Matemática Contemporânea, Vol. 13, 1–19
http://doi.org/10.21711/231766361997/rmc131

ON THE ACTION OF SEMIGROUPS IN FIBER BUNDLES

Carlos José Braga Barros **Luiz A. B. San Martin***

Abstract

This paper studies the behavior of control sets for semigroup actions on principal bundles and their associated bundles. The emphasis is put on the description of those sets from their projections onto the base space and their intersections with the fibers.

Resumo

O objetivo deste artigo é estudar os conjuntos controláveis para a ação de um semigrupo em um fibrado principal e em seus fibrados associados, enfatizando a descrição desses conjuntos através de suas projeções no espaço de base e de suas interseções com as fibras.

1. Introduction

Semigroup actions on homogeneous spaces and fiber bundles have received quite some attention lately. This is in part due to the envisaged applications of this theory in dynamical systems, control theory or spectral theory of flows on vector bundles. One of the central ideas in these applications is the study of transitivity of the semigroup action, that is to say, the analysis of the control sets.

The present paper studies the control sets of semigroups acting of fiber bundles using topological methodology. The main object to be considered are the control sets for this action (see Definition 2.1 below). Our purpose is to describe these sets from their projections onto the base space and their intersections with the fibers. More specifically, let $Q(M,G)$ be a principal bundle with base space M, total space Q and structure group G, and $E(M,F,G,Q)$ a

*Research partially supported by CNPq grant n° 301060/94-0

fiber bundle associated to Q with typical fiber F. Let also S_Q be a semigroup of automorphisms of Q. Then S_Q induces semigroups S_E and S_M acting on E and M respectively. We study here the control sets for the S_E-action on E. We first check that the projection onto M of the S_E-control sets are control sets for the S_M-action so that we are led to the analysis of the S_E-action over a control set. For this we suppose that the orbits $S_Q q$, $q \in Q$ have nonempty interior. With this assumption, the intersection of $S_Q q$ with the fiber through q identifies with a semigroup, say S_q, which has nonvoid interior in the structure group G. Roughly speaking, the intersection with a fiber of a control set for S_E is shown to be a control set for the action of S_q on the typical fiber F. So that we get a description of the control sets on E as soon as we have a procedure of bunching together the control sets on the different fibers. We present such a procedure for the invariant control sets, obtaining these sets as the union of invariant control sets on the fibers (see Theorem 4.4 below).

2. Preliminaries on control sets

In this section we recall the definition of control sets for semigroup actions and present the results that will be used throughout the paper. We refer to [1], [3], for results on control sets for control systems and [4], [9], [10], [11], for control sets for semigroup actions.

Let S be a semigroup of diffeomorphisms acting on a manifold M.

Definition 2.1 *A control set for S on M is a subset $D \subset M$ satisfying*

1. $\operatorname{int} D \neq \emptyset$,
2. $D \subset \operatorname{cl}(Sx)$ *for all* $x \in D$, *and*
3. D *is maximal satisfying these two properties.*

An *invariant* control set is a control set D which besides of 2) satisfies $\operatorname{cl}(Sx) = \operatorname{cl} D$ for all $x \in D$. We note that the term invariant here is not properly used in the sense that invariant control sets are not always S-invariant (see [8] for an example of an invariant control set which is not S-invariant). However, in the context we work here, namely with the assumption that S has the accessibility property (i.e., $\operatorname{int} S(x) \neq \emptyset$ for all $x \in M$) invariant control sets are indeed S-invariant and closed subsets (see [2], [5]).

Suppose that both S and $S^{-1} = \{g^{-1} : g \in S\}$ satisfy the accessibility property. Let D be a control set for S and define

$$D_0 = \{x \in D : x \in \operatorname{int}(Sx) \cap \operatorname{int}(S^{-1}x)\}.$$

Then we have the following result which is a slight extension of Proposition 2.2 of [10].

Proposition 2.2. *Suppose $D_0 \neq \emptyset$. Then*

1. $D \subset \operatorname{int}(S^{-1}x)$ *for every* $x \in D_0$.
2. $D_0 = \operatorname{int}(S^{-1}x) \cap \operatorname{int}(Sx)$ *for every* $x \in D_0$
3. *For every* $x, y \in D_0$ *there exist* $g \in S$ *with* $gx = y$.
4. D_0 *is dense in* D
5. D_0 *is S-invariant on D, in the sense that* $hx \in D_0$ *if* $h \in S$, $x \in D_0$ *and* $hx \in D$.

Proof.

1. Take $y \in D$ and $x \in D_0$. Since $x \in \operatorname{int}(S^{-1}x) \cap D$ there exists $g \in S$ such that $gy \in \operatorname{int}(S^{-1}x)$. Thus $y \in g^{-1}\operatorname{int}(S^{-1}x) \subset \operatorname{int}(S^{-1}x)$.
2. Suppose $x \in D_0$ and $y \in \operatorname{int}(Sx) \cap \operatorname{int}(S^{-1}x)$. We have that $\operatorname{int}(Sx) = \operatorname{int}(Sy)$ and $\operatorname{int}(S^{-1}x) = \operatorname{int}(S^{-1}y)$. Therefore $y \in D_0$. On the other hand, if $x, y \in D_0$ we have by 1), $y \in \operatorname{int}(S^{-1}x)$ and $x \in \operatorname{int}(S^{-1}y)$. Thus, there exists $g \in S$ such that $gx = y$. Since $x \in \operatorname{int}(Sx)$ we have that $y = gx \in g\operatorname{int}(Sx) \subset \operatorname{int}(Sx)$. Therefore $y \in \operatorname{int}(Sx) \cap \operatorname{int}(S^{-1}x)$.

3. Follows immediately from $D_0 \subset \text{int}\,(Sx) \cap \text{int}\,(S^{-1}x)$.

4. Take $x \in D_0$. Since $\text{int}\,(Sx)$ and $\text{int}\,(S^{-1}x)$ are open we have

$$\text{cl}\, D_0 = \text{cl}(\text{int}\,(Sx)) \cap \text{int}\,(S^{-1}x).$$

   Now, $D \subset \text{int}\,(S^{-1}x)$ by 1). On the other hand

$$D \subset \text{cl}(Sx) \subset \text{cl}(S\,\text{int}\,(Sx)) \subset \text{cl}(\text{int}\,(Sx)).$$

   Therefore $D \subset \text{cl}\, D_0$ and D_0 is dense in D.

5. Take $h \in S, x \in D_0$ and suppose that $hx \in D$. Since $x \in \text{int}\,(Sx)$ we have $hx \in \text{int}\,(Sx)$. We also have by 1) that $hx \in \text{int}\,(S^{-1}x)$, and therefore $hx \in D_0$.

□

We say that D_0 is the *set of transitivity* in D, and D is named an *effective* control set if $D_0 \neq \emptyset$.

It is not difficult to show that $D_0 = \{x \in D : x \in (\text{int}\, S)x\}$ if S is a subsemigroup of a Lie group G with interior points in G and M is a homogeneous manifold of G. So that the above proposition implies [10 Prop.2.2].

As a complement to the above proposition we have the following statement which ensures the existence of effective control sets.

Proposition 2.3. *Let $x \in M$ be such that*

$$x \in \text{int}\ (Sx) \cap \text{int}\ \left(S^{-1}x\right).$$

Then there exists a unique control set D such that $x \in D_0$, which is effective.

Proof. The subset $\text{int}\ (Sx) \cap \text{int}\ (S^{-1}x)$ satisfies properties 1) and 2) of Definition 2.1. Therefore if we apply the Principle of Maximality of Hausdorff to the subsets satisfying these properties and containing x, we get a maximal one, say D, which is a control set. This control set clearly contains x in its set of transitivity. Also, the uniqueness follows from the fact that different control sets do not overlap.

□

At this point we mention that for semigroups generated by families of vector fields, which appear in the literature of control theory, there are more precise statements than the above ones: Let Ω be a family of vector fields on M and put

$$S_{\Omega} = \{X^1_{t_1} \circ \cdots \circ X^k_{t_k} : X^i \in \Omega, t_i \geq 0\}$$

where X_t stands for the flow of the vector field X. Then S_{Ω} is a semigroup of (local) diffeomorphisms on M. Under the well known Lie algebra rank condition both S_{Ω} and S_{Ω}^{-1} are accessible, and if this holds, property 2) of the definition of a control set imply that D is effective if it has nonempty interior (see e.g. [3]). In this case we have that $D_0 = \operatorname{int} D$.

3. Fiber bundles

In this section we study the control sets for semigroup actions on principal bundles and their associated bundles.

For the theory of principal bundles and their associated bundles we follow [6], and adhere to its notations. In particular, we consider here only locally trivial bundles.

Let $Q(M, G)$ be a principal bundle having total space Q, base space M and structure group G. We denote by $\pi : Q \to M$ the canonical projection, and the right action of G on Q is denoted by R_a, or simpler by $q.a$ or qa, $q \in Q$, $a \in G$. This action is free and transitive on the fibers, so that if we fix $q \in \pi^{-1}(x) = Q_x$ in the fiber over $x \in M$, this fiber identifies with G by

$$i : a \in G \longmapsto q.a \in \pi^{-1}(x) . \tag{1}$$

We remark that since Q is supposed to be locally trivial, it follows that this identification is actually a diffeomorphism.

Suppose that G acts on the left on a manifold F and consider the associated bundle to the principal bundle $Q(M, G)$ with typical fiber F. This bundle will be denoted by $E(M, F, G, Q)$, or simpler by E, the total space of the bundle.

We recall that the elements of E are equivalence classes with respect to the relation on $Q \times F$ given by $(q, v) \sim (q.a, a^{-1}v)$ for $a \in G$. We use the notation $q \cdot v$, $q \in Q, v \in F$ for the equivalence class of (q, v).

The canonical projection $\pi : E \longrightarrow M$ of the associated bundle is related to that of the principal bundle by $\pi(q \cdot v) = \pi(q)$.

There are the following mappings: A fixed $q \in Q$ induces a bijection $v \in F \longmapsto q \cdot v \in E_x$ where $E_x = \pi^{-1}(x)$ and $x = \pi(q)$, between the fiber over x and the typical fiber. Sometimes we also denote this mapping by q. Note that since the principal bundle Q is locally trivial it follows that this bijection between F and E_x is actually a diffeomorphism. On the other hand, a fixed $v \in F$ defines a map $v' : Q \longrightarrow E$ by $v'(q) = q \cdot v$. If the action of G on F is transitive this map is an onto submersion.

Let S_Q be a semigroup of local diffeomorphisms of Q commuting with the right action, that is, if $\phi \in S_Q$ then $\phi(q.a) = \phi(q).a$ for all $a \in G$. We suppose that the elements of S_Q are defined in subsets of the form $\pi^{-1}(U)$, with U an open set of M. Then S_Q induces a semigroup S_M of local diffeomorphisms of M. We use the same symbol ϕ to denote the diffeomorphism of M induced by $\phi \in S_Q$. It is given by

$$\phi(\pi(q)) = \pi(\phi(q)).$$

Similarly, $\phi \in S_Q$ induces a local diffeomorphism, also denoted by ϕ, of the associated bundle E. It is given by the formula

$$\phi(q \cdot v) = \phi(q) \cdot v.$$

The corresponding semigroup of diffeomorphisms acting in E is denoted by S_E.

In case the semigroup S_Q is generated by a family Ω of vector fields on Q, its elements commute with the right action if and only if the vector fields are right invariant, i.e., $R_{a*}(X) = X$ for all $X \in \Omega$ and $a \in G$, where R_{a*} stands for the differential of the right action. In this situation, the vector fields in Ω are projectable onto M and S_M is generated by these projections.

By the very definitions, the accessibility of S_Q imply that S_M is also accessible. On the other hand, we remark that if S_Q is generated by a family of right invariant vector fields, its accessibility is a consequence of the accessibility of S_M and S_M^{-1}, and the transitivity on Q of the group G_Q generated by S_Q (see [7]). In the sequel our basic assumption is that both S_Q and S_Q^{-1} are accessible. However, since we wish to work over control sets of S_M, the following stronger version of accessibility is needed.

Definition 3.1. *Let D be an effective control set for S_M. The semigroup S_Q is said to be accessible over D if for some, and hence for all, $q \in \pi^{-1}(D_0)$, $\text{int}\,(S_Q q) \cap \pi^{-1}(D) \neq \emptyset$. Similarly, S_E is accessible over D if $\text{int}\,(S_E u) \cap \pi^{-1}(D_0) \neq \emptyset$.*

Given $q \in Q$, the intersection $S_Q(q) \cap Q_x$, $x = \pi(q)$ can be seen as subset of G by putting

$$S_q = \{a \in G : \exists \phi \in S_Q, \phi(q) = q.a\}. \tag{2}$$

It is readily seen that S_q is a subsemigroup of G if $S_q \neq \emptyset$. The subset

$$T_q = \{a \in G : \exists \phi \in S_Q, q.a = \phi(q) \in \text{int}\,(S_Q q)\} \tag{3}$$

which is identifies to $\text{int}\,(S_Q q) \cap Q_x$ is also a semigroup in G. Since the identification (1) is a diffeomorphism we have that $T_q \subset \text{int}\, S_q$, where the interior of S_q is taken in G. The following lemma ensures that in presence of accessibility T_q is not empty.

Lemma 3.2. *Let $D \subset M$ be an effective control set for S_M and suppose that S_Q is accessible over D. Let $q \in \pi^{-1}(D_0)$. Then $T_q \neq \emptyset$ so that $\text{int}\,(S_q) \neq \emptyset$.*

Proof. Take $\phi \in S_Q$ such that $\phi(q) \in \text{int}\,(S_Q q) \cap \pi^{-1}(D)$. Since $x = \pi(q) \in D_0$, there exists $\psi \in S_Q$ such that the corresponding diffeomorphism of M takes $\pi(\phi(q))$ back into x, so that $\psi\phi(q)$ belongs to the same fiber as q. By the choice of ϕ we have that $\psi\phi(q) \in \text{int}\,(S_Q q)$ showing that T_q has interior points as claimed.

□

An immediate consequence of this lemma is that the accessibility of S_Q over D implies that S_Q^{-1} is also accessible over D. In fact, $T_q^{-1} = \{a^{-1} : a \in T_q\}$ identifies with int $\left(S_Q^{-1} q\right) \cap Q_x$ so that int $\left(S_Q^{-1} q\right)$ is not empty if $T_q \neq \emptyset$.

Before proceeding we mention that if q and $q' = q.a$ are in the same fiber then the semigroups $S_{q'}$ and S_q are related by

$$S_{q'} = a^{-1} S_q a.$$

In fact, take $c \in S_{q'}$ and let $\phi \in S_Q$ be such that $\phi(q') = q'.c$. Then $\phi(q).a = \phi(q.a) = q.ac$, that is, $\phi(q) = q.aca^{-1}$. This shows that $aca^{-1} \in S_q$, and hence that $S_{q'} \subset a^{-1} S_q a$. The reverse inclusion is shown the same way by writing $q = q'.a^{-1}$. With the same arguments it follows that $T_{q'} = a^{-1} T_q a$.

Given a fiber bundle $E(M, F, G, Q)$ associated to Q we have that S_Q induces a semigroup S_E of diffeomorphisms of E. We shall assume in the sequel that the action of G on the typical fiber is transitive. In this case, the map $q \longmapsto q \cdot v$ mentioned above is an onto submersion so that the accessibility of S_Q implies that of S_E.

We are interested in the control sets for S_E. For this we note that for each $q \in Q$, the semigroup S_q acts on the typical fiber F. So the idea for studying control sets on E is by breaking down the action of S_E into the action of S_M on the base space and the action of S_q on F with q running through the different fibers of Q. We start by looking at the behavior of the control sets under projections.

Proposition 3.3. *Let E be the above fiber bundle with projection $\pi : E \to M$ and S_Q, S_E and S_M the above semigroups. Let $D \subset E$ be a control set for S_E. Then*

1. *there exists a unique control set $C \subset M$ for S_M and such that $\pi(D) \subset C$. If D is invariant then C is also invariant.*

2. If D is effective then C is also effective and $\pi(D_0) \subset C_0$.

Proof.

1. Let $x, y \in \pi(D)$, and take $u, v \in D$ such that $\pi(u) = x$ and $\pi(v) = y$. By definition there is a sequence $\phi_n \in S_E$ such that $\phi_n(u) \to v$. Therefore $\pi(\phi_n(u)) = \phi_n(\pi(u)) \to \pi(v) = y$. Hence $y \in \mathrm{cl}(S_M x)$. Since x and y were arbitrary this shows that $\pi(D)$ satisfies the second condition for a control set. Moreover, π is an open map hence $\pi(D)$ has nonempty interior so that it is contained in a control C for S_M. The uniqueness of C is due to the fact that the intersection of different control sets is empty and if D is invariant then the invariance of C follows form $\pi(\phi(u)) = \phi(\pi(u))$ for every $u \in E$.

2. Take $v \in D_0$. Then $v \in \mathrm{int}\,(S_E v) \cap \mathrm{int}\left(S_E^{-1} v\right)$. This implies that $\pi(v) \in \mathrm{int}\,(S_M \pi(v)) \cap \mathrm{int}\left(S_M^{-1} \pi(v)\right)$ which shows that $\pi(v) \in C_0$. Hence C_0 is not empty and C is effective.

□

This proposition defines a mapping $D \longrightarrow C$ from the control sets in E into the control sets in M. We show next that in the case E is compact and in presence of accessibility every effective control set in M is in the image of this mapping.

Proposition 3.4. *Let $C \subset M$ be an effective control set for S_M. Suppose that the typical fiber F is compact and that S_Q is accessible over C. Then there exists an effective control set $D \subset E$ for S_E with $\pi(D) \subset C$ and $\pi(D_0) \subset C_0$.*

Proof. Take $x \in C_0$ and $q \in Q_x$. By Lemma 3.2 T_q is an open and nonvoid semigroup in G. On the other hand, the compactness of F implies that there exists an effective control set for T_q on F (see e.g. Section 2 of [9]). Hence there are $a \in T_q$ and $v \in F$ such that $av = v$. Of course $q.a \in \mathrm{int}\,(S_Q q)$, and since the map $q \in Q \mapsto q \cdot v \in E$ is open we have that $qa \cdot v \in \mathrm{int}\,(S_E (q \cdot v))$. But

$qa \cdot v = q \cdot av = q \cdot v$ so that $q \cdot v \in \text{int}\,(S_E\,(q \cdot v))$. Arguing the same way with S_Q^{-1} instead of S_Q, we conclude that

$$q \cdot v \in \text{int}\,(S_E\,(q \cdot v)) \cap \text{int}\,\left(S_E^{-1}\,(q \cdot v)\right).$$

Therefore Proposition 2.3 implies that $q \cdot v$ belongs to an effective control set, say D, for S_E. Since the projection of $q \cdot v$ is $x \in C_0$ we get from the previous proposition that $\pi\,(D) \subset C$ and $\pi\,(D_0) \subset C_0$.

□

Concerning the intersection of a control set with a fiber, we have

Theorem 3.5. *Let $D \subset E$ be an effective control set for S_E. Take $x \in M$ such that $D_0 \cap E_x \neq \emptyset$ and let $q \in Q_x$. Then there exists an effective control set A for S_q in F such that*

$$D_0 \cap E_x = q \cdot A_0.$$

Proof. Define the subset $B_0 \subset F$ by $D_0 \cap E_x = q \cdot B_0$. We have that B_0 is open in F because D_0 is open in E and q defines a diffeomorphism between F and E_x. Take $v_1, v_2 \in B$ and put $u = q \cdot v_1$ and $w = q \cdot v_2$ with $u, w \in D_0 \cap E_x$. Since $u, w \in D_0$ we have by Proposition 2.2 that there is $\phi \in S_E$ such that $\phi(q \cdot v_1) = q \cdot v_2$. Clearly, this map fixes the fiber over x so that in the principal bundle level, $\phi\,(q) = q.a$ for some $a \in G$. Since ϕ is in the semigroup we have that $a \in S_q$. However $\phi(q \cdot v_1) = \phi(q) \cdot v_1 = (q.a) \cdot v_1 = q \cdot av_1$, hence $av_1 = v_2$. This shows that $B_0 \subset S_q x$ for all $x \in B$. Therefore there is a control set A for S_q such that $B_0 \subset A$, i.e., $D_0 \cap E_x \subset q \cdot A$.

We have that A is effective and that $D_0 \cap E_x \subset q \cdot A_0$. In fact, take $v \in B_0$. Then $v \in A$ and for every $v_1 \in B_0$ there are $\phi \in \text{int}\, S_E$ and $\psi \in \text{int}\, S_E^{-1}$ such that $\phi\,(q \cdot v) = q \cdot v_1$ and $\psi\,(q \cdot v) = q \cdot v_1$. Each one of these maps fixes the fiber over x. Hence we can write $\phi\,(q) = q.a$ and $\psi\,(q) = q.b$ with $a \in T_q$ and $b \in T_q^{-1}$. Clearly, both a and b map v into v_1 so that $B_0 \subset \text{int}\, S_q v \cap \text{int}\, S_q^{-1} v$ and

$$v \in \text{int}\,(S_q v) \cap \text{int}\,\left(S_q^{-1} v\right).$$

Hence A is effective and $B_0 \subset q \cdot A_0$.

The reverse inclusion follows if we prove that for any $z \in q \cdot A_0$ there exists $u \in D_0$ such that $z \in S_E u$ and $u \in S_E z$. In fact in this case we have that

$$D \subset \mathrm{cl}\, S_E u \subset \mathrm{cl}\, S_E z$$

because $u \in S_E z$, and reciprocally, for any $w \in D$, $u \subset \mathrm{cl}\, S_E w$ so that

$$z \in S_E u \subset \mathrm{cl}\, S_E w.$$

Therefore the maximality property in the definition of a control set ensures that $z \in D$. Furthermore, the fact that $z \in S_E u$, $u \in D_0$ implies, by Proposition 2.2, that $z \in D_0$.

Now, given $z = q \cdot v \in q \cdot A_0$ take $u \in D_0 \cap E_x$. Then $u \in q \cdot A_0$ so that $u = q \cdot v_1$ with $v_1 \in A_0$. Hence there are $a, b \in S_q$ such that $av = v_1$ and $bv_1 = v$. By the definition of S_q, we have that $\phi(q) = q.a$ and $\psi(q) = q.b$ for some $\phi, \psi \in S_Q$. Therefore if we consider the corresponding mappings $\phi, \psi \in S_E$, we have

$$\phi(z) = \phi(q \cdot v) = \phi(q) \cdot v = q \cdot av = q \cdot v_1 = u$$

and

$$\psi(u) = \psi(q \cdot v_1) = \psi(q) \cdot v_1 = q \cdot bv_1 = q \cdot v = z$$

concluding the proof.

□

Joining together the previous results we can show that control sets for T_q on the fibers are contained in control sets in the bundle.

Proposition 3.6. *Let E be the above fiber bundle and suppose that for $q \in Q$, $T_q \neq \emptyset$. Let also B be an affective control set for T_q in F. Then there exists an effective control set D for S_E in E such that $q \cdot B_0 = D_0 \cap E_x$.*

Proof. Take $q \cdot v \in B_0$ and put $u = q \cdot v$. Then as in the proof of Proposition 3.4 there exists a control set, say D, for S_E such that $u \in D_0$. The above theorem ensures then that $D_0 \cap E_x = q \cdot B_0$.

□

Remark: In the above proposition we are assuming the existence of a control set for the semigroup T_q on the fiber F so we do not require compactness of the fibre, as in Proposition 3.4, where this assumption is made to ensure the existence of a control set. Also, it is not clear that effective control sets for S_q are contained in effective control sets in the bundle. This is because the interior of S_q may contain boundary points of S_Q, i.e., $\mathrm{int}\, S_q$ may be larger than T_q.

With the above proposition we conclude our results about general control sets on fiber bundles. Later on we shall get more precise statements about invariant control sets. In order to finish this section we consider the case where one of the semigroups S_q acts transitively on the typical fiber, or equivalently, the intersection of a control set with the fiber is the whole fiber. In this case we get transitivity of the semigroup above the control sets.

Proposition 3.7. *Assume that S_M is accessible, let $C \subset M$ be an effective control set, and suppose that for some $x \in C_0$ there exists $q \in Q_x$ such that S_q acts transitively on F. Then for any $p \in Q$ such that $\pi(p) \in C_0$ we have that S_p is transitive on F. Moreover, S_E is accessible over C, $D = \pi^{-1}(C)$ is an effective control set on E and S_E is transitive above C_0.*

Proof. Pick $y \in C_0$. Then there are $\phi, \psi \in S_M$ such that $\phi(x) = y$ and $\psi(y) = x$. By considering the corresponding actions on the principal bundle Q put $p = \phi(q)$. Then $\psi(p)$ is in the fiber of q so that there is $a \in G$ such that $\psi(p) = q.a$. Let us show that S_p is also transitive on F. Take $v_1, v_2 \in F$. Then there exists $b \in S_q$ such that $bav_1 = v_2$, because S_q is transitive. This element of S_q is given by $\theta(q) = q.b$. We have that $\phi\theta\psi$ belongs to S_Q. But

$$\phi\theta\psi(p) = \phi\theta(q.a) = \phi(q.ba) = p.ba$$

so that $ba \in S_p$. Hence $v_2 \in S_p v_1$ showing the transitivity of this semigroup. Since $y \in C_0$ was arbitrary and $S_{p.a} = a^{-1} S_p a$ we have the first part of the proposition. From this we get immediately that S_E is transitive on $\pi^{-1}(C_0)$, that is, $\pi^{-1}(C_0) \subset S_E u$ for all $u \in \pi^{-1}(C_0)$. In particular, S_E is accessible over C. Taking closures and using the equalities $\mathrm{cl}\, C = \mathrm{cl}\, C_0$ and $\pi^{-1}(\mathrm{cl}\, C_0) =$

$\mathrm{cl}\,\pi^{-1}(C_0)$ we have

$$\pi^{-1}(C) \subset \pi^{-1}(\mathrm{cl}\,C) \subset \mathrm{cl}\,\pi^{-1}(C_0) \subset \mathrm{cl}\,S_E u$$

for all $u \in \pi^{-1}(C_0)$. On the other hand, if $\pi(v) \in C$ then $S_E v \cap \pi^{-1}(C_0) \neq \emptyset$ so the above inclusion implies that $\pi^{-1}(C) \subset \mathrm{cl}\,S_E v$. This shows that $\pi^{-1}(C)$ is an effective control set, say D', for S_E. However, the previous proposition ensures the existence of a control set D such that $E_x \subset D_0$ and $\pi(D) \subset C$, that is, $D \subset \pi^{-1}(C)$. Hence we have that

$$D \subset \pi^{-1}(C) \subset D'$$

which shows that $D = \pi^{-1}(C)$.

□

This proposition applies in particular to principal bundles, in which case the assumption that S_q is transitive amounts to say that S_q coincides with the structure group G. In case G is compact and connected this assumption is automatic because in such a group G itself is the only of its semigroups with nonempty interior. Hence we have

Corollary 3.8. *Let Q be such that its structure group G is compact and connected. Let C be an effective control set on the base M and assume that S_Q is accessible over C. Then $\pi^{-1}(C)$ is an effective control set for S_Q on Q and S_Q is transitive above C_0.*

In this corollary it is not needed to require that G is connected as soon as we know that the total space of the bundle is itself connected. In fact, we have

Proposition 3.9. *With the set up as in the above corollary, suppose that $\pi^{-1}(C_0)$ is connected instead of assuming that G is connected. Then the same result holds.*

Proof. Let $q \in Q$ be such that $\pi(q) \in C_0$. We have that T_q is a semigroup with nonvoid interior in the compact group G. Hence $G_0 \subset T_q$ where G_0 stands for the component of the identity of G. This implies that $1 \in T_q$ so that

$q \in \text{int}\,(S_Q q)$. Since $q \in \pi^{-1}(C_0)$ was arbitrary we have that $S_Q p \cap \pi^{-1}(C_0)$ is open for every $p \in \pi^{-1}(C_0)$. Therefore if we take into account that $\pi^{-1}(C_0)$ is connected a standard argument shows that $S_Q p \cap \pi^{-1}(C_0) = \pi^{-1}(C_0)$ for all $p \in \pi^{-1}(C_0)$. Hence S_Q is transitive over C_0 and the result follows as above.

□

4. Invariant control sets

For the invariant control sets the results of the previous section can be highly improved so we can get a clear picture of the construction of the invariant control sets on E from those on the base space and on the fibers.

We consider as above a semigroup S_Q acting on the principal bundle Q, which induces semigroups S_E acting on an associated bundle and S_M acting on the base space.

Any invariant control set for S_E projects down onto an invariant control set on M. So in order to get the invariant control sets on E it remains to analyze the structure of the invariant control sets over a given control set $C \subset M$. We shall do this with the further assumption that the typical fiber is compact.

Given such a set, we assume that S_Q is accessible over C. At this regard, it is convenient to mention that accessibility over C is equivalent to the accessibility of S_Q from the points above C, because of the invariance of this set. Also, this accessibility implies that C is closed.

The semigroup $S_q \subset G$ given by the intersection of $S_Q q$ with the fiber through $q \in Q$ has nonempty interior in G. Let us assume that there are a finite number of invariant control sets for S_q on the typical fiber F. Denote this number by $\text{ic}\,(S_q)$ and enumerate the invariant control sets by C_q^j, $j = 1, \ldots, \text{ic}\,(S_q)$. These subsets are contained in F. However by applying q we can map them into the fiber E_x over $x = \pi(q)$ and obtain the subsets

$$q \cdot C_q^j \subset E_x, \; j = 1, \ldots, \text{ic}\,(S_q).$$

These subsets will be the building blocks for the construction of the invariant control sets over C. We first point out that $q \cdot C_q^j$ is actually independent of the specific $q \in Q$ in the fiber over x. Indeed, if $q' = qa$ then $S_{q'} = a^{-1}S_q a$. Hence the invariant control sets for $S_{q'}$ on F are the subsets $C^j_{q'} = a^{-1}C^j_q$. So that

$$q' \cdot C^j_{q'} = q' \cdot a^{-1}C^j_q = q \cdot C^j_q.$$

Therefore we are allowed to put

$$C^j_x = q \cdot C^j_q, \;\; j = 1, \ldots, \mathrm{ic}\,(S_q)\,.$$

We have now the following lemmas, which we state with the assumption that the fiber F is compact, although this condition is not always required.

Lemma 4.1. *For $x, y \in C_0$ take $\phi \in S_Q$ such that $\phi(Q_x) = Q_y$, and given $q_x \in Q_x$ put $q_y = \phi(q_x)$. Let $C^j_{q_x}$ be an invariant control set for S_{q_x}. Then for every $v \in C^j_{q_x}$, we have that $C^j_{q_x} \subset \mathrm{cl}\left(S_{q_y}(v)\right)$.*

Proof. Pick an arbitrary $w \in C_{q_x}$. We wish to find a sequence $(b_k)_{k\geq 1}$ in S_{q_y} with $b_k v \to w$ as $k \to \infty$. Since S_M is transitive on C_0 there exists $\psi \in S_Q$ such that $\psi \circ \phi(q_x) \in Q_x$. Let $a \in G$ defined by $\psi \circ \phi(q_x) = q_x a$. Then $a \in S_{q_x}$ so that $av \in C^j_{q_x}$. Since $w \in C^j_{q_x}$, there exists a sequence $(a_k)_{k\geq 1}$ in S_{q_x} with $a_k a v \to w$. Let $b_k = a_k a$ and let us show that $b_k \in S_{q_y}$. We have that a_k is defined by $\psi_k(q_x) = q_x a_k$ with $\psi_k \in S_Q$. Now

$$\begin{aligned} \phi \circ \psi_k \circ \psi(q_y) &= \phi \circ \psi_k \circ \psi \circ \phi(q_x) = \phi \circ \psi_k(q_x a) \\ &= \phi(q_x a_k a) = q_y a_k a \end{aligned}$$

so that $a_k a \in S_{q_y}$ proving the lemma.

□

Lemma 4.2. *With the notations as in the lemma above, $C^j_{q_x}$ is contained in an invariant control set for S_{q_y}.*

Proof. Let $v \in C^j_{q_x}$. The assumption that F is compact ensures the existence of an invariant control set for S_{q_y} contained in $\mathrm{cl}(S_{q_y}(v))$ (see Lemma 3.1 in [2]). Denote it by C_{q_y}. Since S_{q_y} has interior points in G, C_{q_y} has nonempty interior, hence $C_{q_y} \cap S_{q_y}(v) \neq \emptyset$ and there exists $b \in S_{q_y}$ with $bv \in C_{q_y}$. We claim that there exists $b' \in S_{q_y}$ with $b'bv \in C^j_{q_x}$. In fact, let $\psi' \in S_Q$ be such that $\psi'(q_y) = q_y b$ and take $\psi \in S_Q$ that maps Q_y into Q_x. Then $\phi \circ \psi(q_y) = q_y b'$ with $b' \in S_{q_y}$. Also,

$$\phi \circ \psi \circ \psi'(q_y) = q_y b'b = \phi(q_x)b'b = \phi(q_x b'b),$$

so that $\psi \circ \psi' \circ \phi(q_x) = q_x b'b$ and $b'b \in S_{q_x}$. Hence $b'bv \in C^j_{q_x}$ because $v \in C^j_{q_x}$. This proves the claim. From it and the choice of b it follows that $b'b \cdot v \in C^j_{q_x} \cap C_{q_y}$. Putting $w = b'b \cdot v$ the previous lemma implies that

$$C^j_{q_x} \subset \mathrm{cl}\left(S_{q_y} w\right) = C_{q_y},$$

proving the existence of the invariant control set. The uniqueness follows from the fact that the intersection of different invariant control sets for S_{q_y} is empty.

□

Recall that we are using the same notation for the Q-bundle map and the induced E-bundle map. They are related by $\psi(q \cdot v) = \psi(q) \cdot v$ for $q \in Q$, $v \in F$ and $q \cdot v \in E$. With this notation in mind we translate the statement of the above lemma, in terms of the subsets C^j_x, as follows:

Put $C^j_x = q_x \cdot C^j_{q_x}$ and $C_y = q_y \cdot C_{q_y}$ with C_{q_y} the S_{q_y}-invariant control set ensured by the lemma. If ϕ is as in the proof then

$$\phi(C^j_x) = \phi(q_x \cdot C_{q_x}) = \phi(q_x) \cdot C_{q_x} \subset q_y \cdot C_{q_y}.$$

Therefore ϕ maps the subset C^j_x inside one of the subsets $C^i_y = q_y \cdot C^i_{q_y}$ for $i = 1, ..., \mathrm{ic}(S_{q_y})$. We show now that this $C^i_y \subset E_y$ is the same for every ϕ' that maps E_x into E_y.

Lemma 4.3. *Let ϕ be as before with $q_y = \phi(q_x)$, take C^j_x and suppose that C^j_y is such that $\phi(C^j_x) \subset C^j_y$. Then*

i) If $\psi \in S_E$ maps E_y into E_x then $\psi(C^j_y) \subset C^j_x$.

ii) If $\phi' \in S_E$ maps E_x into E_y then $\phi'(C^j_x) \subset C^j_y$.

Proof. As above we take q_x and q_y in the fiber over x and y respectively such that $\phi(q_x) = q_y$. If ψ is as in ii) then it maps the fiber over y into the fiber over x so that $\psi \circ \phi$ maps the fiber over x into itself. Since both maps are in S_Q we have that there exists $a \in S_{q_x}$ such that $\psi \circ \phi(q_x) = q_x.a$. Now, take $w \in C^j_{q_x}$. Then

$$\psi(q_y \cdot w) = \psi(q_y) \cdot w = \psi \circ \phi(q_x) \cdot w = q_x \cdot aw.$$

Therefore $\psi(q_y \cdot w) \in C^j_x$. However $q_y \cdot w \in C^j_y$ because $q_y \cdot w = \phi(q_x \cdot w)$ and by assumption $\phi(C^j_x) \subset C^j_y$. Hence $\psi\left(C^j_y\right) \cap C^j_x \neq \emptyset$ and i) follows from Lemma 4.2 (and the above comments) applied to ψ.

Concerning ii), we note that it is just a rephrasing of i) with ψ in place of ϕ and ϕ' instead of ψ.

□

As a consequence of the above lemmas we get the following description of the invariant control sets over C.

Theorem 4.4. *Let $C \subset M$ be an invariant control set for S_M, and assume that S_Q is accessible over C. Assume also that the fiber F is compact. Then*

1. $\mathrm{ic}(S_q)$ *is constant as a function of* $q \in \pi^{-1}(C_0)$.
2. *There are invariant control sets for S_E over C and its number equals* $\mathrm{ic}(S_q)$.
3. *For every invariant control set $D \subset \pi^{-1}(C)$ for S_E and $q \in Q$,*

$$C \cap E_x = q \cdot B,$$

where $x = \pi(q)$ and B is an invariant control set for S_q in F.

Proof. Lemma 4.3 implies that $\mathrm{ic}(S_{q_x}) = \mathrm{ic}(S_{q_y})$ so 1) follows from the transitivity of S_M on C_0 and the invariance of $\mathrm{ic}(S_q)$ with q varying in a fixed fiber.

Now, pick $x \in C_0$ and some $C^j_x \subset E_x$. If $q \cdot v \in C^j_x$ then by Lemma 4.3 ii) there is defined for each $y \in C_0$ a unique C_y^j with $C_y^j \cap S_E(q \cdot v) \neq \emptyset$. Put

$$C^j = \bigcup_{y \in C_0} C_y^j.$$

Then C^j is S_E-invariant as follows from Lemma 4.3 i). Hence $\mathrm{cl}\, C^j$ is also S_E-invariant and a fortiori $\mathrm{cl}(S_E(u)) = \mathrm{cl}\, C^j$ for every $u \in \mathrm{cl}\, C^j$. This shows that $\mathrm{cl}\, C^j$ is an invariant control set. By Theorem 3.5 its intersection with a fiber E_x, $x \in C_0$ is the image under $v \to q \cdot v$ of a control set for S_q. However, by construction, C_x^j is contained in $\mathrm{cl}\, C^j$ if $x \in C_0$. Hence

$$\mathrm{cl}\, C^j \cap E_x = C_x^j.$$

We have thus constructed $\mathrm{ic}\,(S_q)$ different control sets for S_E.

To see that these are the only possibilities, let D be an invariant control set for S_E such that $\pi\,(D) \subset C$. The invariance of D implies that $\pi\,(D) = C$. Take $u \in D$ such that $\pi\,(u) \in C_0$, and put $u = q \cdot v$, $v \in F$ and $q \in Q$. Then the compactness of F ensures the existence of an invariant control set for S_q which meets $S_q v$. But $qS_q v \subset S_E(q \cdot v)$, which implies that for some C^j as above $S_E(q \cdot v) \cap C^j \neq \emptyset$, i.e., $C \cap C^j \neq \emptyset$ so that $C = C^j$. This proves 2), 3) and the theorem.

□

References

[1] Albertini, F. and E. D. Sontag, *Discrete time transitivity and accessibility: analytic systems*, SIAM J. Control Optim., 31 (1993), 1599-1622.

[2] Arnold, L., W. Kliemann and E. Oeljeklaus, *Lyapunov exponents of linear stochastic systems*, In Lyapunov Exponents: Proceedings of a Workshop in Bremen (L. Arnold and V. Wihstutz, eds.), Lecture Notes in Mathematics 1186, Springer-Verlag, Berlin (1986), 85-125.

[3] Colonius, F. and W. Kliemann, *Linear control semigroups acting on projective spaces*, Journal of Dynamics and Differential Equations 5(3) (1993),495-528.

[4] Hilgert, J. and K.H. Neeb, Lie semigroups and their applications, Lecture Notes in Mathematics 1552 Springer-Verlag (1993).

[5] Kliemann, W., *Recurrence and invariant measures for degenerate diffusions*, Ann. Probab. 15 (1987), 690-707.

[6] Kobayashi, S. and K. Nomizu, Foundations of differential geometry, John Willey & Sons (1963).

[7] San Martin, L. and L. Arnold, *A control problem related to the Lyapunov spectrum of stochastic flows*, Mat. Aplic. Comput. 5 (1986), 31-64.

[8] San Martin, L., *Controllability of families of measure preserving vector fields*, Systems & Control Letters 8 (1987), 459-462.

[9] San Martin, L., *Invariant control sets on flag manifolds*, Mathematics of Control, Signals and Systems 6 (1993), 41-61.

[10] San Martin, L.A.B, and P.A. Tonelli, *Semigroup actions on homogeneous spaces*, Semigroup Forum 50 (1995) 59-88.

[11] San Martin, L.A.B., *Control sets and semigroups in semi-simple Lie groups*, In Semigroups in algebra, geometry and analysis. Gruyter Verlag (1994)

[12] Steenrod, N., The Topology of fibre bundles, Princeton University Press, Princeton (1951).

Departamento de Matemática
Universidade Estadual de Maringá
Caixa Postal 331
87.020-900 Maringá-Pr, Brasil

Instituto de Matemática
Universidade Estadual de Campinas
Caixa Postal 6065
13.081-970 Campinas-SP, Brasil

Received October 16, 1996
Revised April 24, 1997