}$, which contains D_ζ, contradicting the fact that $F^{-1}(D_\zeta)$ is contained in D_δ. In order to transfer the result from $\tilde{\zeta}$ to ζ, consider a smooth non-zero vector field in the annulus contained between ζ and $\tilde{\zeta}$ transversal to these two curves, coming in through ζ and going out through $\tilde{\zeta}$ and tangent to $F(\beta_j)$. The pullback of this vector field by F defines a smooth non-zero vector field on the pre-image of the said annulus. Each connected component of this pre-image must therefore be a disk with boundary consisting of an arc in ω_j, a connected component of $F^{-1}(\tilde{\zeta})$, an arc in α_{j+1} and a connected component of $F^{-1}(\zeta)$, in this order. Item (ii) is therefore proved for ζ.

□

Notice that it follows from the above proof that the inverse image of the disk D_ζ bounded by ζ is a disk with finitely many holes. The outer boundary γ_∞ of this pre-image is simply the pre-image of ζ if A is of type II. If A is of type III, γ_∞ is formed by arcs which are alternatedly connected components of

$F^{-1}(\zeta)$ and segments of β_j; these segments come in the order indicated by the indices. In either case, the inner boundaries are the γ_i. We orient γ_∞ so that it is positively oriented iff $\sigma_A > 0$; define $\tau(\gamma_\infty) = \tau(F|_{\gamma_\infty})$.

Let F and A be as in the previous lemma. If A is of type II, let d be the number of pre-images of an arbitrary point on an enveloping curve; clearly, this number is independent of the point or the curve. We define the *degree at infinity* of F by $\deg(F) = \operatorname{sgn}(\det(DF))d$. Any continuous function extending F to the plane has topological degree equal to the degree of F at infinity. Also, $\deg(F) = \tau(\gamma_\infty)$.

If A is of type III, we assign a non-negative integer d_j (somewhat similar to d) to each end E_j of A, where E_j is enclosed between ω_j and α_{j+1}. Consider an arbitrary enveloping curve ζ in the image and let p be the only intersection of ζ with $F(\omega_j)$. The number d_j is defined to be the number of pre-images of p in the interior of the connected component of the pre-image of ζ corresponding to the end E_j. Clearly, d_j thus defined does not depend on the choice of ζ since there are d_j pre-images of $F(\omega_j)$ (i.e., connected components of $F^{-1}(F(\omega_j))$) in the interior of E_j. We call $\operatorname{sgn}(\det(DF))d_j$ the *partial degree of F at infinity* associated to the end E_j.

Lemma 3.2. *Let A be an oriented region of type III and $F : A \to \mathbb{R}^2$ be an immersion with sgn$(\det(DF)) = \sigma_A$ such that $F|_{\partial A}$ is an adequate immersion. Let ζ be an enveloping curve in the image and consider γ_∞ as above. Then*

$$\tau(\gamma_\infty) = \sum_j \tau(\beta_j) + \sum_j d_j + \rho_A.$$

Proof. If all d_j and $\tau(\beta_j)$ are zero the curve $F(\gamma_\infty)$ can be deformed without changing turning numbers to the orientation preserving immersion from γ_∞ to ζ with minimum degree taking α_j and ω_j to $F(\alpha_j)$ and $F(\omega_j)$, respectively, hence the formula in this special case. Clearly, changing a d_j amounts to introducing extra turns to the above immersion. Finally, the additive property of turnings takes care of the $\tau(\beta_j)$.

□

Lemma 3.3. *Let $F : A \to \mathbb{R}^2$ be an immersion with $sgn(\det(DF)) = \sigma_A$ and such that $F|_{\partial A}$ is an adequate immersion. Also, let ζ be an enveloping curve in the image and consider γ_∞ as above. Then*

$$\tau(\gamma_\infty) + \sum_i \tau(\gamma_i) = \chi(A),$$

where $\chi(A)$ is the Euler characteristic of A.

Proof. This is the corollary to Proposition 2.0.

□

In order get generalizations of Theorem 2.7 to regions of type II or III we will make use of Blank words. Thus, we first generalize the notion of systems of rays for an adequate immersion f defined on the boundary of an oriented region A.

For regions of type II, a system of rays for f is defined exactly as in Section 2. For regions of type III, a system of rays for f is again a finite set of embeddings of the closed positive half-line, with the same properties of disjointness and transversality to the image of the boundary curves as described in Section 2 plus the additional condition that a ray may only intersect $f(\partial A)$ finitely many times. Also, there must be one ray with origin at each of the (finitely many) *bounded* tiles for the image of the curves. In each case, Blank words for each bounded curve in ∂A are constructed exactly as in Troyer's theorem but, since we are now dealing with unbounded regions, we need a new ingredient which will be called the *word at infinity*.

Consider a system of rays for f as described above. Let ζ be an enveloping curve surrounding all intersections of rays with $f(\partial A)$ crossing each ray transversally and exactly once. Tracing ζ counterclockwise and keeping track of intersections with rays and images of boundary curves, we build a cyclic word $\mathcal{W}^\star$ with the letters used for the rays, $f(\alpha_j)$ and $f(\omega_j)$, describing the order of arrival at infinity of rays and images of unbounded boundary curves (if any).

It can easily be verified that this word is independent of the choice of ζ (with the properties above). Notice that, for A of type III, by ommiting the letters for rays we obtain the word for the image $\mathcal{W}$.

We first consider regions of type II. Let d be a positive integer (the absolute value of the degree at infinity of the desired extension of f). The *word at infinity* is made up of d juxtaposed copies of the word of $\mathcal{W}^\star$ giving all letters a positive sign and height index equal to ∞ (indicating that such heights are always greater than those of intersections with $f(\partial A)$).

For A of type III, given non-negative integers d_j (the absolute value of the partial degrees at infinity of the desired extension), we first construct auxiliary strings $\mathcal{S}_j$ and $\mathcal{R}_j$. The string $\mathcal{S}_j$ is obtained by following $f(\beta_j)$, keeping track of oriented intersections with the rays as before. The string $\mathcal{R}_j$ is constructed by following $\mathcal{W}^\star$ starting at ω_j and reaching α_{j+1} (or α_1 if $j = \ell$) after making d_j full turns around $\mathcal{W}^\star$, ignoring α's and ω's. Finally, the word at infinity is obtained by concatenating $\mathcal{S}_1, \mathcal{R}_1, \mathcal{S}_2, \mathcal{R}_2, \ldots, \mathcal{S}_\ell, \mathcal{R}_\ell$, in this order. Letters in the strings $\mathcal{S}_j$ receive height indices as before while letters in the strings $\mathcal{R}_j$ have height indices equal to ∞. An example is provided immediately after the statement of Theorem 3.5.

Thus, for unbounded regions, we have, given an adequate function and a system of rays, a Blank word for each bounded boundary curve and the word at infinity. The notions of adjunction and grouping are exactly as in Section 2 taking into account all of the above words.

Theorem 3.4. *Let A be an oriented region of type II and $f : \partial A \to I\!R^2$ be an adequate immersion. Given $d > 0$ and a system of rays consider the associated Blank words. Then, there exists a proper immersion $F : A \to \mathbf{r}^2$ extending f, with $sgn(\det(DF)) = \sigma_A$ and degree at infinity equal to $\sigma_A d$ if and only if*

(a) $d + \sum_{1 \leq i \leq k} \tau(\gamma_i) = \chi(A)$,

(b) *the Blank words together with the word at infinity group positively.*

Theorem 3.5. *Let A be an oriented region of type III and $f : \partial A \to \mathbb{R}^2$ be an adequate immersion. Given non-negative integers d_j, $j = 1, \ldots, \ell$, and a system of rays consider the associated Blank words. Then, there exists a proper immersion $F : A \to \mathbf{r}^2$ extending f with $sgn(\det(DF)) = \sigma_A$ and partial degree at E_j equal to $\sigma_A d_j$ if and only if*

(a) $\rho_A + \sum_j d_j + \sum_j \tau(\beta_j) + \sum_i \tau(\gamma_i) = \chi(A)$,

(b) *the Blank words together with the word at infinity group positively.*

Before proving the theorems, we provide an example. Consider the region A of type III in Figure 3.2(a) with image of the boundary under an adequate immersion f shown in 3.2(b). We assign to the two ends of A partial degrees $d_1 = 0$ and $d_2 = 2$; orientation and ends of curves are indicated in the figures. We then have $\tau(\gamma_1) = -5$, $\tau(\beta_1) = +1$ and $\tau(\beta_2) = 0$. The word at the image $\mathcal{W}$ is $f(\omega_1)f(\alpha_2)f(\alpha_1)f(\omega_2)$ and the corresponding permutation is $\pi(1) = 3$, $\pi(2) = 1$ and $\pi(3) = 2$, whence $\rho_A = 2$. Condition (a) in Theorem 3.5 holds, since $\chi(A) = 0$. For the rays in the picture, the Blank word for $f(\gamma_1)$ is $b_0^+ c_0^- d_0^- e_0^- f_0^- g_0^- h_0^-$ and the substrings which concatenate to yield the word at infinity are

$$\begin{aligned}
\mathcal{S}_1 &= a_0^+ b_1^+ c_1^+ d_1^+ e_1^+ f_1^+ g_1^+ h_1^+,\\
\mathcal{R}_1 &= \text{(empty word)},\\
\mathcal{S}_2 &= a_1^+ b_2^+ c_2^+ d_2^+ e_2^+ f_2^+ g_2^+ h_2^+,\\
\mathcal{R}_2 &= a_\infty^+ b_\infty^+ c_\infty^+ d_\infty^+ e_\infty^+ f_\infty^+ g_\infty^+ h_\infty^+ a_\infty^+ b_\infty^+ c_\infty^+ d_\infty^+ e_\infty^+ f_\infty^+ g_\infty^+ h_\infty^+ a_\infty^+ b_\infty^+ c_\infty^+ d_\infty^+ e_\infty^+ f_\infty^+ g_\infty^+ h_\infty^+.
\end{aligned}$$

The reader is invited to check that these words indeed satisfy condition (b) in Theorem 3.5. An immersion F extending f therefore exists and the proofs of the previous theorems show how to construct it; we shall see a related example in Section 4.

Figure 3.2

Proof of Theorem 3.4. We apply Theorem 2.7 to a bounded region $A_0 \subseteq A$, with the same interior boundaries as A and exterior boundary given by a simple closed curve γ_0, which we now construct.

Consider a closed regular curve $\tilde{\gamma}_0$ in the image, parametrized by $\tilde{g}_0 : [0,1] \to \mathbf{r}^2$ with $\tilde{g}_0(0) = \tilde{g}_0(1)$ satisfying the following properties (see Figure 3.3, where $d = 3$ and $\partial A = C$).

- The curve $\tilde{\gamma}_0$ turns d times conterclockwise around a fixed enveloping curve in the image.

- The curve $\tilde{\gamma}_0$ intersects each ray transversally exactly d times.

- The curve $\tilde{\gamma}_0$ has exactly $d-1$ self-intersections, all transversal.

Figure 3.3

We initially prove that if an immersion exists, then items (a) and (b) hold. By continuation, there is a $g_0 : [0,1] \to \mathbf{r}^2$ with $\tilde{g}_0 = F \circ g_0$ for any choice of $g_0(0)$, a pre-image of $\tilde{g}_0(0)$. We want to show that g_0 is the parametrization of a simple closed curve. Indeed, let us first prove that there are no proper loops. A loop has to surround ∂A since its image surrounds an enveloping curve and therefore the turning of the restriction of F to the loop has to be equal to the turning of $F|_{\gamma_\infty}$, which is d. The image of a proper loop must of course have smaller turning than d, a contradiction. If $g_0(0) \neq g_0(1)$, continue the inversion process until g_0 comes back to $g_0(0)$, which it must eventually do since F is proper. The turning of $F \circ g_0$ is greater than d since $F \circ g_0$ traces $\tilde{\gamma}_0$ more than once. Let γ_0 be the image of g_0.

In order to obtain a system of rays for the new problem, it suffices to add to the old system a few rays starting from the tiles created by $\tilde{\gamma}_0$ with positive intersections only. The Blank word for γ_0 in the new problem is therefore, up to irrelevant new letters, equal to the word at infinity. Conditions (a) and (b) of our proposition follow from the corresponding conditions in Theorem 2.7.

To prove the converse, let γ_0 be any enveloping curve in the domain of f. We define $f_0 : \gamma_0 \to \mathbf{r}^2$ as an orientation preserving regular parametrization of $\tilde{\gamma}_0$. Let A_0 be as above and apply to it Theorem 2.7. Clearly, heights for letters in the Blank word of γ_0 are greater than the height of any other intersection on the same ray with any other curve, so that the existence of a positive grouping is unaffected by the change of word. Therefore, conditions (a) and (b) of the proposition imply conditions (a) and (b) in Theorem 2.7. There exists then an F_0 extending simultaneously f and f_0 to A_0. The existence of an F_∞ extending f_0 to $A - A_0$ is trivial. By gluing F_0 and F_∞ we obtain an extension F of f to A which may be non smooth, but which can easily be rendered smooth.

□

Proof of Theorem 3.5. The proof is similar to that of Proposition 3.4: we apply Theorem 2.7 to a bounded region $A_0 \subseteq A$ with the same interior boundaries as A and with exterior boundary a curve γ_0 similar to γ_∞, i.e., composed of big chunks of β_j's together with arcs ξ_j (to be constructed) connecting $\omega_j \subseteq \beta_j$ to $\alpha_{j+1} \subseteq \beta_{j+1}$.

Let ζ be an arbitrary enveloping curve in the image. For each j take an oriented arc $\tilde{\xi}_j$ outside D_ζ with the following properties (see Figure 3.4):

- $\tilde{\xi}_j$ goes from $f(\omega_j)$ to $f(\alpha_{j+1})$,
- $\tilde{\xi}_j$ intersects any $f(\omega)$ transversally from right to left and any $f(\alpha)$ transversally from left to right,
- $\tilde{\xi}_j$ intersects any ray transversally from right to left,
- the intersection of the closed arc $\tilde{\xi}_j$ with $f(\omega_j)$ has exactly $d_j + 1$ points,
- the arcs $\tilde{\xi}_j$ are simple and disjoint.

Figure 3.4

First we prove that the existence of F implies (a) and (b). For each j take by continuation a pre-image ξ_j of $\tilde{\xi}_j$, starting at the only pre-image of the beginning of $\tilde{\xi}_j$ in ω_j; the process can get started since F preserves orientation. This will of course produce a simple arc entirely contained in the end E_j. We have to prove that this arc ends at α_{j+1} and that its image is the entire arc $\tilde{\xi}_j$. The continuation process cannot fail before we reach the end of $\tilde{\xi}_j$ since we would then have less than d_j pre-images of $f(\omega_j)$ in the interior of E_j, contradicting the fact that d_j is the corresponding partial degree. On the other hand, the process must end by reaching α_{j+1}: otherwise, extend $\tilde{\xi}_j$ respecting sense of intersections until we reach ω_j and we have $d_j + 1$ pre-images of $f(\omega_j)$ in the interior of E_j, again a contradiction. We thus have the curve γ_0 constructed from these arcs $\tilde{\xi}_j$ and chunks of β_j's, oriented consistently with the β_j (and thus, automatically, consistently with the $\tilde{\xi}_j$). As usual, let $\tau(\gamma_0) = \tau(F|_{\gamma_0})$. Clearly, since F is an immersion, $\tau(\gamma_0) = \tau(\gamma_\infty)$, where γ_∞ is obtained from ζ and F, and so Lemmas 3.2 and 3.3 imply item (a). Again, in order to apply Theorem 2.7 to A_0 we need to add extra rays but these can be taken with positive intersections only and are therefore irrelevant when considering grouping. The Blank word for γ_0 is, except for these new positive letters, the word at infinity; item (b) follows.

In the other direction, let ξ_j be arbitrary simple arcs contained in E_j joining ω_j to α_{j+1}. We have thus defined the curve γ_0 and f is defined on those parts of it coming from β_j. In order to extend f to γ_0, define homeomorphisms from ξ_j onto $\tilde{\xi}_j$ respecting endpoints. In order to apply Theorem 2.7 to A_0 we first observe that the curve γ_0 and f on it were constructed in order to guarantee that

$$\tau(\gamma_0) = \rho_A + \sum_j (\tau(\beta_j) + d_j),$$

similarly to Lemma 3.2. Item (a) in Theorem 2.7 now follows from our item (a). Item (b) follows from our item (b) since again we need to introduce a few irrelevant rays and then the word for γ_0 is essentially the word at infinity. As in the previous theorem, the extension to each end is trivial; overall smoothing is again done by classical methods.

□

4. The main theorem

In the main theorem, presented in this section, we assemble the previous results in order to give criteria similar to Blank's (and its generalizations) which are appropriate for functions with cusps.

Recall that for a closed subset X of $\mathbb{R}^2$ the connected components of $\mathbb{R}^2 - X$ are called *tiles* for X. Let C be the union of disjoint oriented curves γ_i, $1 \leq i \leq k$ and β_j, $1 \leq j \leq \ell$, where γ_i and β_j are images of proper embeddings of S^1 and $\mathbb{R}$ respectively. We assume that these curves are *consistently oriented*, i.e., that any tile S for C can be oriented so that the induced orientation in $\partial S \subseteq C$ is consistent with that of C: we always use this orientation for S and we say that S is *consistently oriented* with C. Notice that if $\ell = 0$, we have bounded tiles (regions of type I) and a single unbounded one (of type II). On the other hand, if $\ell > 0$, all unbounded tiles are regions of type III and there are at least two of them.

As we saw in Section 1, cusps of a Whitney function can be effective to the right or to the left of their critical curves. The function f alone does not determine to which side of a critical curve a cusp will be effective and it is thus natural to consider this information as another given of the problem. We therefore assign indices $\ll$ or $\mathbf{r}$ to cusps indicating if they are to be effective to the left or right, respectively, for the desired Whitney function F. Thus, a cusp labeled $\ll$ (resp., $\mathbf{r}$) will be called *effective in* S if S is the tile for C immediately to the left (resp., right) of the cusp.

We shall now discuss the construction of Blank words. In our new context, a *ray* for $f|_{\partial S}$ is a proper embedding $r : [0, +\infty) \to \mathbb{R}^2$ with the previous properties and the extra requirements that, except possibly at the origin, the ray does not meet images of cusps. Similarly, a *system of rays* for $f|_{\partial S}$ (or, for simplicity, a system of rays for S) is a finite system of disjoint rays with the following properties:

- Given a bounded connected component of $\mathbb{R}^2 - f(\partial S)$, there exists exactly one ray such that its origin lies in this connected component.
- Given a effective cusp in S, there exists exactly one ray such that its origin is the image of this cusp; furthermore, the ray leaves the cusp to the right of $f(\partial S)$.
- Rays start either in a bounded component of $\mathbb{R}^2 - f(\partial S)$ or at the image of an effective cusp in S.

For each S, given a system of rays for S and $d > 0$, if S is of type II, or $d_j \geq 0$ $(1 \leq j \leq \ell^S)$, if S is of type III, we construct one word for each γ_i in the boundary of S and, if S is unbounded, an additional *word at infinity*. The words for each such γ_i are constructed by following $f(\gamma_i)$, keeping track of oriented intersections with the rays, as before. Letters corresponding to intersections at images of cusps receive a minus sign and a height index 0, by definition. The word at infinity is constructed as before, again assigning to letters corresponding to cusps a minus sign and a height index 0.

Given a tile S for C, let $\kappa(S)$ be the total number of effective cusps in S and $\tau(S)$ be the sum of the turning of f at each boundary curve of S.

Theorem 4.1. *Let $f : C \to I\!R^2$ be an adequate function and let labels $\ll$ and $\mathbf{r}$ be assigned to the cusps of f. Consider a system of rays for each tile S for C. Then there exists a proper Whitney function F extending f to $I\!R^2$ with critical set C, sense of folding corresponding to the given orientation of C and such that a cusp is effective to the left (resp., right) if its label is $\ll$ (resp., $\mathbf{r}$) if and only if for each tile S (consistently oriented with C) the following condition holds:*

- *For S of type I, the identity*

$$\tau(S) - \kappa(S) = \chi(S)$$

holds and the Blank words group positively.

- *For S of type II, we have that the number d, defined by*

$$\tau(S) - \kappa(S) + d = \chi(S)$$

is strictly positive and the associated Blank words group positively (in this case, $d = |\deg(F)|$).

- *For S of type III, with ends E_j, $1 \le j \le \ell^S$, there exist non-negative integers d_j, $1 \le j \le \ell^S$, with*

$$\rho_S + \tau(S) - \kappa(S) + \sum_{1 \le j \le \ell^S} d_j = \chi(S),$$

such that the associated Blank words group positively (d_j is the absolute value of the partial degree of F at the end E_j).

As an example, consider C and $f(C)$ as in Figure 4.1, (a) and (b), respectively. The tiles in the domain are S_1, S_2, S_3 and S_4, the first three being of type III and the last one of type I. The orientations of C and $f(C)$ (which are to be the sense of folding) are indicated, as are the labels for cusps and partial degrees. We now check the condition in Theorem 4.1 for tile S_1. It is easy to see that $\kappa(S_1) = 6$ and that $\tau(\gamma_1) = 1$, $\tau(\beta_1) = 1$ and $\tau(\beta_2) = 0$, whence $\tau(S_1) = 2$. Since the behaviour at infinity of this example for tile S_1 is identical to that of the example shown in Figure 3.2, we have $\rho_{S_1} = 2$ and the same word at infinity. The expression for $\chi(S_1) = 0$ therefore holds. Also, the Blank word for γ_1 is identical to that for the example in Figure 3.2, and again the words group positively. The reader will easily check that the remaining tiles also satisfy the appropriate conditions in the above theorem. Thus, there is a proper Whitney function F extending f with prescribed critical set C, partial degrees and senses of cusps. Later, we shall give a more explicit description of this function.

Figure 4.1

Remember, however, that we are often interested in computing the critical set of a nice function F. Assuming that some critical curves for F are known we may use Theorem 4.1 to check if it is at least consistent that these are all the critical curves. Information about F such as sense of folding, sense of cusps and partial degrees makes the application of the theorem more efficient.

For example, given a generic proper polynomial functions from $I\!R^2$ to $I\!R^2$, a nice function, its behaviour at infinity can be determined by methods such as Newton polygons. This gives us the number of unbounded critical curves, the extrema of such curves, their order of arrival at infinity and the partial degrees at infinity. Thus, using Theorem 4.1 as a test to see if it is reasonable to assume that we have already found the full critical set of such a polynomial function, the criterion as in item (c) has to work with the d_j being the known partial degrees.

Proof of Theorem 4.1. Since f is adequate, Proposition 1.1 tells us that it can be extended to a Whitney function $\tilde{f}$ defined on a thin tubular neighborhood U of C. In order to prove the existence of F, it therefore suffices to extend to each tile S immersions which are already defined near the boundary.

Let S be a tile and let its boundary be composed of bounded curves γ_i^S, $1 \leq i \leq k^S$, and unbounded curves β_j^S, $1 \leq j \leq \ell^S$. We denote by $\kappa^S(\gamma_i^S)$ (resp., $\kappa^S(\beta_j^S)$) the number of cusps on γ_i^S (resp., β_j^S) which are effective in S. Thus,

$$\tau(S) = \sum_{1 \leq i \leq k^S} \tau(\gamma_i^S) + \sum_{1 \leq j \leq \ell^S} \tau(\beta_j^S)$$

and

$$\kappa(S) = \sum_{1 \leq i \leq k^S} \kappa^S(\gamma_i^S) + \sum_{1 \leq j \leq \ell^S} \kappa^S(\beta_j^S).$$

Let $S^\star \subseteq S$ be a closed set whose boundary is composed of smooth curves $\gamma_i^\star$ and $\beta_j^\star$ which are sufficiently near the corresponding curves in ∂S and contained in U. Orient $\gamma_i^\star$ and $\beta_j^\star$ as the nearby γ_i^S and β_j^S, so that they form the oriented boundary of $S^\star$, which is oriented positively or negatively according to the constant sign of $\det(D\tilde{f})$ in $S \cap U$. We apply to $S^\star$ Theorem 2.7 if S is of type I, Theorem 3.4 if S is of type II or Theorem 3.5 if S is of type III. We now show that the conditions in our theorem are precisely what is necessary in order to guarantee the hypothesis of these theorems.

From the local behaviour of cusps, each effective cusp creates a little loop in $f(\gamma_i^\star)$ or $f(\beta_j^\star)$, always negatively oriented (see the paragraph immediately following the proof for an example) and if $\gamma_i^\star$ and $\beta_j^\star$ are sufficiently near γ_i^S and β_j^S these are the only new intersections. This implies

$$\tau(\tilde{f}(\gamma_i^\star)) = \tau(\gamma_i^S) - \kappa^S(\gamma_i^S)$$

and

$$\tau(\tilde{f}(\beta_j^\star)) = \tau(\beta_j^S) - \kappa^S(\beta_j^S).$$

Also, the connected components of $I\!R^2 - (\bigcup_i \tilde{f}(\gamma_i^\star) \cup \bigcup_i \tilde{f}(\gamma_i^\star))$ coincide with those of $I\!R^2 - (\bigcup_i f(\gamma_i^S) \cup \bigcup_i f(\gamma_i^S))$ except that one small disk has been created by the loop around the image of each effective cusp. The Blank words (with cusps) as constructed for our theorem are therefore the Blank words for $S^\star$. Thus, the hypothesis of the appropriate theorem for the type of $S^\star$ hold and the desired immersion exists. If we take the immersions for all tiles and smooth out the resulting function we obtain the proper Whitney function F.

Conversely, if F exists, its restriction to $S^\star$ as above is an immersion and the same theorems show that the conditions in our theorem hold.

□

With the notation of the above proof, we explain the relation between the examples in Figures 3.2 and 4.1. Indeed, taking $S_1^\star$ for S_1 in Figure 4.1 as described in the proof above, we obtain precisely the example in Figure 3.2. Notice that the six cusps on γ_1 are effective in S_1 and thus generate six small negatively oriented loops, whereas the cusp in β_1, not being effective in S_1, creates no loop.

The next two results give formulae for the topological degree of F and the number of pre-images of an arbitrary regular value. Clearly, the number of pre-images is constant on tiles for $F(C)$ and changes by 2 from a tile to another neighbouring one, the tile to the left (according to sense of folding) having more pre-images. In the first theorem, we use information about the behaviour of the function near infinity (such as ρ_S or d_j). In the second one, we express these results in terms of finite information (such as τ or κ).

For a closed oriented parametrized curve $\alpha : S^1 \to I\!R^2$ and a point p not on the image of α let $w(\alpha, p)$ be *winding number* of α around p (i.e., the topological degree of $\theta \mapsto \alpha(\theta) - p/|\alpha(\theta) - p|$). We extend this concept for parametrized curves of the form $\alpha : I\!R \to I\!R^2$, where α is a proper continuous function injective in the complement of a compact interval I. Consider a circle around the origin such that $\alpha(I)$ and p are in its interior. Taking the maximal arc of α which contains $\alpha(I)$ and is in the interior of the circle, we construct two oriented closed curves by using two complementary arcs in the circle. We define the winding number $w(\alpha, p)$ as the average of the two winding numbers of these auxiliary closed curves around p. Notice that $w(\alpha, p)$ independs on the choice of the circle and is always in $\mathbf{Z} + 1/2$. In Figure 4.2, we have $w(\alpha, p_1) = 1/2$, $w(\alpha, p_2) = -1/2$ and $w(\alpha, p_3) = -3/2$.

Figure 4.2

Let F be a nice function with bounded critical curves γ_i and unbounded critical curves β_j, always oriented by sense of folding. Consider a connected component T of $\mathbb{R}^2 - \bigcup \beta_j$. Let $\sigma(T)$ be the sign of $\det(DF)$ in the unbounded tile S contained in T, $\rho_T = \rho_S$ and d_T be the sum of the absolute values of the partial degrees of F on the ends of S.

Theorem 4.2. *The topological degree of F is*

$$\deg(F) = \sum_T \sigma(T)(\rho_T + d_T)$$

and the number of pre-images of an arbitrary regular value p is

$$\#F^{-1}(p) = \sum_T (\rho_T + d_T) + 2\sum_j (w(F(\beta_j), p) - 1/2) + 2\sum_i w(F(\gamma_i), p)$$

where T ranges over all connected components of $I\!R^2 - \bigcup_{1 \leq j \leq \ell} \beta_j$.

Let us apply this result to the example in Figure 4.1: we must have $\deg(F) = 2$ ($\sigma(T)$ being obtained from the orientations of critical curves) and $\#F^{-1}(p) = 2$ for p in the uppermost tile. This can be visualized by considering the behaviour of F on a pre-image δ of a simple closed curve ζ in the image which is enveloping for every unbounded tile S. From Lemma 3.1, δ is a simple closed curve (with the same properties of an enveloping curve in the domain) and we orient it counterclockwise. The behaviour of F on δ is schematically shown in Figure 4.3: actually, the image of δ is contained in ζ and in order to render the graph visible we lifted it away from ζ; the true value of $F(p)$, $p \in \delta$ is obtained by radial projection onto ζ. The diagram is constructed from the sign of $\det(DF)$, the word in the image and the partial degrees.

Figure 4.3

Proof of Theorem 4.2. We use the same notation for the boundary of a tile S as in the proof of Theorem 4.1.

Take a curve ζ in the image which is enveloping for every tile and consider its pre-image by F, a curve δ which is, up to orientation, an enveloping curve for every tile in the domain. For each unbounded tile S construct γ_∞^S as in Section 3. The turning of $F(\gamma_\infty^S)$, $\tau(\gamma_\infty^S)$, can be interpreted as the degree of a function ϕ^S from γ_∞^S to ζ. In order to construct ϕ^S, first smooth out the curves $F(\beta_j^S)$ near cusps without introducing new intersections and bend them near ζ so that they become tangent to ζ, with counterclockwise orientation in ζ corresponding to the sense of folding in $F(\beta_j^S)$. Clearly, $\tau(\gamma_\infty^S)$ coincides with the turning of this deformed curve. Now define $\phi^S(p) = F(p)$ for $p \in \delta$ and $\phi^S(p)$ as the direction of the normal vector to the auxiliary curve above near $F(p)$ for $p \in \beta_j^S$. By construction, the degree of ϕ^S is the turning of the deformed curve and so $\deg(\phi^S) = \tau(\gamma_\infty^S)$.

Clearly, the topological degree of F is the degree of $F|_\delta$ as a function from the oriented curve δ to the oriented curve ζ. This last degree can be written as

$$\sum_{\text{unbounded } S} \deg(\phi^S),$$

if each γ_∞^S is oriented counterclockwise or

$$\sum_{\text{unbounded } S} \sigma(S) \deg(\phi^S),$$

if each γ_∞^S is oriented, as usual, according the sign of $\det(DF)$, which is the same as the orientation induced by the sense of folding in β_j. Thus,

$$\deg(F) = \sum_{\text{unbounded } S} (\sigma(S)\tau(\gamma_\infty^S)).$$

On the other hand, we have, by Lemma 3.2,

$$\tau(\gamma_\infty^S) = \rho_T + d_T + \sum_{1 \le j \le \ell^S} \tau(\beta_j)$$

and thus our formula for $\deg(F)$, since for each j the term $\tau(\beta)$ appears twice with opposite signs.

In order to count the number of pre-images of a point p, we again consider γ_∞^S as above. The number of pre-images of p in a bounded tile S is given by the sum of $w(F(\gamma_i), p)$ over all boundary components γ_i of S (each γ_i being oriented, as usual, by sense of folding), as the sign of $\det(DF)$ inside S is constant equal to σ_S. Similarly, the number of pre-images of p in an unbounded tile S is given by the sum of all $w(F(\gamma_i), p)$ (again over all bounded boundary components) with $w(F(\gamma_\infty^S), p)$. We have, however,

$$w(F(\gamma_\infty^S), p) = \rho_T + d_T + \sum_{1 \leq j \leq \ell^S} (w(F(\beta_j), p) - 1/2);$$

the proof of this identity is similar to that of Lemma 3.2 and is left to the reader. By adding all these terms, we get the desired formula for the number of pre-images of a regular point.

□

Let κ be the total number of cusps of F and $\tau(C)$ be the sum of all $\tau(\beta_j)$ and $\tau(\gamma_i)$. For T as above, we define $\kappa(T)$ to be the number of cusps which are in the interior of T or are effective in the unbounded tile of T. Similarly, let $\tau(T) = \sum_{\gamma_i \subseteq T} \tau(\gamma_i)$; notice that only bounded critical curves are taken into account.

Theorem 4.3. *The topological degree of F is*

$$\deg(F) = \sum_T \sigma(T)(\kappa(T) - 2\tau(T) + 1)$$

and the number of pre-images of an arbitrary regular value p is

$$\#F^{-1}(p) = 1 + \kappa - 2\tau(C) + 2\sum_j w(F(\beta_j), p) + 2\sum_i w(F(\gamma_i), p)$$

where T ranges over all connected components of $\mathbb{R}^2 - \bigcup_{1 \leq j \leq \ell} \beta_j$.

Proof. Let T be a component of $\mathbb{R}^2 - \bigcup_{1 \leq j \leq \ell} \beta_j$ and S be the unbounded tile contained in T. With the notation of Theorem 4.2, we prove that

$$\tau(\gamma_\infty^S) = \kappa(T) - 2\tau(T) + 1.$$

From Lemma 3.2 and Theorem 4.1,

$$\tau(\gamma_\infty) = \chi(S) + \kappa(S) - \sum_{1 \le i \le k^S} \tau(\gamma_i^S).$$

On the other hand, for an arbitrary bounded tile S' contained in T we have, again from Theorem 4.1,

$$0 = \chi(S') + \kappa(S') - \sum_{1 \le i \le k^{S'}} \tau(\gamma_i^{S'}).$$

Adding all these equations,

$$\tau(\gamma_\infty) = \chi(S) + \kappa(S) - \sum_{1 \le i \le k^S} \tau(\gamma_i^S) + \sum_{S' \subseteq T} \left(\chi(S') + \kappa(S') - \sum_{1 \le i \le k^{S'}} \tau(\gamma_i^{S'}) \right).$$

Notice that

$$\chi(S) + \sum_{S' \subseteq T} \chi(S') = \chi(T) = 1$$

and observe that for each bounded critical curve $\gamma \subseteq T$, $\tau(\gamma)$ appears twice in the above expression. Finally, each cusp is counted exactly once since it is effective in one tile only. Thus, the above identity reduces to $\tau(\gamma_\infty^S) = \kappa(T) - 2\tau(T) + 1$, as desired, yielding the degree formula.

For the other formula, by Theorem 4.2,

$$\#F^{-1}(p) = 1 + \sum_T (\rho_T + d_T - 1) + 2\sum_j w(F(\beta_j), p) + 2\sum_i w(F(\gamma_i), p)$$

and by Theorem 4.1,

$$\rho_S + d_S - 1 = \chi(S) - \tau(S) + \kappa(S) - 1,$$

for the unbounded tile S contained in T and

$$0 = \chi(S') - \tau(S') + \kappa(S')$$

for bounded tiles S' contained in T. Adding the identity for S and the identities for all S', we have

$$\rho_T + d_T - 1 = -\sum_i \tau(\beta_j^S) - 2\sum_i \tau(\gamma_i^S) + \kappa(T)$$

whence

$$\sum_T (\rho_T + d_T - 1) = -2\tau(C) + \kappa,$$

yielding our theorem.

□

Figure 4.4

We now return to the example in Figure 4.1. Figure 4.4(a) shows $F^{-1}(F(C))$, the *flower* of F. It is easy to see ([MST1], [MST2]) that the restriction of F to a tile for the flower is covering map for a tile for $F(C)$. Five of the six tiles for $F(C)$ are simply connected and the restrictions are thus diffeomorphisms from tiles for the flower to such tiles for $F(C)$. Notice that the tile for $F(C)$ surrounding $F(\gamma_1)$ has three pre-images restricted to which F is a diffeomorphism and one additional connected pre-image in which F is a five-fold covering map. The flower, thus, provides an explicit, geometric representation of a nice function F with the prescribed behaviour on the critical set.

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