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GLOBAL EXISTENCE OF L^2 SOLUTIONS IN DYNAMICAL ELASTO-PLASTICITY

M. Rascle

Abstract

In previous papers we studied a classical dynamical model of elasto-plasticity, with so-called *isotropic hardening* and we proved that this model has a unique *generalized solution*, which is globally defined. Here we prove independently that the Riemann Problem has a unique solution, which (fortunately !)coincides with the above generalized solution.

1. Introduction

In [1], we have considered dynamical models of elastic-plastic material, with *isotropic hardening* and established the global existence and uniqueness of *solutions* in the Sobolev space H^1. In [6] is defined a notion of *generalized solutions* in L^2, which naturally coincides with the previous one for smooth solutions. The generalized solution to the initial-boundary value problem is shown to be unique and globally defined. Moreover, either notion of solution is associated to a semi-group of pseudo-contractions in L^2. This property, very surprising in the context of conservation laws, is closely related to the absence of shock waves in this model, which itself is implied by the convexity assumptions on the plastic yield curves, as stated below. For instance these very nice properties of the model are not at all satisfied in the class of models considered by B. Plohr, [5]. Here we solve the Riemann problem in the classical way and we show that the unique solution to that problem is also the unique generalized solution introduced in [6]. We also show, as in the above paper that this generalized solution, although only piecewise smooth, satisfies the same evolution variational inequality (2.17) below as does a solution.

The outline of the paper is the following. First the precise model, and the two notions of solutions are recalled in Section 2 below. Then in Section 3 we study the simple waves in the elastic and in the plastic regime, before solving the Riemann Problem in Section 4 and comparing with the generalized solution in Section 5.

2. Basic facts

A. The model.

In [1], [6] we have considered the following dynamical model of a long thin bar of elastic-plastic material, with *isotropic hardening*:

$$\begin{array}{llcl} \partial_t v - \partial_x \sigma & & = & 0 \\ \partial_t \sigma - \partial_x v & +\lambda \,\mathrm{sgn}(\sigma) & = & 0 \\ \partial_t \gamma & +\lambda & = & 0, \ \lambda \geq 0 \,. \end{array} \tag{2.1}$$

In these equations, sgn denotes the sign function, and λ is an unknown Lagrange multiplier, which vanishes in the elastic regime.

Since this kind of model, although classical, is not necessarily familiar to the reader, we first briefly recall the main assumptions. For more details, we refer to the references in [1] and [6].

The first equation in (2.1) is the momentum equation, in which v is the velocity and σ the stress. For simplicity, the density ρ and the Young modulus E are assumed to be equal to 1. We assume that the strain ε can be decomposed into an elastic and a plastic part

$$\varepsilon := \partial_x u = \varepsilon_e + \varepsilon_p \ ,$$

where u is the displacement.

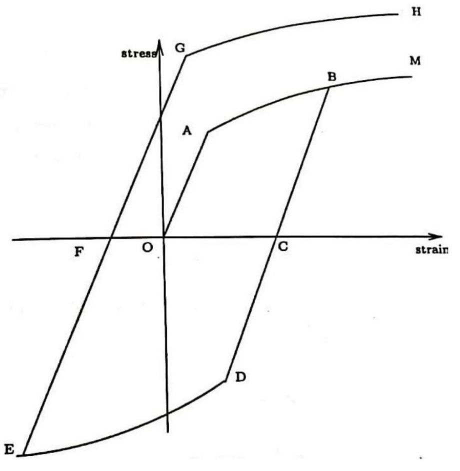

Figure 1: the coordinates of A are (σ_y, σ_y).

For instance in Figure 1 the abscissa of point C is the plastic strain of any point on the straight line BCD. The underlying constitutive relation is compatible with Clausius-Duhem inequality and satisfies the classical *normality* assumptions. Essentially, it can be described as follows:

(i) We assume (since $E = 1$) that

$$\sigma = \varepsilon_e \ , \tag{2.2}$$

(ii) We rewrite the second equation in (2.1) under the form

$$\partial_t \varepsilon_e - \ \partial_x v = \ -\partial_t \varepsilon_p \ , \tag{2.3}$$

which is nothing but the compatibility equation

$$\partial_t \varepsilon = \partial_x v = \partial_x \partial_t u.$$

(iii) We introduce the *accumulated plastic deformation*

$$p := -\gamma := \int_0^t |\partial_t \varepsilon_p(x, s)| \, ds \ . \tag{2.4}$$

(iv) Moreover, we impose that the the solution to satisfies the constraint

$$|\sigma| + g(\gamma) \leq \sigma_y \ . \tag{2.5}$$

In order to do that, we have introduced in (2.1) a nonnegative Lagrange multiplier λ corresponding to this constraint. This parameter is determined dynamically by (2.1)and by the yield condition (2.5) and in turn determines the dynamics of the plastic deformation by

$$\partial_t \varepsilon_p(x,t) = \lambda \mathrm{sgn}(\sigma) \ . \tag{2.6}$$

The function g in (2.5) is implicitly defined in Figure 1, in which the equation for the first yield-curve ABM is

$$\sigma = \ \sigma_y - \ g(-\varepsilon + \sigma_y) \ . \tag{2.7}$$

In other words, g is a smooth non-decreasing convex function, such that $g(0) = 0$, which satisfies for some positive constants c_1 and c_2

$$(H_1) \ \forall \gamma \leq 0, \ 0 < c_1 \leq g'(\gamma) \leq c_2, \ g''(\gamma) \geq 0 \ .$$

(v) Now we introduce

$$U := (v, \sigma, \gamma) \ \text{ and } G(U) := (v, \sigma, g(\gamma))$$

and let us define the *convex sets of plasticity* by

$$C := \{U / \ G(U) \in K\}, \quad K := \{V = (v, \sigma, \beta) / \ |\sigma| + \beta \leq \sigma_y\} \ . \tag{2.8}$$

Notice that K is a cone, and that the velocity v does not play any role in its definition and is only introduced here for mathematical convenience. In fact, we also want $\gamma \leq 0$, but that will be implied by the initial data $\gamma(.,0) = 0$ and the last equation in (2.1).

(vi) We say that U is in the *elastic regime* if

$$U \in Int(C) : \ |\sigma| + g(\gamma) - \sigma_y \ < \ 0, \tag{2.9}$$

or

$$U \in \partial C : \ |\sigma| + g(\gamma) - \sigma_y \ = \ 0, \text{ and } \partial_t|\sigma| \ = \ \mathrm{sgn}(\sigma).\partial_t(\sigma) \ \leq \ 0. \tag{2.10}$$

On the contrary, U is in the *plastic regime* if

$$U \in \partial C : \ |\sigma| + g(\gamma) - \sigma_y \ = \ 0, \ \text{ and } \ \partial_t|\sigma| \ = \ \mathrm{sgn}(\sigma).\partial_t(\sigma) \ > \ 0\,. \tag{2.11}$$

We also notice that the unknown vector

$$(0, \lambda\, \mathrm{sgn}(\sigma), \lambda), \ \ \lambda \geq 0$$

appearing in (2.1) vanishes in the elastic regime and always belongs to the normal exterior cone to K. In convex analysis, see e.g. [3], [2], the set of such vectors is called the *subdifferential* of the indicator function χ_K of the convex K at point $G(U)$ and is denoted by $\partial\chi_K(G(U))$.

Finally, if the problem is set on the domain $\Omega \subset I\!R$, we introduce the Hilbert space $H := (L^2(\Omega))^3$, equipped with its natural scalar product $(.\ ,\ .)$, and we define the unbounded operator A in H by

$$AU := \ (-\partial_x\sigma, -\partial_x v, 0)\ . \tag{2.12}$$

We now can rewrite system (2.1) under the form of an abstract dynamical constrained problem:

$$\partial_t U + \ AU + \ \partial\chi_K(G(U)) \ \ni 0\ . \tag{2.13}$$

In particular, in the elastic regime the subdifferential $\partial\chi_K(U)$ vanishes, and system (2.13 is simply

$$\begin{array}{rcl} \partial_t v - \partial_x\sigma & = & 0\ , \\ \partial_t\sigma - \partial_x v & = & 0\ , \\ \partial_t\gamma & = & 0\ . \end{array} \tag{2.14}$$

In contrast, in the plastic regime the solution U (resp. $G(U)$) " would like" to leave the convex set C (resp. K), so that we need the additional (unknown) nonnegative function λ in system (2.1) to remain in K. All we know about this Lagrange multiplier is its direction and orientation.

B. Solutions and generalized solutions.

Therefore we are left with system (2.13), which is an evolution variational inequality. Naturally, we add initial data

$$U(x,0) = (v,\sigma,\gamma)(x,0) = U_0(x) := (v_0(x),\sigma_0(x),0) \tag{2.15}$$

and boundary conditions, e.g. on $\Omega := (0,L)$

$$v(0,t) = v(L,t) = 0\,. \tag{2.16}$$

We could replace either of these conditions by any other natural boundary condition.

In [1] we defined a **solution** $U = (v,\sigma,\gamma)$ to the initial-boundary value problem (2.13), (2.15), (2.16)as a function with values in the domain $D(A)$ such that

$$\begin{aligned} &\forall U^* = (v^*,\sigma^*,\gamma^*) \in L^1(0,T;H) |\, U^*(x,t) \in C\, a.e., \\ &\int_0^T (\partial_t U + AU,\, G(U) - G(U^*))\, dt \le 0, \end{aligned} \tag{2.17}$$

and which satisfies in an appropriate sense (2.15), (2.16). Due to the nonlinearity of function g, the problem is not that easy, since A is a skew-symmetric operator in H, and therefore is monotone with respect to U, whereas the subdifferential term is monotone with respect to $G(U)$, but *not* with respect to U. Nevertheless, using the Yosida regularization technique for maximal monotone operators in Hilbert spaces, see eg. [2], we proved the global existence and uniqueness of such a solution, and even a nice property of pseudo-contraction in L^2, provided that the initial data satisfy $U_0 \in D(A)$. Note that here

$$D(A) = H^1_0(\Omega) \times H^1(\Omega) \times L^2(\Omega)\,,$$

where the above spaces are the classical Sobolev spaces.

The obvious limitation of this result is of course the above assumptions of regularity and compatibility between initial data and boundary conditions. In particular, this theory is unable to handle the case of a Riemann problem. However, if we want to deal with discontinuous solutions on $\Omega = I\!R$, two difficulties arise:

(i) the solution to the Riemann problem is not even in $H = (L^2(I\!R))^3$! This is a minor problem, since we can always replace $I\!R$ by a large bounded interval Ω and use the finite speed of propagation to describe - locally in time- the solution to the Riemann problem as an element of $H = (L^2(\Omega))^3$.

(ii) the second problem, much deeper, is that the scalar product

$$(\partial_t U + AU, G(U))$$

in (2.17) has no meaning if U is a L^2-solution.
For this reason, we defined in [6] a **generalized solution** $U = (v, \sigma, \gamma)$ to the initial-boundary value problem (2.13), (2.15), (2.16) as a function with values in $H = (L^2)^3$ such that, for almost all t in $(0, T)$, and for all **smooth** function $U^* = (v^*, \sigma^*, \gamma^*)$ with values in C, we have

$$\begin{aligned} &[\int_\Omega (v^2/2 + \sigma^2/2 + \ \mathbf{G}\ (\gamma))\,(x,s)\,dx\,]_{s=0}^{s=t} \\ &- [\int_\Omega (v.v^* + \sigma.\sigma^* + \gamma.g(\gamma^*))\,(x,s)\,dx\,]_{s=0}^{s=t} \\ &+ \int_0^t [\,(\sigma.v^*)(x,s)\,]_{x=0}^{x=L}\,ds \\ &+ \int_0^t \int_\Omega (v.\partial_t v^* + \sigma.\partial_t\sigma^* + \gamma.\partial_t g(\gamma^*) - \sigma.\partial_x v^* - v.\partial_x\sigma^*)\,(x,s)\,dx\,ds \le 0, \end{aligned} \tag{2.18}$$

where

$$\mathbf{G}\ (\gamma) := \int_0^\gamma g(s)\,ds. \tag{2.19}$$

In this formula, the absence of contributions $v.\sigma$ and $v.\sigma^*$ in the boundary term expresses the boundary condition (2.16). Integrating by parts the terms containing the smooth function U^* and applying the chain-rule formula in those which only involve U, it is easy to see that solutions and generalized solutions are the same if they lie in the domain $D(A)$. We can now start studying the Riemann problem, i.e. the initial value problem with initial data

$$U(x,0) = U_-(resp. U_+)\ for\ x < \ 0\ (resp. >\ 0).$$

3. Simple waves

A. The elastic regime

We have already seen that in this case the system (2.1) reduces to (2.14), which is *linear*. Its three characteristic speeds are

$$\lambda_1 = \ -1 < \ \lambda_2 = \ 0 < \ \lambda_3 = \ 1$$

and the corresponding Rankine-Hugoniot relations are

$$\begin{array}{rcl} s[v] & = & -[\sigma] \ , \\ s[\sigma] & = & -[v] \ , \\ s[\gamma] & = & 0 \ . \end{array} \tag{3.1}$$

Moreover, since the system is linear(ly degenerate), for any entropy-flux pair (η, q) we have the entropy *equality*

$$s[\eta] = [q] \ . \tag{3.2}$$

Note that in particular the "physical" pair

$$(\eta, q) = \ (v^2/2 + \sigma^2/2 + \mathbf{G}(\gamma), \ q = -v.\sigma) \tag{3.3}$$

appears in the boundary terms in (2.18).
Therefore U_- and U_+ can be connected by a contact discontinuity of the first (resp. third) family if and only if

$$[v - \sigma] = [\gamma] = 0 \ , \ s = -1; \ (\text{ resp. } [v + \sigma] = [\gamma] = 0 \ , \ s = 1)) \ , \tag{3.4}$$

and by a (stationary) contact discontinuity of the second family if and only if

$$[v] = [\sigma] = 0 \ , \ s = 0. \tag{3.5}$$

B. The plastic regime

First we note that any line $\{x = \mathit{constant}\}$ corresponds to the same material particle, whose state evolves with respect to time. In particular in Figure 1 the particle at point B is "older" than it was at point A. Since the slope of the tangent to the curve ABM is the square of the characteristic speed,the concavity of this curve rules out the possibility of shock waves in the plastic regime. The same holds for the lower family of compression plastic yield curves like curve DE. In contrast, the models considered in [7], [4] or in [5] admit concave plastic compression curves, and therefore admit plastic shock waves, which then raises (for multi-dimensional problems) the difficult question of giving a meaning to products of distributions in non conservative equations. This difficulty is totally avoided here.

To connect U_- and U_+ by a plastic wave, we first note that using (2.4) it is equivalent to determine the evolution of the accumulated plastic deformation, in fact one of γ or ε_p. Knowing either one of these variables at time $t = 0$ is sufficient to specify the plastic yield curve on which each particle is going to evolve. Consequently for $x < 0$ or $x > 0$, the stress σ is a known function of the strain:

$$\sigma = \sigma(\varepsilon) := \sigma_\pm(\varepsilon)$$

Therefore, using (2.3) we rewrite (2.1) under the form

$$\begin{array}{lcl} \partial_t v - \partial_x \sigma(\varepsilon) & = & 0\ , \\ \partial_t \varepsilon - \partial_x v & = & 0\ , \\ \partial_t \gamma & = & -|\partial_t \varepsilon_p|\ . \end{array} \tag{3.6}$$

The first two equations in the above system, uncoupled from the last one, are nothing but the system of one-dimensional elasticity. Here the two eigenvalues of this reduced system are

$$\lambda_\pm := \pm\sqrt{\sigma\prime(\varepsilon)}.$$

They are genuinely nonlinear, and for the reason we already mentioned, the corresponding waves are rarefaction waves, across which the corresponding Riemann invariant

$$w_\pm := v \pm \int_0^\varepsilon \sqrt{\sigma\prime(\varepsilon)} \tag{3.7}$$

is constant, and $|\sigma|$ is increasing with respect to time.

In addition to these two classical waves, we see that the last equation in (3.6) can be rewritten as

$$\partial_t \gamma = -|\partial_t(\varepsilon - \sigma(\varepsilon))| \tag{3.8}$$

Knowing ε determines the time evolution of γ on either side of the line $\{x = 0\}$, and allows any discontinuitiy of γ across this line. Therefore any stationary contact discontinuity (or slip surface in the multi-dimensional case) which satisfies (3.5) is also permitted in the plastic regime.
We are now ready to solve the Riemann problem.

4. The Riemann Problem

We are going to give a graphical proof of existence and uniqueness, using what is known in French as *géométrie descriptive* to depict curves in $I\!R^3$, namely the left-hand part of Figure 2 below describes the projection of the wave curves on the plane (v, ε) and the right-hand part the projection of the *same* curves on the plane $(\varepsilon_p, \varepsilon)$.
We first note that the plastic rarefaction waves are much slower than the (fast) elastic precursor waves, since the slope of the plastic yield curves is much smaller than the Young modulus (a typical ratio is 1/2500). Therefore in Figure 2, the curves in the right-hand part, i.e. the projections of the wave curves on the plane $(\varepsilon_p, \varepsilon)$, have a vertical part, as long as the solution remains in the elastic regime, and the other part of these curves has a slope *strictly bigger* than 1. We also see that stationary contact discontinuities correspond to vertical straight lines on the left-hand part of the picture, and - with appropriate units - to parallels to the bissector of the first quadrant in the right-hand part, since

$$[v] = [\sigma] = [\varepsilon - \varepsilon_p] = 0.$$

In the typical case depicted in Figure 2, the solution U is defined as follows:

(i) $U(x/t) = U_-$ for $x/t < -1$ and then U_- is connected to U_1 by a "fast" (backward) elastic precursor wave, of speed $\lambda_1 = -1$.

(ii)$U(x/t) = U_1$ for $-1 < x/t < \lambda_-(U_1)$, and then U_1 is connected to U_2 by a slow (backward) plastic rarefaction wave.

(iii) $U(x/t) = U_2$ for $\lambda_-(U_2) < x/t < 0$, and then U_2 is connected to U_3 by a stationary contact discontinuity.

(iv) $U(x/t) = U_3$ for $0 < x/t < \lambda_+(U_3)$, and then U_3 is connected to U_4 by a slow (forward) plastic rarefaction wave.

(v) $U(x/t) = U_4$ for $\lambda_+(U_4) < x/t < +1$, and then U_4 is connected to U_+ by a fast (forward) elastic precursor of speed $+1$.
This solution involves five different waves, or more accurately one composite backward (resp. forward)wave, corresponding to the increasing (resp. decreasing) curve Γ_- (resp. Γ_+) in the left-hand part of Figure 2 (and to C_- (resp. C_+) in the right-hand part of Figure 2) , and a stationary contact discontinuity. We see that the crucial part in the above description is this contact discontinuity between U_2 and U_3. In the (v, ε) plane, we must have $v_2 = v_3 := v$, whereas in the $(\varepsilon_p, \varepsilon)$ plane, we must have

$$F(v) := [\sigma] = \varepsilon_2 - \varepsilon_3 - \varepsilon_{p,2} - \varepsilon_{p,3} = 0. \tag{4.1}$$

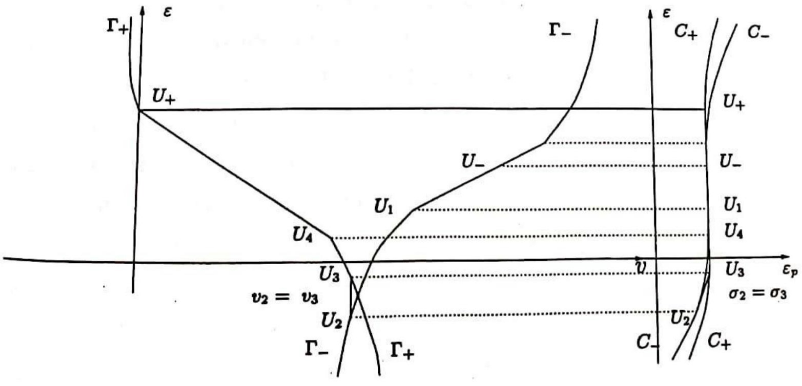

Figure 2: The Riemann Problem: a typical example.

Since the slopes of curves $C_\pm$ are strictly bigger than 1, we easily see that F is a *strictly* increasing function. Therefore

Theorem 4.1
For all Riemann data U_- and U_+, there exists a unique solution to the corresponding Riemann problem.

We are now ready to check that this solution is also the unique generalized solution defined in Section 2.

5. Generalized solution to the Riemann Problem

As we already said, any solution to (2.13), (2.15), (2.16) is a generalized solution. Conversely, integrating by parts, it is easy to see that any generalized solution with values in the domain $D(A)$ satisfies (2.17) and therefore is a solution. More generally, as in [6], let us consider a piecewise-smooth generalized solution U and let us assume that the distribution

$$\partial_t U + AU$$

does not charge the curves of discontinuity of U. In particular this is the case for the above solution to the Riemann Problem , which is piecewise-smooth and does not involve any shock wave in the plastic regime. For this solution the only possible discontinuities are elastic contact discontinuities or stationary contact discontinuities in the plastic regime. In either case, the wave is linearly degenerate. Therefore, integrating (2.18) by parts on each side of the curves of discontinuity and using (3.3) and the Rankine-Hugoniot conditions (3.1) (3.2) as in [6], we easily see that there is no contribution on these curves of discontinuity, so that we recover (2.17). Conversely the same calculation shows that under the same regularity assumptions we would recover (2.17) from (2.18). Therefore

Theorem 5.2
(i) Any solution to the initial-boundary value problem (2.13), (2.15), (2.16) is a generalized solution.
(ii) Conversely, any piecewise-smooth generalized solution to the initial-boundary value problem (2.13), (2.15), (2.16) satisfies (2.17) for each smooth $U^* \in C$, provided that the distribution

$$\partial_t U + AU$$

does not charge the curves of discontinuity.
(iii) In particular, the above solution to the Riemann Problem satisfies (2.17) and the above regularity assumptions, and therefore is the unique generalized solution to (2.13), (2.15), (2.16).

In conclusion, on one hand the definition of generalized solution introduced in [6] is quite natural from the point of view of functional analysis, and on the other hand it recovers the natural solution to the Riemann Problem. Therefore this theory is quite satisfactory for the class of models considered here. Now modeling *realistic* impact problems with *"real"* materials is quite an ambitious program, see e.g. [5] for an attempt in this direction, and definitely exceeds the purpose of this work, whose main advantage - not negligible ! - is simplicity.

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Laboratoire de Mathématiques, U.R.A. C.N.R.S. 168
Université de Nice, Parc Valrose B.P. 71, F 06108 Nice Cédex 2

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