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# STEREOGRAPHIC PROJECTIONS AND GEOMETRIC SINGULARITIES

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**Abstract**

We relate the contacts of k-spheres and submanifolds of $\mathbb{R}^n$ with those of (k+1)-planes and their inverse images through the stereographic projection in $\mathbb{R}^{n+1}$ by defining $\mathcal{K}$-equivalences between their respective families of contact maps.

## 1. Introduction

One of the facts that led to the consideration of four-vertex theorems for closed space curves was the observation that the inverse of the stereographic projection, $\xi : \mathbb{R}^2 \to S^2 - \{(0,0,1)\}$, takes vertices of plane curves into zero torsion points of their images considered as space curves (see [5]). It was also classically known that $\xi : \mathbb{R}^3 \to S^3 - \{(0,0,0,1)\}$ transforms umbilical points of surfaces in 3-space into inflection points (in the sense of [6]) of their corresponding images in 4-space.

From the singularity theory viewpoint, these facts amount to say that

a) If $t$ is a singularity of type $A_k,\ k > 2$ (in Arnold's notation [1]) for some distance squared function on a plane curve $\alpha : \mathbb{R} \to \mathbb{R}^2$, then it is a singularity of the same type $A_k,\ k > 2$ for some height function on the curve $\alpha^* = \xi \circ \alpha : \mathbb{R} \to \mathbb{R}^3$.

b) If $p$ is a singular point of type $D_k,\ k \geq 3$ (notation of [1]) for some distance squared function on a surface $S$ immersed in $\mathbb{R}^3$, so it is for some height function on the surface $S^* = \xi(S)$ immersed in 4-space.

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It is natural to ask whether, in general, the map $\xi : \mathbb{R}^n \to S^n - \{(0, ..., 0, 1)\}$ transforms singularities of distance squared functions of a given type on some submanifold $M$ of $\mathbb{R}^n$ onto the same type of singularities for adequate height functions on $M^* = \xi(M)$ in $\mathbb{R}^{n+1}$ considered as a submanifold of $\mathbb{R}^{n+1}$. The answer is affirmative as shown in [15] and [18], where this fact has been explored in particular to deduce some new geometrical consequences for plane curves from known results on convex space curves.

It has actually been shown by Sedykh ([18]) that the manifold of singularities of any simple class for the evolute (focal set) of a smooth submanifold $M$ of $\mathbb{R}^n$ is isomorphic to the manifold of singularities of the same class for the front of hyperplanes in $\mathbb{R}^{n+1}$ that are tangent to the image of $M$ under $\xi$ and do not pass through the north pole of this projection. To prove this, he related through $\xi$ the germ of the generating family of the lagrangian submanifold of $T^*\mathbb{R}^n$ (with its natural symplectic structure) defined by the normal bundle of the manifold $M$, to that of the generating family of the legendrian submanifold of the contact manifold given by the projectivization, $PT^*\mathbb{R}^{n+1}$, of the bundle $T^*\mathbb{R}^{n+1}$ defined by the tangent elements (hyperplanes) of $\xi(M)$ at corresponding points.

It can be seen that these two generating families can be respectively identified with those of distance squared functions on $M$ and of height functions on $\xi(M)$. Then the relation between their germs, obtained by Sedykh, is precisely $\mathcal{K}$-equivalence (or V-equivalence, as in [8]). We use here an alternative approach, based on the idea of contact between pairs of submanifolds of euclidean spaces, developed by J. Montaldi ([12]), in order to prove this fact. This viewpoint allows us not only to relate the contacts of $M$ with hyperspheres of $\mathbb{R}^n$ with those of $\xi(M)$ with hyperplanes of $\mathbb{R}^{n+1}$ (by exhibiting an explicit $\mathcal{K}$-equivalence between the two families), but also to analyze the relations between the contacts of $M$ with lower dimensional spheres of $\mathbb{R}^n$ with those of $\xi(M)$ with lower dimensional linear subspaces of $\mathbb{R}^{n+1}$.

We conclude, in particular, some facts about contacts of surfaces and planes in $\mathbb{R}^4$, from the results obtained by Montaldi in [13] on contacts of surfaces with circles in 3-space.

## 2. Height functions and distance squared functions and the geometry of submanifolds

Given submanifolds $X$ and $Y$ of $\mathbb{R}^n$, locally defined by $X = g(\mathbb{R}^m)$ and $Y = f^{-1}(0)$, where $g : \mathbb{R}^m \to \mathbb{R}^n$ is an embedding, and $f : \mathbb{R}^n \to \mathbb{R}^q$ is a submersion, we can "measure" their contact at a common point $p \in X \cap Y$ by analyzing the singularities of the composed map $f \circ g : \mathbb{R}^m \to \mathbb{R}^q$ ("contact map"). In fact, Montaldi [12] proved that the contact class of $X$ and $Y$ at $p$ depends only on the $\mathcal{K}$-singularity type of $f \circ g$ (which in turn, does not depend on the maps $g$ and $f$ chosen to represent $X$ and $Y$ respectively). Whenever we say that a couple of manifolds $X$ and $Y$ has the same contact as another couple of manifolds $X'$ and $Y'$ has at some other point, we mean that their respective "contact maps" are $\mathcal{K}$-equivalent as germs at the given points. Two map-germs $f_i : (\mathbb{R}^m, x_i) \to (\mathbb{R}^q, y_i)$, $i = 1, 2$ are said to be contact-equivalent or $\mathcal{K}$-equivalent if there is a diffeomorphism-germ (contact-equivalence), $H : (\mathbb{R}^m \times \mathbb{R}^q, (x_1, y_1)) \to (\mathbb{R}^m \times \mathbb{R}^q, (x_2, y_2))$ of the form $H(x, t) = (h(x), \theta(x, t))$, with $\theta(x, y_1) = y_2$ for all $x$ in a neighbourhood of $x_1$ in $\mathbb{R}^m$, such that $H(x, f_1(x)) = (h(x), f_2(h(x)))$. Notice that this amounts to say that $H$ takes the graph of $f_1$ to the graph of $f_2$, mapping the linear subspace $y = y_1$ onto the linear subspace $y = y_2$. We refer to [4] or [8] for the definition and details on $\mathcal{K}$-equivalence.

When we study the geometry of an m-submanifold $X$ of $\mathbb{R}^n$ we are led to consider its contacts with hyperspheres and hyperplanes of the ambient space. These are thus described by the respective behaviour of the families

a) of distance squared functions on $X$

$$\begin{array}{ccccccc} \phi & : & \mathbb{R}^m \times \mathbb{R}^n & \overset{g\times 1_{\mathbb{R}^n}}{\longrightarrow} & \mathbb{R}^n \times \mathbb{R}^n & \overset{d}{\longrightarrow} & \mathbb{R} \\ & & (x, y) & \longmapsto & (g(x), y) & \longmapsto & \|g(x) - y\|^2 \end{array}$$

b) of height functions on $X$

$$\begin{array}{ccccccc} \lambda & : & \mathbb{R}^m \times S^{n-1} & \overset{g\times 1_{S^{n-1}}}{\longrightarrow} & \mathbb{R}^n \times S^{n-1} & \overset{i}{\longrightarrow} & \mathbb{R} \\ & & (x, v) & \longmapsto & (g(x), v) & \longmapsto & < g(x), v > \end{array}$$

We shall denote by $\phi_y$ and $\lambda_v$ the functions obtained when fixing the parameters $y$ and $v$ respectively.

The generic singularities of $\phi$ were initially studied by Porteous [16], who observed that its singular set,is precisely, the normal bundle of $X$ in $\mathbb{R}^n$. On the other hand, the singular set of $\lambda$,

$$\Sigma(\lambda) = \{(x,v) \in \mathbb{R}^m \times S^{n-1} | \frac{\partial \lambda_v}{\partial x} = Dg(x) \cdot v = 0\}$$

is the unit normal bundle of $X$ and was studied in [14].

The bifurcation sets of $\phi$ and $\lambda$ can be geometrically interpreted as the set of focal centers (or caustic) of $X$ in $\mathbb{R}^n$ and the set of singular values of the Gauss map on $X$ (or of the Gauss map on the unit normal bundle of $X$, or *canal hypersurface* of $X$, considered as a hypersurface embedded in $\mathbb{R}^n$, when the codimension of $X$ is higher than 1) respectively.

It is well known (see [7], [11]) that for a generic embedding $X = g(\mathbb{R}^m) \subset \mathbb{R}^n$, the families $\phi$ and $\lambda$ are generic families of functions on $\mathbb{R}^m$. For a detailed description of the term "generic family of functions" we refer to [7] and [19]. This means, in particular, that these families are topologically stable, and for $n \leq 5$, smoothly stable too.

Some geometrically relevant subsets are:

The *symmetry set*,

$$M(\phi) = \{y \in \mathbb{R}^n | \phi_y \text{has at least 2 critical points at the same level}\}$$

corresponding to the set of centres of hyperspheres touching $X$ at least twice. The generic structures of the symmetry sets of submanifolds of $\mathbb{R}^2$ and $\mathbb{R}^3$ have been described by Bruce, Giblin and Gibson in [3].

The *central set* is the subset of $M(\phi)$ composed of centres of spheres n-tangent ($n \geq 2$) to $X$, whose radius is the minimal distance from the centre to the submanifold.

The *Maxwell set* of $X$ is defined as the set of unit vectors corresponding to a height function whose absolute minimum is either degenerate or attained at more than one point. This set was studied in detail in [14] for the case of hypersurfaces embedded in euclidean space. It is not difficult to see that the

Maxwell set of a submanifold of codimension higher than 1 coincides with that of its canal hypersurface (defined as its unit normal bundle smoothly embedded in $\mathbb{R}^n$).

## 3. $\mathcal{K}$-equivalence and stereographic projection

For a family $F : \mathbb{R}^m \times P \to \mathbb{R}^p$ of mappings $\mathbb{R}^m \to \mathbb{R}^p$, set $Z(F) = \{(x, c) \in \mathbb{R}^m \times P | F(x, c) = 0\}$. Given two families of maps $F_1$ and $F_2$ from $\mathbb{R}^m$ to $\mathbb{R}^p$, with parameter spaces $P_1$ and $P_2$ respectively, we say that two germs $F_1$ at $(x_1, c_1) \in Z(F_1)$ and $F_2$ at $(x_2, c_2) \in Z(F_2)$ are $\mathcal{K}$-*equivalent* if there are smooth map-germs,

i) $V : (\mathbb{R}^m \times P_1, (x_1, c_1)) \to (\mathbb{R}^m, x_2)$ such that, for some representative of $V$, the map $V_c : \mathbb{R}^m \to \mathbb{R}^m$ is a local diffeomorphism, for all $c$ in a neighbourhood of $c_1$ in $P_1$,

ii) a diffeomorphism-germ $\theta : (P_1, c_1) \to (P_2, c_2)$ , and

iii) $\mu : (\mathbb{R}^m \times P_1 \times \mathbb{R}^p, (x_1, c_1, 0)) \to (\mathbb{R}^p, 0)$ such that, for some representative of $\mu$, $\mu_{(x,c)}$ is a diffeomorphism of $\mathbb{R}^p$ taking 0 to 0 for all $(x, c)$ in a neighbourhood of $(x_1, c_1)$ in $\mathbb{R}^m \times P_1$, satisfying

$$F_1(x, c) = \mu(x, c, F_2(V(x, c), \theta(c)))$$

Notice that if the germ of $F_1$ at $(x_1, c_1)$ and the germ of $F_2$ at $(x_2, c_2)$ are equivalent in the sense of the above definition, then the diffeomorphism-germ defined by $(x, c) \mapsto (V(x, c), \theta(c))$ maps the germ of $Z(F_1)$ at $(x_1, c_1)$ to the germ of $Z(F_2)$ at $(x_2, c_2)$, preserving the decomposition of $Z(F_1)$ and $Z(F_2)$ by $\mathcal{K}$-equivalence classes respectively.

We shall say that the family $F_1$ is (locally) $\mathcal{K}$-equivalent to the family $F_2$ if there is a (local) diffeomorphism $\mathbb{R}^m \times P_1 \to \mathbb{R}^m \times P_2$ of the form $(x, c) \mapsto (V(x, c), \theta(c))$, taking $Z(F_1)$ to $Z(F_2)$, such that the germ of $F_1$ at $(x, c)$ is $\mathcal{K}$-equivalent to the germ of $F_2$ at $(V(x, c), \theta(c))$ in the above sense, for all $(x, c) \in Z(F_1)$.

We shall show now that there is a $\mathcal{K}$-equivalence between the extensions

$$\begin{array}{rcll} \Phi & : & \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ & & (x,(a,r)) & \longmapsto & \|g(x)-a\|^2 - r^2 \end{array}$$

and

$$\begin{array}{rcll} \Lambda & : & \mathbb{R}^m \times S^n \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & (x,v,\rho) & \longmapsto & <\xi(g(x)),v> -\rho \end{array}$$

of the families $\phi$ and $\lambda$ when we restrict ourselves to convenient subsets of their parameter spaces.

Let $\mathbf{H}$ be the set of all the affine hyperplanes of $\mathbb{R}^{n+1}$ whose distance $\rho$ to the origin $O$ is $< 1$ and denote by $\mathbf{H^*}$ the subset of $\mathbf{H}$ made by all the hyperplanes that do not pass through the north pole $P$ of $S^n$ and that are not parallel to the vector $\vec{OP}$. Clearly $\mathbf{H^*}$ is an open and dense subset of $\mathbf{H}$. Any hyperplane $l \in \mathbf{H}^*$ can be represented by its unit normal vector $v$, chosen in such a way that the real number $< v,(0,...,0,1) >$ is positive, and its distance to the origin, $\rho =< v,\vec{OA} >$, where $\vec{OA}$ denotes any vector of $\mathbb{R}^{n+1}$ with end point $A \in l$. We can consider $\mathbf{H^*}$ as a subset of $S^n \times \mathbb{R}$ through the above association of any $l$ with a pair $(v,\rho)$. And we have then a height function, $h_l(x) =< v,x > -\rho$, associated to each hyperplane $l \in \mathbf{H}^*$, and hence a family of height functions

$$\begin{array}{rcll} \Lambda & : & \mathbb{R}^m \times \mathbf{H}^* & \longrightarrow & \mathbb{R} \\ & & (x,v,\rho) & \longmapsto & <\xi(g(x)),v> -\rho \end{array}$$

Notice that any hyperplane $l \in \mathbf{H}$ may have an associated distance function, but there is an ambiguity in the choice of sign when the hyperplane $l$ is parallel to the vector $\vec{OP}$.

Given a k-sphere, $S^k$, in $\mathbb{R}^n$, we know that its image, $\xi(S^k)$, by $\xi$ is a k-sphere in $S^n - \{(0,...,0,1)\}$, for stereographic projection is a conformal map. Then, if $S(a,r)$ denotes a hypersphere of radius $r$ centered at a point $a \in \mathbb{R}^n$, we can denote by $l(a,r)$, the unique affine hyperplane of $\mathbb{R}^{n+1}$ whose intersection with $S^n$ is $\xi(S(a,r))$. We have in this way an injective map,

$$\begin{array}{rcll} \theta & : & \mathbb{R}^n \times \mathbb{R}_+ & \longrightarrow & \mathbf{H} \\ & & (a,r) & \longmapsto & l(a,r). \end{array}$$

Moreover, it can be shown that

$$\begin{array}{lllll} \theta & : & \theta^{-1}(\mathbf{H}^*) & \longrightarrow & \mathbf{H}^* \\ & & (a,r) & \longmapsto & (u(a,r),\rho(a,r)) \end{array}$$

is a diffeomorphism, where $u(a,r)$ and $\rho(a,r)$ are, respectively, the unit vector and real number associated to the hyperplane $l(a,r)$ in the above way.

We can also define another smooth map

$$\begin{array}{lllll} \mu & : & (\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+) \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & (x,a,r,t) & \longmapsto & \mu_{(x,a,r)}(t) \end{array}$$

where the value of the function $\mu_{(x,a,r)}(t) : \mathbb{R} \longrightarrow \mathbb{R}$ at $t$ is given by the height of the point $s(\frac{t}{r}a + \frac{r-t}{r}(a + r\frac{g(x)-a}{||g(x)-a||}))$ over the hyperplane $l(a,r)$ of $\mathbb{R}^{n+1}$. This is well defined and smooth except when $a = g(x)$.

And now it is not difficult to verify that

$$\Lambda(x,(v,\rho)) = \mu(x,\theta^{-1}(v,\rho),\Phi(x,\theta^{-1}(v,\rho))).$$

Consequently, if we denote

$$P_1 = \{(a,r) \in \mathbb{R}^n \times \mathbb{R}_+ : \theta(a,r) \in \mathbf{H}^* \wedge a \notin \mathrm{Image}(g)\}$$

and $P_2 = \theta(P_1)$, we can state the following

**Theorem 1.** *The families of functions*

$$\begin{array}{lllll} \Phi & : & \mathbb{R}^m \times P_1 & \longrightarrow & \mathbb{R} \\ & & (x,(a,r)) & \longmapsto & ||g(x)-y||^2 - r^2 \end{array}$$

*and*

$$\begin{array}{lllll} \Lambda & : & \mathbb{R}^m \times P_2 & \longrightarrow & \mathbb{R} \\ & & (x,v,\rho) & \longmapsto & <\xi(g(x)),v> -\rho \end{array}$$

*are $\mathcal{K}$-equivalent.*

**Remark:** Observe that $P_1$ and $P_2$ are open and dense respectively in $\mathbb{R}^n \times \mathbb{R}_+$ and $S^n_+ \times ]-1,1[$.

We can extend the set of hyperspheres of $\mathbb{R}^n$ in order to include the degenerate ones, i.e. the hyperplanes, so that the map $\theta$ defined above becomes a bijection over $\mathbf{H}$. In this case we must consider, instead of $\Phi$, the following family of functions

$$\begin{aligned} D \ : \ \mathbb{R}^m \times \mathbb{RP}^{n+1} &\longrightarrow \mathbb{R} \\ (x, [a:t:r]) &\longmapsto \ < a, g(x) > -\tfrac{1}{2}t||g(x)||^2 + r \end{aligned}$$

so that, if $t \neq 0$, the function $D_{[a:t:r]}$ gives the contact of $M = g(\mathbb{R}^m)$ with the hypersphere of $\mathbb{R}^n$ defined by the equation

$$|y - \frac{a}{t}|^2 - \frac{r}{t} - |a|^2 t^2 = 0,$$

whereas, if $t = 0$, it measures the contact of $M$ with the hyperplane of equation

$$< a, y > \ + r \ = \ 0.$$

The problem now to define a $\mathcal{K}$-equivalence between the families $\Phi$ and

$$\Lambda : \mathbb{R}^m \times S^n \times ]-1, 1[ \longrightarrow \mathbb{R}$$

resides in the ambiguity of the choice of sign in the association of a height function to any hyperplane parallel to the vector $\vec{OP}$, as mentioned before. But observe that this difficulty can be overcome by considering the local situation, in such a way that the direction of the unit vector and the sign of the number $\rho$ are chosen consistently on a sufficiently small neighbourhood of any of these hyperplanes.

On the other hand, although we have had to avoid the subset $\{(a, r) \in \mathbb{R}^n \times \mathbb{R}_+ : \exists x \in \mathbb{R}^m \text{ with } g(x) = a\}$ in order to define globally the $\mathcal{K}$-equivalence between $\Phi$ and $\Lambda$ in the theorem 1, it is not difficult to see that this can be locally defined at every point of the parameter space $\mathbb{R}^n \times \mathbb{R}_+$ for, generically, the focal centers of the submanifold at any point $g(x)$ lie off some neighbourhood of $g(x)$. So, by working in small enough neighbourhoods of $(x, a, r)$ we should not have any problems in defining the local diffeomorphism $\mu_{(x,a,r)}$. Consequently we can state the following,

**Theorem 2.** *Given $x \in \mathbb{R}^m$ and $[a : t : r] \in \mathbb{RP}^{n+1}$, the germ of $D$ at $(x, [a : t : r])$ is $\mathcal{K}$-equivalent to that of $\Lambda$ at $(x, \xi^*([a : t : r]))$, where $\xi^*([a : t : r])$ represents any of the two opposite parameters in $S^n \times ]-1, 1[$ corresponding to the hyperplane of $\mathbb{R}^{n+1}$ associated through the stereographic projection to the hypersphere of $\mathbb{R}^n$ determined by the point $[a : t : r]$.*

Given an embedding, $g : \mathbb{R}^m \to \mathbb{R}^n$, of a manifold $M$ in $\mathbb{R}^n$, the wavefront of $M$ at time $r$ is obtained as the envelope of families of hyperspheres of radius $r$ centered on $M$, or equivalently, as the set of centers of hyperspheres of radius $r$ tangent to $M$. This is given, in terms of the map $\Phi$, as $M_r = \{a \in \mathbb{R}^n : \exists x \in \mathbb{R}^n \text{ with } \Phi_{(a,r)} = \frac{\partial \Phi_{(a,r)}}{\partial x_1}(x) = ... = \frac{\partial \Phi_{(a,r)}}{\partial x_m}(x) = 0\}$. We define the *big front*, $W$, as the union of all the wavefronts of $M$. This can be viewed as the set of all the tangent hyperspheres to $M$. Genericity conditions on the embedding $g$ imply that the map $\phi$ is a versal unfolding of the germ of the distance squared function $\phi_a$ at the point $x$, for all $(x, a)$ in $\mathbb{R}^m \times \mathbb{R}^n$. we can then look $M_r$ as the discriminant set of this unfolding. Now, for $n \leq 5$, only simple singularities of functions of Arnol'd's list ([1]) may occur, and as a consequence we have a finite number of local models for wavefronts and big fronts ([2]). Similar arguments apply to bifurcation sets, singular sets and symmetry sets for both $\phi$ and $\lambda$. In the case of the last one, instead of the big front, we have what is usually called the *front* of $M$ (or the *dual* when $M$ is a hypersurface), that is the set ,$F$, of all the tangent hyperplanes to $M$.

We can add to $W$ the tangent hyperspheres of infinite radius by considering the discriminant sets of the map $D : \mathbb{R}^m \times \mathbb{RP}^{n+1} \to \mathbb{R}$ defined above , instead of those of $\phi$. We denote their union by $W_D$ and call it *compactified big front.*

An immediate consequence of the Theorem 2 is that for any generic embedding of a n-dimensional manifold $M$ with $n \leq 5$:

i) The compactified big front of $M$ is locally diffeomorphic to the front of $\xi(M)$. Observe that , since the map $\theta$ takes $W_D$ bijectively onto the front of $\xi(M)$, we actually have a *global diffeomorphism* between these two sets.

ii) The caustic of $M$ is locally diffeomorphic to the set of singular values of the generalized Gauss map on $\xi(M)$.

iii) The Maxwell set of $\xi(M)$ is locally diffeomorphic to a double covering of the closure of the central set of $M$.

## 4. Contacts with higher codimensional spheres and affine subspaces

Let us denote by $\mathcal{S}_k$ (resp.$\mathcal{L}_k$) the set of all the k-dimensional spheres (resp. affine subspaces) in $\mathbb{R}^n$. Any k-sphere (k-dimensional affine subspace) can be put as the intersection of $n-k$ hyperspheres (hyperplanes), but observe that we cannot consider all the possible intersections between all the possible hyperspheres (or hyperplanes) of $\mathbb{R}^n$ in order to describe all the elements in $\mathcal{S}_k$ (resp.$\mathcal{L}_k$), for on one hand not all the pairs of hyperspheres intersect and on the other hand there is a (finite dimensional) redundancy in this representation, for different subsets of hyperspheres may give, when intersecting them, the same k-sphere. Nevertheless, given a submanifold $M = g(\mathbb{R}^m)$ of $\mathbb{R}^n$ and any k-sphere, $S$, we can make a choice of $n-k$ hyperspheres centered at $c_1, ..., c_{n-k}$ and with radii $r_1, ..., r_{n-k}$ respectively, so that the contact of $M$ and $S$ is given by the $\mathcal{K}$-class of the map

$$\begin{array}{rccl} \varphi : & \mathbb{R}^m & \longrightarrow & \mathbb{R}^{n-k} \\ & x & \longmapsto & (||c_1 - x||^2 - r_1^2, ..., ||c_{n-k} - x||^2 - r_{n-k}^2) \end{array}$$

This $\mathcal{K}$-class is independent on this choice ([11,12]).

By working locally, we can make consistent choices on appropriate open subsets, $\mathcal{U}_j$ of $\mathcal{S}_k$, so that it can be parametrized by coordinates of the type $(c_1, ..., c_{n-k}, r_1, ...r_{n-k}) \in \mathbb{R}^{n(n-k)+n-k}$. In this case the family

$$\begin{array}{rlcl} \Psi_k : & \mathbb{R}^m \times \mathcal{U}_j & \longrightarrow & \mathbb{R}^{n-k} \\ & (x, (c_1, ..., c_{n-k}, r_1, ...r_{n-k})) & \longmapsto & (||c_1 - x||^2 - r_1^2, ..., ||c_{n-k} - x||^2 - r_{n-k}^2) \end{array}$$

contains all the informations about the contacts of $M$ with the k-spheres of $\mathbb{R}^n$ lying on $\mathcal{U}_j$. Montaldi showed in [11] that there is a residual subset of embeddings of $M$ in $\mathbb{R}^n$ for which the corresponding families of maps $\Psi_k$ are "well-behaved" in the sense that their r-jets satisfy certain convenient transversality conditions. We shall refer to these as generic embeddings.

If we want to describe instead the contacts of $M$ with k-dimensional affine subspaces, we must consider a family

$$\begin{array}{rlcl} \Omega_k : & \mathbb{R}^m \times \mathcal{W}_j & \longrightarrow & \mathbb{R}^{n-k} \\ & (x, (n_1, ..., n_{n-k}, \rho_1, ...\rho_{n-k})) & \longmapsto & (< n_1, x > -\rho_1, ..., < n_{n-k}, x > -\rho_{n-k}) \end{array}$$

with $\{(n_i, \rho_i)\}_{i=1}^{n-k}$ representing the subset of hyperplanes whose intersection gives the considered k-plane of $\mathbb{R}^n$.

The map $\xi : \mathbb{R}^n \to S^n$ transforms any k-sphere $S$ of $\mathbb{R}^n$ into a k-sphere $C$ in $S^n$, which in turn defines a unique $(k+1)$-dimensional affine subspace, $l$, of $\mathbb{R}^{n+1}$. Moreover, if $S = S^{n-1}(a_1, r_1) \cap ... \cap S^{n-1}(a_{n-k}, r_{n-k})$, we have that $l = l^n(a_1, r_1) \cap ... \cap l^n(a_{n-k}, r_{n-k})$. Now, similar arguments to those used in the previous section for the case of distance squared and height functions, lead us to the following:

**Theorem 3.**

Given any embedding $g : \mathbb{R}^m \to \mathbb{R}^n$, the families $\Psi_k$ and $\Omega_k$ are locally $\mathcal{K}$-equivalent, for all $k = 1, ..., n-1$.

**Proof:** The diffeomorphism $\theta : C_1 \to C_2$ of the definition of $\mathcal{K}$-equivalence between the families is given by the assignation $S^k = S(c_1, r_1) \cap ... \cap S(c_{n-k}, r_{n-k}) \mapsto l^{k+1} = l(c_1, r_1) \cap ... \cap l(c_{n-k}, r_{n-k})$. And the map $\mu : \mathbb{R}^m \times \mathcal{S} \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ can be defined as

$$\mu_{x,c_1,r_1,...,c_{n-k},r_{n-k}}(t_1, ..., t_{n-k}) = (\mu_{x,c_1,r_1}(t_1), ..., \mu_{x,c_{n-k},r_{n-k}}(t_{n-k}))$$

with each $\mu_{x,c_i,r_i}$ constructed as in the Theorem 1, for all $i = 1, ..., n-k$.

We give next, as an application of the above considerations, some conclusions about contacts of spherical surfaces with planes in $\mathbb{R}^4$ obtained from the results of J. Montaldi ([13]) on contacts of circles with surfaces in 3-space.

Given a surface $M$ in 4-space, its contacts with planes are measured by the singularity types of maps $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ as exposed above. These may have corank 1 or 2. In the first case, the singularities will only be of type $A_n$. We shall say that a plane has a *k-point contact* with the surface at a point $x$ if the corresponding contact map has a singularity of type $A_{k-1}$ at $x$. Notice that the only plane for which the contact map has corank 2 is the tangent plane of the surface at the given point. By using stereographic projections, as in the previous section, we can translate contacts of surfaces with circles in 3-space into contacts of surfaces with planes in 4-space. We must point out that since circles are parametrized by a single variable, their contacts will always give rise to singularities of type $A_n$, and so will those of the planes corresponding to them through the stereographic projection in $\mathbb{R}^4$.

Any tangent direction $w$ in $T_xM$ defines a "normal hyperplane" , $N_w$, to $M$ at $x$ (obtained as the direct sum of the tangent line generated by $w$ and the normal plane to $M$ at $x$). The intersection of $N_w$ with $M$ is a curve $\gamma_w$ contained in this hyperplane (3-space), and its curvature vector, $\eta(w)$, lies in the normal plane of $M$ at $x$. As $w$ varies, the vector $\eta(w)$ describes an ellipse in this normal plane, known as the curvature ellipse of $M$ at $x$ ([6]). At some points (generically isolated) in $M$ this ellipse may degenerate to a segment such that the line for it determined passes through the origin $x$ of the normal plane. These are known as the *inflection points* of $M$, and they were characterized in [9] as the umbilic points of height functions on $M$. A tangent direction $w_o$ is said to be *asymptotic* if the vectors $\eta(w_o)$ and $\frac{\partial \eta}{\partial w}(w_o)$ are collinear. It has been shown in [9] that each asymptotic direction $w$ at $x$ is tightly related to a binormal direction $b$ at $x$. Moreover, it is possible to prove that $w$ must be in the kernel of the Hessian of the height function corresponding to $b$.

Given any tangent direction $w$ at a pont $x$ of $M$, we call *w-plane* to any affine plane of $\mathbb{R}^4$, distinct from the tangent plane of $M$ at $x$, that passes through $x$ and contains the direction $w$.

In what follows we shall consider spherical surfaces in $\mathbb{R}^4$, that is, surfaces contained in $S^3$. Given $M \subset S^3$, the stereographic projection, $\xi^{-1} : S^3 - \{P\} \to \mathbb{R}^3$, takes $M$ to a surface $M'$ in 3-space, and it can be shown that $d\xi$ transforms principal directions of curvature of $M'$ into asymptotic directions of $M$ (a proof of this fact is given in a forthcoming paper [10] on 2-codimensional submanifolds of $\mathbb{R}^n$). Now, from Montaldi's theorems 1 (pg. 112) and 2 (pg. 116) in [13], we can conclude:

1) If $w$ is not an asymptotic direction, then there is a unique $w$-plane with at least 4-point contact with the surface at $x$.

2) If $x$ is not an inflection point and $w$ is not an asymptotic direction then

   a) if $X$ is not an $A_3$ point for any height function on $M$, it does not exist any $w$-plane with 4-point contact with $M$ at $x$.

   b) if there is some height function on $M$ havind a cusp singularity at $x$, there is a 1-parameter family of $w$-planes having at least 4-point contact with $M$ at $x$.

3) If $x$ is an inflection point, there are exactly 3 or 1 circles with 5-point contact with $M$ at $x$ (for different directions), according to $x$ being elliptic or hyperbolic umbilic for the height function in the unique binormal direction of $M$ at $x$ (see [9]).

4) For a generic $M$, there are at most 10 planes at any point, with at least 5-point contact with the surface at $x$ (which reduces to 3 or 1 if $x$ is an inflection point).

We conjecture that the above results are true not only for spherical surfaces, but also for all the convex, i.e. those all whose points are of hyperbolic or parabolic type, ([9]) surfaces in 4-space.

**Some final comments:**

It is not difficult to see that the techniques developed in this paper for the stereographic projection can be adapted to the case of a map given by the restriction to $\mathbb{R}^n\times\{0\}$ of an arbitrary inversion with respect to some hypersphere of $\mathbb{R}^{n+1}$, whose center does not lie in $\mathbb{R}^n \times \{0\}$. This is due to the fact that such an inversion would also preserve contacts between pairs of submanifolds of $\mathbb{R}^n \times \{0\}$, taking them to another pair of submanifolds of some hypersphere of $\mathbb{R}^{n+1}$ with the same contact map. And on the other hand, inversions take k-spheres of $\mathbb{R}^n \times \{0\}$ into k-spheres which determine (k+1)-planes of $\mathbb{R}^{n+1}$. We would like to thank Ricardo Uribe for pointing out this fact to us.

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