Matemática Contemporânea, Vol. 12, 67–82

http://doi.org/10.21711/231766361997/rmc124

REAL SINGULARITIES AND DIHEDRAL REPRESENTATIONS

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Abstract

The Euler characteristics of some *real* algebraic hypersurfaces determine and are determined by natural dihedral representations on the cohomology of their associated *complex* Milnor fibers.

1. Introduction

This paper can be seen as one in a series of papers, see for instance [P], [DHP], in which we intend to investigate the topology of real algebraic varieties by using the extensive existing knowledge about complex singularities. Let $f \in \mathbb{R}[x_0, \cdots, x_n]$ be a weighted homogeneous polynomial of degree d with respect to the weights $\deg(x_j) = w_j$, where w_j are strictly positive integers for $j = 0, \cdots, n$.

Consider the smooth affine hypersurfaces

$$F_{\mathbb{R}} = F_{\mathbb{R}}(f) = \{x \in \mathbb{R}^{n+1}; f(x) = 1\}$$

$$F = F(f) = \{x \in \mathbb{C}^{n+1}; f(x) = 1\}$$

which we call the *real* (resp.*complex*) *Milnor fiber* of the polynomial f.

The monodromy homeomorphism

$$h : F \to F,\ h(x) = (t^{w_0}x_0, \cdots, t^{w_n}x_n)$$

*We thank Dr. Hillman and Dr. Kuo for encouraging interest in our project. We also note that the results in this paper were used and extended by McCrory-Parusiński in [MP]. Our research was completed thanks to the Australian Research Council grant A 695 30267.
1991*Mathematics Subject Classification*: 14B05, 14P25

where $t = \exp(2\pi i/d)$ and the induced *monodromy operator* $h_* : \widetilde{H}_*(F) \to \widetilde{H}_*(F)$, where $\mathbb{C}$-coefficients are used for (co)homology if not explicitly mentioned otherwise, are familiar objects of study, see for instance [M], [M Or], [AGV], [D2]. In particular, when f has an isolated singularity at the origin, then the pair $(\widetilde{H}_*(F), h_*)$ is completely determined by the weighted homogeneity type $(\underline{w}; d) = (w_0, \cdots, w_n; d)$, see [M Or].

On the other hand, the topology of the real Milnor fiber $F_{\mathbb{R}}$ is much more complicated to describe, e.g. cannot be derived just from the type $(\underline{w}; d)$.

Example 1. Any quadratic polynomial f ($w_0 = \cdots = w_n = 1$, $d = 2$) is linearly equivalent over $\mathbb{R}$ to one of the normal forms f_p, $p = -1, 0, \cdots, n$, where $f_p(x) = x_0^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$, see [D1] p.46. It is easy to show that $F_{\mathbb{R}}(f_p)$ has the homotopy type of the sphere S^p, compare to Remark 11 below. Here we use the convention $S^{-1} = \emptyset$.

The aim of this paper is to show that there is a natural dihedral representation on the reduced homology $\widetilde{H}_*(F)$ of the complex Milnor fiber of f combining the action of the monodromy with the complex conjugation.

Let

$$K_{\mathbb{R}} = K_{\mathbb{R}}(f) = \{x \in S^n \subset \mathbb{R}^{n+1}; f(x) = 0\}$$

be the *real link* of the polynomial f. When f is a homogeneous polynomial (i.e. $w_0 = \cdots = w_n = 1$) we consider also the real projective hypersurface defined by f, namely

$$V_{\mathbb{R}} = V_{\mathbb{R}}(f) = \{x \in \mathbb{R}P^n; f(x) = 0\}.$$

The understanding of the topology of such hypersurfaces $V_{\mathbb{R}}$ is a central problem in mathematics, going back to Hilbert 16^{th} problem, see for instance the excellent survey by Gudkov [G] and also [BCR], [Si].

Assuming the zeta function of the monodromy operator h_* known, as is often the case by [AC], [MOr], [D2] $\cdots$, it turns out that the equivariant Euler characteristic of F with respect to the dihedral representation introduced here determines (and is determined by) the Euler numbers $\chi(F_{\mathbb{R}})$ and $\chi(K_{\mathbb{R}})$, see Cor. 19.

In the case of an isolated singularity f, the equivariant Euler characteristic of F satisfies some additional properties and, as a result, is determined just by $\chi(F_{\mathbb{R}})$.

One of the main advantages of our method is that this dihedral representation behaves well under the Thom-Sebastiani construction, in a similar way to $\chi(F)$, $\chi(F_{\mathbb{R}})$ but unlike $\chi(K_{\mathbb{R}})$ or $\chi(V_{\mathbb{R}})$, see Example 24 for some explicit computations along this line.

1. Review of dihedral representations

In this section we recall the basic facts on dihedral groups and their representations following Serre [Se], pp. 36-38. The dihedral group D_m is a finite group of order $2m$ and has the following presentation

$$D_m = \langle r, s; r^m = s^2 = srsr = 1\rangle.$$

To discuss the irreducible representations of the dihedral group D_m we have to consider two cases according to the parity of m.

Case A: (m even ≥ 2)

In this case there are 4 irreducible representations of degree 1, given by the following obvious table :

Table 2

	r	s
ψ_1	1	1
ψ_2	1	-1
ψ_3	-1	1
ψ_4	-1	-1

Next there are $m/2 - 1$ irreducible representations of degree 2 given by

$$\rho^h(r) = \begin{pmatrix} \lambda^h & 0 \\ 0 & \overline{\lambda}^h \end{pmatrix}, \quad \rho^h(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $0 < h < m/2$ and $\lambda = \exp(2\pi i/m)$.

Actually the above formulas define a D_m-representation for any $h \in \mathbb{Z}$ such that

$$\begin{cases} \rho^h = \rho^{m-h} = \rho^{h+m} \\ \rho^0 = \psi_1 \oplus \psi_2 \\ \rho^{m/2} = \psi_3 \oplus \psi_4 \end{cases} \tag{3}$$

where "=" means isomorphic representations and " $\oplus$ " denotes the direct sum of representations.

Case B: (m odd)

In this case there are only 2 irreducible representations of degree 1, namely ψ_1 and ψ_2 from Table 2.

And there are $(m-1)/2$ irreducible representations of degree 2, given by ρ^h above for $0 < h < m/2$.

The following result has a trivial proof using Ex. 5.2 in (5.3) from [Se]. Here $\otimes$ denotes the tensor product of two representations.

Lemma 4.

(i) $\psi_j \otimes \psi_j = \psi_1, \ \psi_1 \otimes \psi_j = \psi_j$ *for* $j = 1, 2, 3, 4$

$\psi_2 \otimes \psi_3 = \psi_4, \ \psi_2 \otimes \psi_4 = \psi_3, \ \psi_3 \otimes \psi_4 = \psi_2$

(ii) $\rho^h \otimes \rho^k = \rho^{h+k} \oplus \rho^{h-k}$ *for any* $h, k \in \mathbb{Z}$;

(iii) *If* V *and* W *are irreducible* G*-representations and* $\dim V = 1$ *then* $V \otimes W$ *is an irreducible* G*-representation for any group* G.

2. Dihedral representations on the Milnor fiber

Since f is a polynomial with real coefficients, we can consider the complex conjugation map

$$c : F \to F, c(x) = \overline{x}.$$

The monodromy homeomorphism h and the complex conjugation homeomorphism c satisfy the following easy-to-check relations

$$h^d = c^2 = chch = 1.$$

In other words, there is a group homomorphism $\psi : D_d \to Homeo(F)$ defined by $\psi(r) = h$, $\psi(s) = c$ where D_d is the d-dihedral group with the presentation given in section 1, and $Homeo(F)$ is the group of all the homeomorphisms of the complex Milnor fiber F.

Using the natural group homomorphism $Homeo(F) \to Aut(\widetilde{H}_*(F))$ we get thus the following basic result.

Proposition/Definition 5. The complex Milnor fiber F of a real weighted homogeneous polynomial f of degree d has a natural D_d-action. This induces a dihedral representation on the graded vector space $\widetilde{H}_*(F)$ called the *dihedral Milnor representation.*

Example 6.

(i) $f(x) = x_0^d, (n = 0)$. It is well-known that the monodromy operator h_* has in this case the following characteristic polynomial

$$\Delta(t) = \det(tI - h_*) = (t^d - 1)/(t - 1) \tag{7}$$

see for instance [M], [M Or].

To describe the corresponding dihedral Milnor representation on $\widetilde{H}_*(F) = \widetilde{H}_0(F) = \mathbb{C}^{d-1}$ we have to consider two cases.

(a) d odd, $d \geq 3$.

Using (7) it follows that the dihedral Milnor representation in this case is (isomorphic to) the direct sum

$$\rho^1 \oplus \rho^2 \oplus \cdots \oplus \rho^{(d-1)/2}$$

(b) d even, $d \geq 2$.

In this case h_* has a real eigenvalue, namely -1. The corresponding dihedral Milnor representation is the sum

$$\psi_3 \oplus \rho^1 \oplus \cdots \oplus \rho^{d/2-1}.$$

More precisely, let $\epsilon_k = \exp(2\pi i k/d)$ be the d-th roots of unity. Then a basis for the vector space $\widetilde{H}_0(F)$ is given by $\alpha_k = \epsilon_k - \epsilon_{k+1}$ for $k = 0, 1, \cdots, d-2$.

The vector corresponding to the representation ψ_3 above is

$$\alpha = \alpha_0 + \alpha_2 + \cdots + \alpha_{d-2}.$$

Indeed, one has $h_*(\alpha) = -\alpha$ and $c_*(\alpha) = \alpha$.

(ii) $f(x) = -x_0^d$

As in (i) above, there two cases to consider

(a) d odd, $d \geq 3$. The corresponding dihedral Milnor representation is again

$$\rho^1 \oplus \cdots \oplus \rho^{(d-1)/2}.$$

This can be proved as above, or by using the real homeomorphism

$$\sigma : F(x_0^d) \to F(-x_0^d), \sigma(x) = -x$$

(b) d even, $d \geq 2$.

The corresponding dihedral Milnor representation is now

$$\psi_4 \oplus \rho^1 \oplus \cdots \oplus \rho^{d/2-1}.$$

Indeed, the elements $\beta_k = \tilde{\epsilon}_k - \tilde{\epsilon}_{k+1}$ form a basis for $\widetilde{H}_0(F)$ in this case, where $\tilde{\epsilon}_k = \exp((2k+1)\pi i/d)$. The vector

$$\beta = \beta_0 + \beta_2 + \cdots + \beta_{d-2}$$

satisfies the relations $h_*(\beta) = c_*(\beta) = -\beta$. This explains why the representation ψ_4 occurs here.

Let $f \in \mathbb{R}[x_0, \cdots, x_n]$ and $g \in \mathbb{R}[y_0, \cdots, y_m]$ be two weighted homogeneous polynomials of the same degree d (note that this last property can always be achieved just by changing the weights of f and g by suitable factors!).

Then Oka's version [O] of the Thom-Sebastiani construction [ST] tells us that the Milnor fiber $F(f+g)$ is homotopy equivalent to the join $F(f)*F(g)$ of the "partial" Milnor fibers $F(f)$ and $F(g)$.

Passing to reduced homology, this gives the following formula (up-to a shift in grading!)

$$\widetilde{H}_*(F(f+g)) \simeq \widetilde{H}_*(F(f)) \otimes \widetilde{H}_*(F(g)) \tag{8}$$

Using the explicit form of the homotopy equivalence $j : F(f)*F(g) \to F(f+g)$ given by Oka [O], it follows that j_* commutes with both the monodromy operators (fact already stated in [O]) and complex conjugation morphisms c_*. This gives us the following

Proposition 9. The dihedral Milnor representation on $\widetilde{H}_*(F(f+g))$ is isomorphic to the tensor product of the dihedral Milnor representations on $\widetilde{H}_*(F(f))$ and $\widetilde{H}_*(F(g))$.

Example 10. Consider again the polynomial f_p from Example 1. Using Example 6 and Prop. 9 repeatedly, we get that the dihedral Milnor representation on the space $\widetilde{H}_*(F(f_p)) = \widetilde{H}_n(F(f_p)) = \mathbb{C}$ is isomorphic to

$$\underbrace{\psi_3 \otimes \cdots \otimes \psi_3}_{(p+1)-times} \otimes \underbrace{\psi_4 \otimes \cdots \otimes \psi_4}_{(n-p)-times}$$

which is ψ_1 when $p+1$ and $n-p$ are both even, ψ_3 when $p+1$ odd and $n-p$ even, ψ_4 when $p+1$ even and $n-p$ odd and, finally, ψ_2 when $p+1$ and $n-p$ are both odd. (Use Lemma 4, (i)).

Remark 11. The real Milnor fibers behave in a similar way under the Thom-Sebastiani construction.

For f and g as above, we introduce the notations

$$F^+(f) = F_{\mathbb{R}}(f), F^-(f) = F_{\mathbb{R}}(-f)$$

(and similarly for g).

Then using similar arguments to [O] one can prove the following.

(i) If $F^+(f) \neq \emptyset$ and $F^+(g) \neq \emptyset$, then there is a homotopy equivalence

$$F^+(f) * F^+(g) \to F^+(f+g)$$

(ii) If $F^+(f) \neq \emptyset$ and $F^+(g) = \emptyset$, then there is a homotopy equivalence

$$F^+(f) \to F^+(f+g)$$

(iii) If $F^+(f) = F^+(g) = \emptyset$, then $F^+(f+g) = \emptyset$.

Note that there are similar properties for $F^-(f)$.

As an example, using (ii) for $f = x_0^2 + \cdots + x_p^2$ and $g = -x_{p+1}^2 - \cdots - x_n^2$ it follows that the real Milnor fiber $F_{\mathbb{R}}(f_p)$ has the homotopy type of S^p as claimed in Example 1.

3. Equivariant Euler characteristics

To simplify the notation, let G denote the dihedral group D_d. Then we can consider the equivariant Euler characteristic of the Milnor fiber F given by

$$\chi_G(F) = \sum_{j=0}^{n} (-1)^j [H_j(F)] \tag{12}$$

the classes [ ] being evaluated in the complex representation ring $R(G)$, see Wall [W].

It is convenient to introduce also the reduced equivariant Euler characteristic $\widetilde{\chi}_G(F)$ defined by the equality

$$\chi_G(F) = \psi_1 + (-1)^n \widetilde{\chi}_G(F) \tag{13}$$

Using Prop. 9 it follows that

$$\widetilde{\chi}_G(F(f+g)) = \widetilde{\chi}_G(F(f)) \otimes \widetilde{\chi}_G(F(g)) \tag{14}$$

i.e. $\widetilde{\chi}_G$ has a nice multiplicative property with respect to the Thom-Sebastiani construction.

Consider now the decomposition of $\widetilde{\chi}_G(F)$ into irreducible G-representations

$$\widetilde{\chi}_G(F) = a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + a_4\psi_4 + \sum_h b_h\rho^h \tag{15}$$

where a_j, b_h are integers. The usual Lefschetz fixed point formula [Sp] applied to c gives the following

$$\chi(F_{\mathbb{R}}) = 1 + (-1)^n(a_1 - a_2 + a_3 - a_4) \tag{16}$$

Let $X = \{x \in \mathbb{R}^{n+1}; f(x) = 0\}$, S^n be the unit sphere in $\mathbb{R}^{n+1}$ and $K_{\mathbb{R}} = X \cap S^n$ be the real link of our polynomial f as in the introduction.

Proposition 17.

$$\chi(K_{\mathbb{R}}) = 1 + (-1)^{n+1} + 2(a_2 - a_1)$$

Proof. Using the $\mathbb{R}_+^*$-action on $\mathbb{R}^{n+1}$ associated with the weights of f, we see that the spaces $\mathbb{R}^{n+1} \setminus X$ and $S^n \setminus K_{\mathbb{R}}$ have the same homotopy type. Hence, using Alexander duality, [Sp] p.296, we get

$$\begin{aligned}\chi(\mathbb{R}^{n+1} \setminus X) &= \chi(S^n \setminus K_{\mathbb{R}}) = (-1)^n\chi(S^n, K_{\mathbb{R}}) \\ &= (-1)^n(\chi(S^n) - \chi(K_{\mathbb{R}})) = (-1)^{n+1}\chi(K_{\mathbb{R}}) + 1 + (-1)^n.\end{aligned}$$

On the other hand, the space $\mathbb{R}^{n+1} \setminus X$ has the same homotopy type as the disjoint union $F^+(f) \cup F^-(f)$, using the notation from Remark 11. Therefore

$$\chi(K_{\mathbb{R}}) = 1 + (-1)^n + (-1)^{n+1}\Big(\chi\big(F^+(f)\big) + \chi\big(F^-(f)\big)\Big).$$

To prove the result, it is enough to show that

$$\widetilde{\chi}_G(F(f)) = a_1\phi_1 + a_2\phi_2 + a_4\phi_3 + a_3\phi_4 + \sum b'_h\rho^h$$

where $a_1, \cdots, a_4$ are the same integers from (15).

When d is odd, this is very simple. Indeed the homeomorphism

$\varphi : F(f) \to F(-f)$ given by
$\varphi(x) = (-1) \cdot x = ((-1)^{w_0}x_0, \cdots, (-1)^{w_n}x_n)$

induces an isomorphism of G-modules $H_j(F(f)) \simeq H_j(F(-f))$ for all j. Moreover, in this case $a_3 = a_4 = 0$ since the representations ϕ_3 and ϕ_4 do not occur!

When d is even, one can again define a homeomorphism φ as above by the formula $\varphi(x) = \lambda \cdot x$, where $\lambda = \exp(\pi i/d)$.

Then it is easy to check that $\varphi h = h\varphi$, i.e. φ commutes with the corresponding monodromy homeomorphisms. On the other hand, $hc\varphi = \varphi c$. When $h_* = 1$, this gives $c_*\varphi_* = \varphi_* c_*$ i.e. φ_* preserves the representations of type ψ_1 and ψ_2. When $h_* = -1$, we have $-c_*\varphi_* = \varphi_* c_*$ i.e. φ_* interchanges the representations of type ψ_3 and ψ_4 (a special case of this phenomenon is seen in Example 6). This ends the proof of Prop. 17.

It is usual to consider the zeta function $\zeta(h)(t)$ of the monodromy homeomorphism $h : F \to F$ defined by

$$\zeta(h)(t) = \prod_{j=0}^{n} \det\left(tI - h_*|H_j(F)\right)^{(-1)^j} \tag{18}$$

This zeta function is known for large classes of weighted homogeneous singularities f, in particular for all isolated singularities [MOr] and for large classes of homogeneous singularities, see [D2].

It is clear that the knowledge of this zeta function $\zeta(h)$ determines the following integers a_1+a_2, a_3+a_4 and b_k, where a_j and b_k are as in (15). Indeed, one has to consider the multiplicity of 1, -1 and respectively, $\exp(2\pi ik)/d)$ as a root (or a pole) in $\zeta(h)$. This fact, together with (16), (17), gives

Corollary 19. *The monodromy zeta function $\zeta(h)$ and the Euler numbers $\chi(F_{\mathbb{R}})$ and $\chi(K_{\mathbb{R}})$ determine the equivariant Euler characteristic $\chi_G(F)$ and conversely.*

Let us consider now the case when 0 is an isolated singularity for f. Then $\widetilde{H}_j(F) = 0$ except for $j = n$ and $K_{\mathbb{R}}$ is a smooth manifold of dimension $n-1$.

Proposition 20. *When O is an isolated singularity for f, the equivariant Euler characteristic $\widetilde{\chi}_G(F)$ satisfies the conditions*

(i) $a_1 = a_2$ *when n is even, and*

(ii) $a_3 = a_4$ *when* n *is odd.*

Proof. Assume that n is even. Then $K_{\mathbb{R}}$ is an odd-dimensional oriented closed manifold and hence $\chi(K_{\mathbb{R}}) = 0$. This implies $a_1 = a_2$ by (17). Assume now that n is odd. If d is odd as well, $a_3 = a_4 = 0$ and there is nothing to prove. When $d = 2d_1$ is even, consider the polynomial

$$\tilde{f}(x_0, \cdots, x_{n+1}) = f(x_0, \cdots, x_n) + x_{n+1}^2$$

which is weighted homogeneous if we set $wt(x_{n+1}) = d_1$. Using (14) and Example 6, if follows that

$$\begin{aligned} \tilde{\chi}_G\left(F(\tilde{f})\right) &= (a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + a_4\psi_4 + \cdots) \otimes \psi_3 \\ &= a_1\psi_3 + a_2\psi_4 + a_3\psi_1 + a_4\psi_2 + \cdots \end{aligned}$$

where $\cdots$ denotes a sum of 2-dimensional irreducible representations. Applying (i) to the polynomial $\tilde{f}$ we get $a_3 = a_4$.

Example 21. When O is not an isolated singularity, the equalities in Prop. 20 do not hold in general. For example, take $f = x_0^2 x_1^2$. Then it is easy to see that $\tilde{\chi}_G(F(f)) = \psi_2 - \psi_3 + \psi_4$.

Assume now that the weights w_j of the polynomial f have been numbered in such a way that $w_0, \cdots, w_k$ are odd for $0 \le k \le n$ and the rest $w_{k+1}, \cdots, w_n$ are even (if any). Then the group $\mathbb{Z}_2 = \{\pm 1\}$ acts on the sphere S^n by the formula $\pm 1 \cdot x = (\pm 1) \cdot x$ and the fixed points are the $n-k-1$ sphere S^{n-k-1} given by the linear section $x_0 = \cdots = x_k = 0$. The quotient $(S^n \setminus S^{n-k-1})/\mathbb{Z}_2$ can be identified with the product $\mathbb{R}P^k \times D^{n-k}$ where

$$D^{n-k} = \{y \in \mathbb{R}^{n-k}; y_1^2 + \cdots + y_{n-k}^2 < 1\}.$$

If the hypersurface $K_{\mathbb{R}}$ is disjoint from the fixed point set S^{n-k-1}, we can define a hypersurface in $\mathbb{R}P^k \times D^{n-k}$ by taking the quotient

$$\overline{K}_{\mathbb{R}} = K_{\mathbb{R}}/\mathbb{Z}_2 \tag{22}$$

When 0 is an isolated singularity for f, we get $\chi(\overline{K}_{\mathbb{R}}) = 0$ for n even, while for n odd we get from (16), (17) and (20) :

$$\chi(\overline{K}_{\mathbb{R}}) = 1 + a_2 - a_1 = \chi(F_{\mathbb{R}}) \tag{23}$$

In particular, for homogeneous polynomials $\overline{K}_{\mathbb{R}}$ is just the projective hypersurface $V_{\mathbb{R}}$ discussed in the introduction.

Example 24. Let $f_k(x_0, x_1)$ be a homogeneous polynomial of degree d with $d - k$ an even non negative integer such that 0 is an isolated singularity for f_k (over $\mathbb{C}$) and the irreducible factorization of f_k in $\mathbb{R}[x_0, x_1]$ has k distinct linear factors. Since the zeta function $\zeta(h)$ is known [M Or] and $K_{\mathbb{R}}$ consists of $2k$ points it follows easily that

$$a_1 = (d-k)/2, \ a_2 = (d+k-2)/2,$$
$$a_3 = a_4 = \begin{matrix} 0 & \text{for } d \textit{ odd} \\ (d-2)/2 & \text{for } d \text{ even} \end{matrix}$$

and $b_\ell = d - 2$ for $1 \le \ell \le (d-1)/2$.

Using the identification of elements in $R[G]$ to class functions on G (by taking characters), we get $\tilde{\chi}_G(F(f_k))(s) = a_1 - a_2 + a_3 - a_4 = -(k-1)$. Using now the formula (14) we get $\tilde{\chi}_G(F(f_{k_1} + f_{k_2}))(s) = (k_1 - 1)(k_2 - 1)$ or

$$\begin{aligned} \chi(V_{\mathbb{R}}(f_{k_1} + f_{k_2})) &= \chi(F_{\mathbb{R}}(f_{k_1} + f_{k_2})) \\ &= 1 - (k_1 - 1)(k_2 - 1) \end{aligned} \tag{24}$$

Hence we get these possibilities for $d = 4$

(k_1, k_2)	(0,2)	(0,4)	(2,2)	(2,4)	(4,4)
$\chi(V_{\mathbb{R}})$	2	4	0	-2	-8

Take now $g = x_0^2 x_1^2$. Using Example 21 it follows that

$$\tilde{\chi}_G(F(g))(s) = -3.$$

As above we get

$$\chi(F_{\mathbb{R}}(g + f_k)) = 1 - 3(k-1)$$

for $k = 0, 2$ or 4.

On the other hand, to determine $\chi(V_{\mathbb{R}}(g + f_k))$ we have to compute first

$$\begin{aligned}\tilde{\chi}_G(F(g+f_k)) &= (\psi_2 - \psi_3 + \psi_4)(a_1\psi_1 + a_2\psi_2 + a_3\psi_3 + a_4\psi_4 + \cdots) \\ &= (a_2 - a_3 + a_4)\psi_1 + (a_1 + a_3 - a_4)\psi_2 \\ &+(-a_1 + a_2 + a_4)\psi_3 + (a_1 - a_2 + a_3)\psi_4 + \cdots \\ &= \tilde{a}_1\psi_1 + \tilde{a}_2\psi_3 + \tilde{a}_3\psi_3 + \tilde{a}_4\psi_4 + \cdots\end{aligned}$$

where $\cdots$ denotes a sum of 2-dimensional irreducible G-representations. Using (17) and (23) we get

$$\begin{aligned}\chi(V_{\mathbb{R}}(g+f_k)) &= 1 + \tilde{a}_2 - \tilde{a}_1 \\ &= 1 + (a_1 + a_3 - a_4) - (a_2 - a_3 + a_4) \\ &= 1 + a_1 - a_2 + 2(a_3 - a_4) = k.\end{aligned}$$

Note that for the *nonisolated singularities* of the form $f + g$ the computation of $\chi(V_{\mathbb{R}})$ or $\chi(K_{\mathbb{R}})$ cannot be done just by using the topological information contained in the Thom-Sebastiani formula. For such computations the consideration of dihedral representations is essential.

Besides (22), there is another *generalization of a real projective hypersurface*.

Let $\mathbb{P} = \mathbb{P}(w_0, \cdots, w_n)$ denote the complex weighted projective space associated to the weights $w_0, \cdots, w_n$, see [D2]. Let $V = V(f)$ be the hypersurface in $\mathbb{P}$ corresponding to the equation $f = 0$.

Let $V_{\mathbb{R}}$ be the set of real points of the variety V. In other words, $V_{\mathbb{R}}$ is the fixed point set of the involution $c : V \to V$ induced by the complex conjugation. When f is homogeneous, this $V_{\mathbb{R}}$ is just the hypersurface in $\mathbb{R}P^n$ given by $f = 0$, as in the Introduction.

Let $K = f^{-1}(0) \cap S^{2n+1}$ be the complex link of the polynomial f at the origin. Note that the projection $p : K \to V,\ x \mapsto [x]$ induces a surjective map $p_{\mathbb{R}} : K_{\mathbb{R}} \to V_{\mathbb{R}}$, for which each fiber consists of exacty two points. We warn the reader that this map $p_{\mathbb{R}}$ does not come from a $\mathbb{Z}_2$-action on $K_{\mathbb{R}}$. Indeed, the involution of $K_{\mathbb{R}}$ obtained by interchanging the two points in each fiber of $p_{\mathbb{R}}$ is not continuous! (see the example discussed below).

Nevertheless, we can stratify this map $p_{\mathbb{R}}$ in such a way that its restriction over every stratum in $V_{\mathbb{R}}$ is a $\mathbb{Z}_2$-bundle. Since Euler numbers behave additively with respect to stratifications (use Lefschetz duality theorem for relative manifolds, see [Sp], p.297 with $\mathbb{Z}_2$ coefficients), we get the following.

Corollary 25.

$$\chi(V_{\mathbb{R}}) = a_2 - a_1 + (1 + (-1)^{n+1})/2.$$

In particular, when f has an isolated singularity at the origin and $n-1 = \dim_{\mathbb{C}} V$ is odd, then $\chi(V_{\mathbb{R}}) = 0$.

Finally, we note that in general $V_{\mathbb{R}}$ is not a $\mathbb{Q}$-manifold (as is V itself being a quasismooth weighted hypersurface, see for instance [D2]). A very simple example is when $w_0 = \cdots = w_{n-1} = 1,\ \ w_n = 2$ and $f(x) = x_0$. Then the link $K_{\mathbb{R}}$ is just the sphere S^{n-1}. The projection $P_{\mathbb{R}}$ identifies the points $(0, x_1, \cdots, x_{n-1}, x_n)$ and $(0, -x_1, \cdots, -x_{n-1}, x_n)$ for all $(0, x_1, \cdots, x_n) \in S^{n-1}$ as well as the points $(0, \cdots, 0, 1)$ and $(0, \cdots, 0, -1)$. The latter identification comes from the multiplication by i.

Hence the space $V_{\mathbb{R}}$ is obtained from the product $\mathbb{R}P^{n-2} \times [-1, 1]$ by collapsing the boundary $\mathbb{R}P^{n-2} \times \{\pm 1\}$ to a point y. It is clear that $V_{\mathbb{R}}$ is not a $\mathbb{Q}$-manifold at this point y.

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Received November 4, 1996