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HOPF MAPS, TRIALITY AND NON CANCELLATION PHENOMENA

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Abstract

We give an overview of some applications to topology of the classical identifications between low dimensional Lie groups. The setting is that of Spin(8), its subgroups and triality. The applications include sections of bundles, formulas relating Hopf maps, generators of homotopy groups and a suggestion for a different model for the study of non cancellation phenomena related to S^3-bundles. This is an expository article and the details and proofs are in [C-R$_1$, C-R$_2$, C-R$_3$, R-C, B, B-R].

Resumo

Apresentamos aqui algumas aplicações à topologia das identificações cássicas entre grupos de Lie compactos de dimensões baixas. O ambiente é o Simp(8), seus subgrupos e a trialidade. As aplicações incluem seções de fibrados, fórmulas relacionando aplicações de Hopf, geradores de grupos de homotopia e a sugestão para o uso de um novo modelo para o estudo de fenômenos de não cancelamento relacionados a S^3-fibrados. O trabalho é expositório e os detalhes e demonstrações encontram-se em $[C-R_2, C-R_3, R-C, B, B-R]$.

Introduction

Non cancellation phenomena in Topology are usually seen by students at the beginning of introductory undergraduate courses [A, p. 25]. They do not seize, nevertheless, to be a source of fascination for students and researchers alike. In this note we consider problems of this nature related to S^3-bundles as the motivation for introducing a setting for their study.

The problems are related to S^3-bundles, principal and associated ones, first posed by Peter Hilton, Joseph Roitberg and collaborators in the mid 60's [H-R, H-M-R$_4$, H-M-R$_5$]. There is a milder non cancellation phenomenon in the same line involving differetiable structures on the 7-sphere, considered in [R] and [B-R]. The setting we propose is that of Cayley multiplication, Spin (8) its subgroups and the concept of Triality [L-M,P, W, C, R_2].

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1. Some Problems

One of the problems posed by P. Hilton and J. Roitberg [H-R] is to produce an explicit diffeomorphism between $Sp(2) \times S^3$ and $E(7) \times S^3$. Here $Sp(2)$ is the group of quaternionic 2×2 matrices A with $AA^* = A^*A = I, S^3$ can be considered as $Sp(1)$, subgroup of $Sp(2)$ included as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)$ and acting from the right on $Sp(2)$ by matrix multiplication producing the principal S^3 bundle

$$S^3 ... Sp(2) \xrightarrow{p} S^7,$$

where the matrix projects onto its first column.

This bundle generates the S^3-principal bundles over S^7 classified by $\pi_6 S^3$ and $E(7)$ is the pullback of the bundle p through a map of degree 7 from S^7 to itself. It was observed in [H-R] that $E(7)$ is a loop space, not of the same homotopy type as the Lie group $Sp(2)$ and it was proved that their product with S^3 results in diffeomorphic manifolds.

Our first observation is that one can view such a non cancellation phenomenon as an exotic action.

Lemma 1. *There exists a free $Sp(1)$ action on $Sp(2) \times Sp(1)$, such that the quotient space is diffeomorphic to $E(7)$.*

Proof: Consider the trivial principal bundle $S^3 ... E(7) \times S^3 \longrightarrow E(7)$ and replace the total space by its diffeomorphic image $Sp(2) \times S^3$.

□

A milder example of such nature involves the differentiable structures on the 7-sphere: It follows from simple considerations on bundles [R$_1$] that for some homotopy 7-spheres Σ^7 the cartesian product $\sum^7 \times S^3$ is diffeomorphic to $S^7 \times S^3$. For at least some of the $\sum^7$'s it is possible to make the diffeomorphism explicit and consequently to write down a formula for the free S^3-action on $S^7 \times S^3$ with quotient diffeomorphic to $\sum^7$ [R$_1$ and B-R for a correction]. This action is of the form

$$q * \left(\begin{pmatrix} a \\ b \end{pmatrix}, h\right) = \left(\begin{pmatrix} \overline{q}aq \\ \overline{q}bq \end{pmatrix}, \overline{F_\theta \begin{pmatrix} \overline{q}aq \\ \overline{q}bq \end{pmatrix}}\ \overline{q}\ F_\theta \begin{pmatrix} a \\ b \end{pmatrix} h\right)$$

where $\theta = \cos^{-1}|a|$ in $[0, \pi/2]$, $\binom{a}{b}$ in $q, h \in S^3$, unit quaternions and all products are quaternionic multiplications. The map F_θ is a homotopy with parameter θ between the following two maps

$(a, b) \mapsto a^m(b\overline{a})^m\overline{b}^m$ and $(a, b) \mapsto 1$, both from $S^3 \times S^3$ to S^3 for certain integer values of m that depend on the diffeomorphism type of Σ^7. Note that this map factors through $S^6 \sim S^3 \wedge S^3$ (homeomorphic).

We can consider these as examples of exotic S^3 actions on products $P \times S^3$, where P is a parameter space on which S^3 acts in a canonical way, P parametrizes a complicated action of S^3 on itself and the quotient has a structure distinct from that of P.

Still, these actions are not completely explicit since, as far as we know, no formula for the homotopy F above exists in the literature.

A similar treatment can be given to the Hilton - Roitberg problem ([B-R], [B]): Here, to describe the free S^3 - action on $Sp(2) \times S^3$ with quotient E(7), in analogy to the homotopy F above, one has homotopies between high powers of double commutators of maps from $S^3 \times S^3 \times S^3 \to S^3$ and the constant map. One such homotopy, for example,is between $(a, b, c) \mapsto ([c^{-1}, b]^{12})^{325}([b^{-1}, a]^{12})^{5772}$ $([a^{-1}, c]^{12})^{325}([a^{-1}, [b, c^{-1}]]^3)^{301730}([c^{-1}, [a^{-1}, b]]^3)^{7720635}$ and $(a, b) \mapsto 1$, where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$ and $[a, [b, c]]$ denotes the double commutator. Again, there are no explicit formulas in the literature for the homotopy between $(a, b, c) \mapsto [a, [b, c]]^3$ and the constant map. We conjecture, however, that there exist elegant solutions for this and similar problems. We suggest that such solutions should come about through the description of sections of certain principal bundles by closed formulas.

In what follows we will try to convince the reader that the above allegation is reasonable considering the setting of Cayley product, Spinors and Triality, in place of quaternions, Sp(2) and submanifolds of Sp(n) that were employed in [R_1, B-R and B].

2. Triality

This term is used today to describe an external automorphism group, isomorphic to S_3, of certain algebraic structures. It was first described by Study and other algebraic geometers at the end of last century and classified by Elie Cartan in the early 20's. We are going to use only the most elementary manifestation of this phenomenon. Much of the elementary material presented below is contained in [H] and [C-R_2].

Definition 2. Given $\mathbb{K} = \mathbb{H} \oplus \mathbb{H}$ we define the Cayley product between $\binom{a}{b}$ and $\binom{c}{d}$ in $\mathbb{K}$ by

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - \bar{d}b \\ da + b\bar{c} \end{pmatrix}$$

using the quaternionic product between the coordinates. This is a non associative division algebra with 1 and the following is true:

Proposition 3. Given any A in $SO(8)$, the special orthogonal group, there exists a unique pair, modulo common sign, $\pm(B, C)$ in $SO(8) \times SO(8)$, such that for all ξ, η in $\mathbb{K}$ we have

$$\text{(T)} \qquad A(\xi\eta) = B(\xi)C(\eta) \ ,$$

where all products are Cayley multiplications.

The proof is relatively simple, based on reflections and two Moufang identities [M], [R_2]. One can identify Spin(8) with the subgroup of the Cartesian product $SO(8) \times SO(8) \times SO(8)$ composed of all (A, B, C) that obey (T). The natural projection onto $SO(8)$ is $(A, B, C) \mapsto A$ and the triality automorphisms of Spin(8) are:

$$\tau(A, B, C) = (\tilde{A}, \tilde{C}, \tilde{B}) \text{ of order } 2,$$
$$\delta(A, B, C) = (C, \tilde{B}, A) \text{ of order } 2,$$
$$\gamma(A, B, C) = \tau \circ \delta(A, B, C) = (\tilde{C}, \tilde{A}, B) \text{ of order } 3,$$

where we have used,

Definition 4. $\tilde{A}(x) = \overline{A(\overline{x})}$, where the bar denotes the conjugate of a Cayley number.

Proposition 5. The canonical inclusion of Spin(7) in Spin(8) is given by $\{(A, B, C)|A(1) = 1\}$. a) This is equivalent to $C = \tilde{B}$ and to $\tilde{A} = A$. b) The canonical Spin(7) $= \{(A, B, \tilde{B})$ in Spin(8)$\}$ is precisely the fixed point set of the automorphism τ.

Proof. a) $1 = A(1) = B(x)C(\overline{x})$ for any x in S^7, the unit sphere in $\mathbb{K}$. So, $\overline{C(\overline{x})} = B(x)$ or $(A, B, \tilde{B})$ in Spin(8), $A(1) = B(x)B(\overline{x})$, for all x in S^7. Take $x = 1, A(1) = B(1)\overline{B(\overline{1})} = B(1) \cdot \overline{B(1)} = 1$. Suppose now $A(1) = 1$ and let x be imaginary, i.e., $\langle x, 1\rangle = 0$. Then $\langle A(x), A(1)\rangle = 0$ or $A(x)$ is imaginary too and $\overline{A(x)} = -A(x)$. For any y in $\mathbb{K}$, $y = y_0 + y_1, y_0$ real, y_1 imaginary, $A(y) = A(y_0) + A(y_1), \tilde{A}(y) = \overline{A(y_0 - y_1)} = \overline{A(y_0)} - \overline{A(y_1)} = A(y_0) + A(y_1) = A(y_0 + y_1) = A(y)$. So, if $A(1) = 1$ then $\tilde{A} = A$. (b) $\tau(A, B, \tilde{B}) = (\tilde{A}, B, \tilde{B}) = (A, B, \tilde{B})$, so τ fixes the canonical Spin(7) pointwise in Spin(8).

Let now $\tau(A, B, C) = (A, B, C)$, i.e., $(\tilde{A}, \tilde{C}, \tilde{B}) = (A, B, C)$. Then $A = \tilde{A}$ and $\tilde{B} = C$, each of which is equivalent to $A(1) = 1$, i.e., τ fixes precicely the canonical Spin(7). □

By the way, δ fixes another subgroup of Spin(8) isomorphic to Spin(7) and γ fixes G_2, the automoprhism group of $\mathbb{K}$ that consists of all (A, A, A) in Spin(8).

Definition 6. Let Spin(6) be the subgroup of Spin(7) defined by

$\{(A, B, \tilde{B}), A(e_1) = e_1\}$,
Spin(5) $= \{(A, B, \tilde{B})$ in Spin(6) $|A(e_2) = e_2\}$,
Spin(4) $= \{(A, B, \tilde{B})$ in Spin(5) $|A(e_3) = e_3\}$,
Spin(3) $= \{(A, B, \tilde{B})$ in Spin(4) $|A(e_4) = e_4\}$.

Here, we are considering a fixed basis for $\mathbb{K}$ with $1 = \binom{1}{0}, e_1 = \binom{i}{0}, e_2 = \binom{j}{0}$, $e_3 = \binom{k}{0}, e_4 = \binom{0}{1}, e_5 = \binom{0}{i}, e_6 = \binom{0}{j}$ and $e_7 = \binom{0}{k}$, so the above are inclusions associated to this fixed basis.

Observe that $(A, B, \tilde{B}) \mapsto B(1)$ defines the various principal bundles over S^7 depending on which one is the total space. More explicitly, it defines G_2...Spin(7)$\to S^7$ when $(A, B, \tilde{B}) \in$ Spin(7),

$$SU(3) \ldots \text{ Spin}(6) \ \to S^7$$

(or $SU(3) \ldots SU(4) \to S^7$) when $(A, B, \tilde{B}) \in$ Spin(6) and

$$S^3 \ldots \text{ Spin}(5) \ \to S^7$$

(or $Sp(1) \ldots Sp(2) \to S^7$) when $(A, B, \tilde{B}) \in$ Spin(5).

We observe that the Moufang identity

$$\alpha(xy)\overline{\alpha} = (\alpha x \alpha^2)(\overline{\alpha}^2 y \overline{\alpha})$$

was first used in [T-S-Y] to describe explicitly the generator of $\pi_7 SO$ and it provides a section of the pull back f_3^*[Spin(7)] over S^7 through the map $f_3(\alpha) = \alpha^3$ from S^7 to itself [R_3]. It is also used to show that a certain conjugate orbit of G_2, extends, in some sense, the Cartan inclusion of a symmetric space into the group, generates $\pi_6(G_2)$ and provides a direct proof that it is isomorphic to $\mathbb{Z}_3$[C-R_1].

In what follows we will concentrate to the bundle $Sp(1) \ldots Sp(2) \to S^7$ and to its equal S^3 ... Spin(5) $\to S^7$.

The identification between Spin(5) and $Sp(2)$ is classical and well known. We will concern ourselves only with the principal bundle as a whole. Observe that the above bundle can be obtained as a pull back of "the negative" of the Hopf bundle over the Hopf map $h_1 : S^7 \to S^4$.

In order words, the following diagram is commutative, where $-h_1$ is $(-i_4) \circ h_1, -i_4$ denoting the antipodal map of the sphere S^4:

$$
\begin{array}{ccccc}
 & & S^3 & & S^3 \\
 & & \vdots & & \vdots \\
S^3 & \dots & Sp(2) & \xrightarrow[\text{2}^{\text{nd}}\text{Col.}]{} & S^7 \\
 & & \downarrow{\scriptstyle 1^{\text{st}}\text{Col}} & & \downarrow{\scriptstyle -h_1} \\
S^3 & \dots & S^7 & \xrightarrow[h_1]{} & S^4
\end{array}
$$

In fact, $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in Sp(2)$ implies $a\bar{b} = -c\bar{d}, ||a|| = ||d||, ||b|| = ||c||$ and $||a||^2 + ||b||^2 = 1, h_1$ is the classical Hopf map defined by

$$h_1 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ||a||^2 - ||b||^2 \\ 2a\bar{b} \end{pmatrix}$$

and therefore, $-h_1 \begin{pmatrix} c \\ d \end{pmatrix} = h_1 \begin{pmatrix} a \\ b \end{pmatrix}$.

One has the analogous diagram in the Spin notation [$C - R_3$],

$$
\begin{array}{ccccc}
 & & S^3 & & S^3 \\
 & & \vdots & & \vdots \\
S^3 & \dots & \text{Spin}(5) & \longrightarrow & S^7 \\
 & & \downarrow & & \downarrow{\scriptstyle -h} \\
S^3 & \dots & S^7 & \xrightarrow[h]{} & S^4
\end{array}
$$

where the horizontal map sends $(A, B, \tilde{B})$ to $B(e_4)$ and the vertical one to $B(1)$. The analog of the Hopf map h_1 is $h(\alpha) = (e_1\overline{\alpha})(\alpha e_2), h : S^7 \to S^4 \subseteq \mathbb{R}^5 = Span\{e_3, \dots, e_7\}$ and $-h(\beta) = \overline{h(\beta)} = (e_2\overline{\beta})(\beta e_1)$. The diagram is commutative with the diagonal map from Spin(5) to S^4 being equal to $(A, B, \tilde{B}) \mapsto A(e_3)$.

The proofs are quite simple. It is clear that had the Cayley numbers been an associative field h would have been the constant map $\alpha \mapsto e_3$. More is true: h is homotopically non-trivial because the Cayley product is non-associative up to homotopy.

3. Applications

Triality interacts with Hopf maps associacted to h in the following manner:

Observe first that the range of the Hopf map h defined above depends on the choice of the complex units e_1 and e_2. One can generalize it by defining $h(\alpha; J, K) = (J\overline{\alpha})(\alpha K)$ for any element (J, K) of $V_{7,2}$, the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^7 = Im\ \mathbb{K}$. The image will be in the particular $S^4_{J,K} \subseteq \mathbb{R}^5_{J,K} = \{J, K\}^\perp$ in $Im\mathbb{K}$.

In general the image is in S^6, the unit sphere in $Im\mathbb{K}$. One also has the conjugation of Cayley numbers defining $H' : S^7 \times S^6 \to S^6$ by $H'(\alpha, J) = \alpha J \overline{\alpha}$, where S^6 is the unit sphere in $Im\ \mathbb{K}$. A rapid calculation shows this formula to be very close to that of the classical Hopf map h_1. Analogously, we have that $H : S^7 \to V_{7,2}$ defined by $H(\alpha) = \{\alpha e_1 \overline{\alpha}, \alpha e_2 \overline{\alpha}\}$, generates $\pi_7(V_{7,2}) \cong \mathbb{Z}_4$.

Theorem 8 ([R-C]): *For any α in S^7, m, n in $\mathbb{Z}$ and (e_1, e_2) the base point of $V_{7,2}$ the Hopf type maps h', H and H' satisfy*

$$h'(\alpha^m; H(\alpha^n)) = H'(\alpha^n, h'(\alpha^{m+3n}; e_1, e_2)).$$

Corollary 9 ([R-C]): (i) *There exists a smooth map*

$$\varphi : S^7 \to S^4, \text{ such that } \langle h(\alpha^{12}), \varphi(\alpha)\rangle = 0$$

for all α in S^7.

(ii) *A generator of π_7 Spin(5) and $\pi_7 Sp(2)$ can be written explicitly in terms of h and φ.*

(iii) *A section of $S^3 ... E(12) \to S^7$ can be written explicitly in terms of the generator of $\pi_7 Sp(2)$.*

As a consequence, φ will furnish the free actions of S^3 on $S^7 \times S^3$ that give $\sum^7$ as a quotient.

It will also shed some light on the homotopy between $a^m(b\overline{a})^m\overline{b}^m$ and 1 from $S^3 \times S^3 \to S^3$ for the appropriate values of the integer m and also on the related classical problem of J.-P. Serre [Se]: describe the homotopy between $(a, b) \mapsto [a, b]^{12}$ and a constant.

There are some indications that the Hilton - Roitberg problem can be presented in this context as well.

A geometric application of the above setting is a description of the symmetric space $G_2/SO(4)$ embedded in G_2 through the classical Cartan embedding [C-E].

Proposition 10 ([R-C]): *Let* $c : G_2/SO(4) \to G_2$ *be the Cartan embedding. If [B] is in* $G_2/SO(4)$*, then* $c([B])$ *is* $L_{B_1 \cdot B_2} \cdot L_{B_2} \cdot L_{B_1}$ *where* B *is any matrix in* G_2 *whose class modulo* $SO(4)$ *is [B],* B_i *are its columns,* $B_1 \cdot B_2$ *is the Cayley product of the first two and* L_x *denotes Cayley multiplication from the left by* x.

Other applications of triality and of infinitesimal triality [R_2], to geometry include considerations on the geometric structure of non standard 15 spheres. Details will appear elsewhere.

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