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RECENT PROGRESS IN THE TOPOLOGY OF PROJECTIVE STIEFEL MANIFOLDS

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Abstract

In the past two to three years there has been a substantial amount of progress in understanding the topology of projective Stiefel manifolds. This note attempts to describe the main results and techniques involved. In particular, new work (much of it still unpublished) on K-theory, embeddings and immersions, span, almost complex and complex structures, and the order of the canonical line bundle, all pertaining to the projective Stiefel manifolds, will be described.

1. Earlier work on projective Stiefel manifolds

The projective Stiefel manifolds $X_{n,r}$, $1 \le r \le n-1$, were first studied in 1965 by Baum and Browder [6] as well as in 1968 by Gitler and Handel [8], Gitler [9]. For the basic definitions and notations the reader is referred to these papers or to Korbaš and Zvengrowski [13], §3.2. In the pioneering works the cohomology algebra $H^*(X_{n,r};\mathbf{Z}/2)$ was computed and the action of the Steenrod algebra $\mathcal{A}(2)$ partially found. The tangent bundle of $X_{n,r}$ was first determined in 1975 by K.Y. Lam [16] and in 1976 by the author [22]. It will be useful to recall their formula.

Lemma 1.1. $\tau_{n,r} \oplus \binom{r+1}{2}\epsilon \approx nr\xi_{n,r}$, *where* $\tau_{n,r}$, $k\epsilon$, $\xi_{n,r}$ *are respectively the tangent bundle, the rank k trivial bundle, and the canonical (or Hopf) line bundle over* $X_{n,r}$.

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This formula gives the stable tangent bundle and hence also the stable normal bundle as suitable multiples of the canonical line bundle.

In 1977 Antoniano [2] completely determined the $\mathcal{A}(2)$ action. One of the original reasons for studying these manifolds was their relation to immersions of real projective spaces P^m, and this is discussed in a paper of Smith [20] in 1980. We shall need the following result of Smith, which also appears in the earlier paper [8].

Theorem 1.2. *Let ξ be a line bundle over a finite CW-complex Y. Then $n\xi$ admits r linearly independent sections if and only if there exists a map $f : Y \to X_{n,r}$ such that $f^*(\xi_{n,r}) \approx \xi$.*

The 1986 paper of Antoniano, Gitler, Ucci, and the author [3] initiated the computation of the complex K-theory of these spaces, at least for the case n divisible by 4. One immediate application, in the same paper, was the determination of all parallelizable (or stably parallelizable) projective Stiefel manifolds, namely $X_{n,n-1}, X_{2m,2m-1}, X_{4,r}, X_{8,r}$, and $X_{16,8}$ with the single case $X_{12,8}$ remaining undecided. In the much more difficult problem of finding the span of $X_{n,r}$ (excluding of course the already solved cases $X_{n,1} = P^{n-1}$, by Adams [1], and the above mentioned parallelizable cases), some progress was achieved in a 1991 preprint of Korbaš, Sankaran, and the author [12]. However, much stronger results on span$(X_{n,r})$ will be described in §4, §5 below. This brief summary does not touch upon some of the interesting applications of this family of manifolds, we will simply cite as examples [2], [17], and [7], which deal respectively with the generalized vector field problem, skewness of r-fields on spheres, and the immersion conjecture.

2. $K^*(X_{n,r})$ and applications

The ring $K^*(X_{4m,r})$ was computed in [3]. It is beyond the scope of this note to go into the details of this calculation, but we will try to point out the salient features. The main idea is to represent the space as a homogeneous space, i.e.

$X_{n,r} \cong G/H$, where G is a connected Lie group with $\pi_1(G)$ torsion free, H a closed subgroup, and then apply the Hodgkin spectral sequence with E_2 term given by

$$E_2^p = Tor^p_{R(G)}(R(H), \mathbf{Z}).$$

Here $R(G)$, $R(H)$ denote the complex representation rings, RH is an RG-module by restriction, and $\mathbf{Z}$ is the trivial RG-module. In the case at hand $X_{n,r} \cong O(n)/O(n-r) \times (\mathbf{Z}/2)$, where $\mathbf{Z}/2 = \{\pm I_n\}$, but $O(n)$ is not connected and has a fundamental group $\mathbf{Z}/2$ with torsion. For n even one has $X_{n,r} \cong SO(n)/SO(n-r) \times (\mathbf{Z}/2)$ which can be lifted, using the double cover $\mathrm{Spin}(n) \to SO(n)$, to $X_{n,r} \cong \mathrm{Spin}(n)/H$ for a suitable closed subgroup H of $\mathrm{Spin}(n)$. For $n = 4m$, it turns out that $H \approx \mathrm{Spin}(4m-r) \times (\mathbf{Z}/2)$. This fact makes the computation of $RH \approx R\mathrm{Spin}(4m-r) \otimes R(\mathbf{Z}/2)$ trivial, which in turn simplifies the calculation of the restriction homomorphism $RH \to R\mathrm{Spin}(n)$.

As announced in [4], Barufatti and Hacon have recently determined the complex K-theory of all $X_{n,r}$. Presumably, their method involves first expressing $X_{n,r}$ as a homogeneous space $X_{n,r} \cong \mathrm{Spin}(n)/H$, where H is no longer a simple direct product when $n \not\equiv 0\,(mod\,4)$, calculating the restriction homomorphism, and then proceeding along the lines of [3]. One important consequence of this work is the determination of the (additive) order of the class of the complexified canonical line bundle $c\xi_{n,r}$ over $X_{n,r}$ (cf.[4], formulae (3), (4)). We point out here that there is a minor error in (3), where the first term should be $2^{[(m-1)/2]}$ (as usual, $[x]$ denotes the integer part of x). In any case, we now describe this result with slightly different notation. For any integer $t \geq 1$, $\nu_2(t)$ denotes the exponent of the highest power of 2 dividing t, e.g. $\nu_2(48) = 4$. For any $X_{n,r}$, let $n = 2m$ or $n = 2m+1$ and let $k = [(n-r)/2]$.

Definition 2.1.

(a) For n even or r even, $a_1(n,r) := min\{2j - 1 + \nu_2\binom{m}{j} : k+1 \leq j\}$,

(b) For n, r both odd, $a_1(n,r) := min\{2k + \nu_2\binom{m}{k}, 2j - 1 + \nu_2\binom{m}{j} : k+1 \leq j\}$,

(c) For any n, r, $a(n,r) := min\{[(n-1)/2], a_1(n,r)\}$.

Theorem 2.2. *The additive order of* $c\xi_{n,r}$ *equals* $2^{a(n,r)}$.

A well known corollary of this is that in real K-theory (i.e. in $KO^*(X_{n,r})$), the additive order of $\xi_{n,r}$ will be $2^{b(n,r)}$, where $b(n,r) = a(n,r) + \varepsilon$ and ε is either 0 or 1. In §5 we shall have a great deal more to say about ε.

Of course the classical method of finding the Stiefel-Whitney classes of the stable normal bundle (cf.[18]) can be used to find upper bounds for immersions and embeddings of $X_{n,r}$, since the stable normal bundle is $-nr\xi_{n,r}$ (and thus the Stiefel-Whitney classes of the normal or tangent bundles are readily found). In addition, Barufatti applies the γ-operations in KO-theory and the information on $b(n,r)$ given above to obtain KO-theoretic upper bounds for immersion and embedding dimensions. For details cf. [4], Prop. 3 and Prop. 4. Finally, the results on $b(n,r)$ are applied in this paper to give a somewhat simpler proof of the parallelizability theorem found in [3].

3. Almost complex structures on $X_{n,r}$

In the preprint [11] Korbaš and Sankaran have completely determined all $X_{n,r}$ admitting almost complex structures, and also found many that admit complex structures. Their methods, for the almost complex structures, are elementary and based on the descriptions of the tangent bundle found in [17], [22]. To further obtain that some of these admit complex structures, results of Wang [21] are applied. The main theorems are as follows.

Theorem 3.1. *Let* $2 \leq r \leq n-2$. *The manifold* $X_{n,r}$ *is almost complex if and only if* $r \equiv 0\,(mod\,4)$, *or* n *is even and* $r \equiv 3\,(mod\,4)$. *Furthermore no* $X_{n,1} = P^{n-1}$ *is almost complex, and* $X_{n,n-1}$ *is almost complex if and only if* $n \equiv 0, 1\,(mod\,4)$.

It is easy to check that the conclusions of this theorem are equivalent to the following more perspicuous statement (also given in [11]): $X_{n,r}$, for

$1 \leq r \leq n-1$, admits an almost complex structure if and only if it is both orientable and even-dimensional.

Theorem 3.2. *The manifolds* $X_{n,4k}$ *(*$k \geq 1$, $n-4k \geq 1$*) and* $X_{8,3}$ *admit complex structures, and any even-dimensional product of a finite number of* $X_{n,2s}$ *(for possibly different* $s \geq 1$*) admits a complex structure.*

4. The span of $X_{n,r}$

In this section it will be useful to use the notation $d = d_{n,r} = \dim(X_{n,r}) = nr - \binom{r+1}{2}$. It will also be useful to recall the notion of stable span, written span^0 (for a definition cf. [13]), and the basic property $\text{span} \leq \text{span}^0$. It is not difficult, using 1.1 and 1.2 above, to prove the following result.

Lemma 4.1. *One has* $span^0(X_{n,r}) \geq d-k$ *if and only if there exists a map* $f: X_{n,r} \to X_{nr,nr-k}$ *such that* $f^*(\xi_{nr,nr-k}) \approx \xi_{n,r}$.

This lemma, or more precisely its contrapositive, can then be used together with any suitable cohomology theory to obtain upper bounds for $\text{span}^0(X_{n,r})$, and hence also for $\text{span}(X_{n,r})$. As in the work in §2 on immersions and embeddings, both $H^*(X_{n,r};\mathbf{Z}/2)$ and $K^*(X_{n,r})$ can be used in this way to furnish good upper bounds. For example, just using the height of the one-dimensional cohomology class $x \in H^1(X_{n,r};\mathbf{Z}/2)$ is already equivalent to the use of the Stiefel-Whitney classes to obtain upper bounds for the span.

Obtaining good lower bounds for the span appears to be more difficult. The lower bound $\text{span}(X_{n,r}) \geq \binom{r}{2}$ is obvious from the isomorphism (cf.[16],[22])

$$\tau_{n,r} \approx r(\beta_{n,r} \otimes \xi_{n,r}) \oplus \binom{r}{2}\epsilon,$$

where $\beta_{n,r}$ is the evident "orthogonal complement" bundle of rank $n-r$. However this lower bound is generally too weak to be of interest. On the other hand, there is an excellent lower bound $k_{n,r} := -\binom{r+1}{2} +$ span$(nr\xi_{n,1})$ for the stable span, as is easily proved using 1.1 and the fibre map $p: X_{n,r} \to X_{n,1} = P^{n-1}$ (noting that $p^*(\xi_{n,1}) \approx \xi_{n,r}$). This lower bound is both strong and computable, thanks to the work of K.Y. Lam [15] (and others) on the generalized vector field problem. Unfortunately it is a priori only a lower bound for span^0, but J. Korbaš and the author have succeeded in showing that it is also a lower bound for span [14].

Theorem 4.2. *For any projective Stiefel manifold $X_{n,r}$, one has span$(X_{n,r}) \geq k_{n,r}$, with the possible exception of $X_{2m+1,2}$, $m \geq 4$.*

The proof of this theorem is surprisingly easy in most cases, and uses the stability properties of vector bundles together with explicit bundle isomorphisms. This portion of the proof can be found in [14] ,p.100, and the cases $X_{5,2}$, $X_{7,2}$ in [12]. The undecided cases in Theorem 4.2 can be settled by the computation of a suitable Browder-Dupont invariant, but thus far attempts to carry out this calculation have not succeeded. The following examples will give some idea of the power of the above methods. They are selected more or less at random.

Examples 4.3.

(a) span$(X_{14,4}) = 38$,

(b) span$(X_{16,5}) = 58$,

(c) $1,618 \leq$span$(X_{58,51}) \leq 1,625$,

(d) $36,897 \leq$span$(X_{314,159}) \leq 37,056$.

5. The real order of the line bundle $\xi_{n,r}$

As we have seen in §2, the (real) additive order of the canonical line bundle $\xi_{n,r}$ is $2^{b(n,r)}$, where $b(n,r) = a(n,r) + \varepsilon$, $a(n,r)$ is given by Definition 2.1, and ε can equal 0 or 1. This determines $b(n,r)$ completely except for the small but somewhat annoying ε. However, finding the value of ε can be a highly non-trivial task. For example, such a problem occurred as part of the Adams conjecture (cf. [10] ,p.227). P. Sankaran and the author (cf. [19]) have given two rather different proofs of a theorem which determines ε in about 70% of all cases (in an asymptotic sense). Before stating the theorem it will be necessary to make a definition. Noticing from Definition 2.1 that as r increases from 1 to $n-1$, the minimum in 2.1 (c) starts at $[(n-1)/2]$ for low values of r but eventually is given by $a_1(n,r)$ once r gets large enough (roughly $n/2$), we proceed as follows.

Definition 5.1. For a given n, we say that r is in the upper range when $a(n,r) = a_1(n,r)$, and otherwise that r is in the lower range.

Theorem 5.2. *Whenever r is in the upper range, or* $n \equiv 0, \pm 1 \ (mod\, 8)$, *one has* $b(n,r) = a(n,r)$ *or equivalently* $\varepsilon = 0$.

It is worth noting that $\varepsilon = 1$ is certainly possible, indeed it follows from [1] that $b(n,1) = a(n,1) + 1$ whenever $n \equiv 2,3,4,5,6 \ (mod\, 8)$. This and other evidence suggest the following.

Lower Range Conjecture 5.3. For $n \equiv 2,3,4,5,6 \ (mod\, 8)$ and r in the lower range, $\varepsilon = 1$.

We close with the observation that Theorem 5.2 and the Lower Range Conjecture have implications for multiples of a line bundle ξ over any finite CW-complex Y, because of Theorem 1.2. Indeed, using this theorem it is clear that if $n\xi$ admits r linearly independent sections, then the additive order of ξ must be a divisor of $2^{b(n,r)}$. Similarly, Theorem 5.2 gives an immediate sharpening of the results of Barufatti (in particular [4], Proposition 3.3).

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