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THE INITIAL VALUE PROBLEM FOR A GENERALIZED BOUSSINESQ MODEL: REGULARITY AND GLOBAL EXISTENCE OF STRONG SOLUTIONS

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Abstract

We study the global existence in time, as well as the regularity, of strong solutions of the partial differential equations of evolution type corresponding to a generalized Boussinesq model for thermically driven flows. The model includes the case in which the fluid viscosity and thermal conductivity depend on the temperature.

Resumo

Estudamos a existência global no tempo, bem como a regularidade, de soluções fortes das equações diferenciais parciais de evolução correspondentes a um modelo do tipo Boussinesq generalizado para escoamentos convectivos termicamente induzidos. O modelo permite que a viscosidade do fluido e o coeficiente de condutividade térmica dependam da temperatura.

1. Introduction

In this work we study global existence and regularity of strong solutions of the equations governing the coupled mass and heat flow of a viscous incompressible fluid in a generalized Boussinesq approximation by assuming that viscosity and heat conductivity are temperature dependent. The equations are

$$\begin{cases} \partial_t u \ - \operatorname{div}(\nu(\varphi)\nabla u) + u.\nabla u - \alpha\varphi g + \nabla p = h, \\ \operatorname{div}\, u = 0, \\ \partial_t\varphi - \operatorname{div}(k(\varphi)\nabla\varphi) + u.\nabla\varphi = f \quad \text{in} \quad (0,T]\times\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $I\!R^N$, $N = 2$ or 3. Here $u(t,x) \in I\!R^N$ denotes the velocity of the fluid at point $x\in\Omega$ at time $t\in[0,T]$; $p(t,x)\in I\!R$ is the hydrostatic pressure; $\varphi(t,x) \in I\!R$ is the temperature; $g(t,x)$ is the external force by unit of mass, $\nu(\,\cdot\,) > 0$ and $k(\,\cdot\,) > 0$ are the kinematic viscosity and thermal conductivity, respectively; α is a positive constant associated to the coefficient of volume expansion. The expression ∇, Δ and div denote the gradient, Laplace and divergence operator, respectively; the $i^{\underline{\text{th}}}$ component in cartesian coordinates of $u\nabla u$ is given by $(u\nabla u)_i = \sum_{j=1}^{N} u_j \dfrac{\partial u_i}{\partial x_j}$; also $u\nabla\varphi = \sum_{j=1}^{n} u_j \dfrac{\partial \varphi}{\partial x_j}$.

The boundary conditions and initial data are as follows

$$\begin{cases} u = 0\ , \quad \varphi = 0 \quad \text{on} \quad (0,T] \times \partial\Omega, \\ u(0,x) = u_0(x) \quad \text{and} \quad \varphi(0,x) = \varphi_0(x) \quad \text{for} \quad x \in \Omega, \end{cases} \tag{1.2}$$

where u_0, φ_0 are given functions on Ω. For simplicity, we will consider homogeneous conditions on $\partial\Omega$ (the boundary of Ω); the general case can be reduced to this one by assuming suitable smoothness on the boundary data (concerning this point, see [7]). Such a reduction leads only to change in the right-hand sides of (1.1), by addition of certain linear and nonlinear terms, which do not influence the proofs of the final results in a essential way. For derivation of the equations (1.1), see for instance Drazin and Reid [3].

The classical Boussinesq equations correspond to the special case where ν and k are positive constants (see Morimoto [9], Óeda [10] and Hishida [5].) For certain fluids, we can not disregard the variation of the viscosity (and thermal conductivity) with temperature, this being important in the determination of the details of the flow. In particular, it is believed that the temperature dependence of the viscosity is responsible for the fact that the direction of the flow in the middle of a convection cell is usually different for gases and liquids (see [6] and the references there in). Thus, it is important to know well the properties of equations (1.1), if one intends to understand details of thermal convection phenomena.

However, from the mathematical point of view, equations (1.1) have been less studied than the ones in the usual Boussinesq approximations, maybe due to the stronger nonlinear coupling between the equations. In fact, a rigorous mathematical analysis is more difficult in this situation than that in the case of the classical Boussinesq equations. Concerning the existence of solution of (1.1), the constructive spectral Galerkin method was used by Lorca and Boldrini [7] to obtain a local strong solution (see [7] and the next section for the precise statements of the results.) Global existence and regularity of solutions were open questions, however, that should be answered if one intends to do further analysis, like the one involved in the study of error bounds for the approximations, which is important from the practical point of view, and the bifurcation analysis done to understand the onset and characteristics of thermal convection. The analysis of error bounds was done in [8], while the study of bifurcation is under way.

In this paper, we describe our results about global existence and regularity of solutions of (1.1). For this, we observe that in [4] Heywood gave an interesting variant of the analysis for the Galerkin method, showing that the solution is regular for positive times. The main point in his method, is to show how the obtaining of estimates for the approximations, used in the Galerkin approach to existence theorems, can be pushed further, to give the classical regularity of the solution directly and easily, with minimal reliance on the regularity theory for the Stokes' equations. We will combine those arguments with additional specific estimates for our Galerkin approximations. With these further estimates, one can infer the existence of a regular solution; then, with this degree of regularity in hand, the fact that the solution has classical regularity follows by the L^p-estimates for the steady generalized Boussinesq equations (see [6]).

Concerning the global existence we obtain a result by assuming that h and f belong to $L^\infty(0,\infty;L^2(\Omega))$, that certain other regularity assumptions to be detailed later on hold, and, as it is usual, that the data have small enough norms. We observe that we do not require any sort of decay in time of the associated external forces.

2.Regularity of the solution

We begin by recalling certain definitions and facts to be used later on in this paper.

In what follows the functions are either $I\!R$ or $I\!R^N$ valued ($N = 2$ or 3), and to easy the notation, sometimes we will not distinguish them in our notation; this will be clear from the context. The $L^2(\Omega)$-product and norm are denoted by $(\ ,\ )$ and $|\ |$, respectively; the $L^p(\Omega)$ norm by $|\ |_p, 1 \leq p \leq \infty$; the $H^m(\Omega)$ norm are denoted by $||\ ||_m$ and the $W^{k,p}(\Omega)$-norm by $|\ |_{k,p}$. Here $H^m(\Omega) = W^{m,2}(\Omega)$ and $W^{k,p}(\Omega)$ are the usual Sobolev Spaces (see Adams [1]; for their properties); $H^1_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in the H^1-norm.

If B is a Banach space, we denote $L^q(0,T;B)$ the Banach space of the B-valued functions defined in the interval $(0,T)$ that are L^q-integrable in the sense of Bochner.

Let $C^\infty_{0,\sigma}(\Omega) = \{v \in C^\infty_0(\Omega)^N;\ \text{div}\ v = 0 \ \text{ in } \ \Omega\}$; $V =$ closure of $C^\infty_{0,\sigma}(\Omega)$ in $(H^1_0(\Omega))^N$, and $H =$ closure of $C^\infty_{0,\sigma}(\Omega)$ in $(L^2(\Omega))^N$.

Let P be the orthogonal projection from $(L^2(\Omega))^N$ onto H obtained by the usual Helmholtz decomposition. Then the operator $\tilde{\Delta} : H \to H$ given by $\tilde{\Delta} = -P\Delta$ with domain $D(\tilde{\Delta}) = (H^2(\Omega))^N \cap V$ is called the Stokes operator. It is well known that $\tilde{\Delta}$ is a positive definite self-adjoint operator and is characterized by the relation

$$(\tilde{\Delta}w, v) = (\nabla w, \nabla v) \quad \text{for all} \quad w \in D(\tilde{\Delta}), \quad v \in V\ .$$

In order to obtain regularity properties of the Stokes operator we will assume that Ω is of class $C^{1,1}$ (see Amrouche and Girault [2]). This assumption implies, in particular, the equivalence of the norm given by the Stokes operator and the $V \cap H^2(\Omega)$ norm.

We will denote by v^k and α_k ($k \in I\!N$) respectively the eigenfunctions and eigenvalues of the Stokes operator. We know that v^k are orthogonal in the inner products $(\ ,\ ), (\nabla, \nabla)$ and $(\tilde{\Delta}, \tilde{\Delta})$ and are complete in the spaces H, V and $(H^2(\Omega))^N \cap V$ (see Temam ([12]).

Similar considerations are true for the Laplace operator Δ; we will denote by ψ^k and λ_k ($k \in I\!N$) respectively the eigenfunctions and eigenvalues of the operator $-\Delta$ defined on $H_0^1(\Omega) \cap H^2(\Omega)$.

For each $n \in I\!N$, we denote P_n the orthogonal projection from $(L^2(\Omega))^N$ onto $V_n = \text{span}\{v^1, \dots, v^n\}$ and we denote $\tilde{P}_n$ the orthogonal projection from $L^2(\Omega)$ onto $W_n = \text{span}\{\psi^1, \dots, \psi^n\}$.

Throughout the paper, we will suppose that

$$\begin{cases} 0 < \nu_0 < \nu(\sigma) < \nu_1 < +\infty \ , \ 0 < k_0 < k(\sigma) < k_1 < +\infty \\ |\nu'(\sigma)| < \nu_1' < +\infty \ , \ |k'(\sigma)| < k_1' < +\infty \quad \text{for all} \quad \sigma \in I\!R \ . \end{cases} \tag{2.1}$$

Remark. We note that if we have a maximum principle for φ in (1.1)-(1.2), then we can relax the assumption for ν, k. In this case it is sufficient to suppose $\nu(\ \cdot\ ) > 0$ and $k(\ \cdot\ ) > 0$ and then we can transform problem (1.1)-(1.2) into an equivalent one in the case of strong solutions (see [6], [7], for details). For example, we have this kind of maximum principle when $f \leq 0$ and $\varphi_0 \in L^\infty(\Omega)$.

Now, we rewrite problem (1.1)–(1.2) as follows: find $u \in L^\infty(0,T;V) \cap L^2(0,T;(H^2(\Omega))^N)$, $\partial_t u \in L^2(0,T;(L^2(\Omega))^N)$ and $\varphi \in L^\infty(0,T;H^2(\Omega) \cap H_0^1(\Omega))$, $\partial_t \varphi \in L^\infty(0,T;L^2(\Omega))$ ($0 < T \leq +\infty$) such that

$$\begin{cases} (\partial_t u, v) - (\text{div}(\nu(\varphi)\nabla u), v) + (u.\nabla u, v) = (\alpha\varphi g - h, v), \quad \forall v \in V \ , \\ (\partial_t \varphi, \xi) - (\text{div}(k(\varphi)\nabla\varphi), \xi) + (u.\nabla\varphi, \xi) = (f, \xi) \ , \quad \forall \xi \in H_0^1(\Omega) \\ u/_{t=0} = u_0 \quad ; \quad \varphi/_{t=0} = \varphi_0 \quad , \quad a.e. \quad x \in \Omega \ . \end{cases} \tag{2.2}$$

The spectral Galerkin approximation for (u, φ) are defined for each $n \in I\!N$ as the solution $(u^n, \varphi^n) \in C^2([0,T], V_k \times W_k)$ of

$$\begin{cases} (\partial_t u^n, v) - (\text{div}(\nu(\varphi^n)\nabla u^n), v) + (u^n.\nabla u^n, v) = (\alpha\varphi^n g - h, v), \quad \forall v \in V_k \ , \\ (\partial_t \varphi^n, \xi) - (\text{div}(k(\varphi^n)\nabla\varphi^n), \xi) + (u^n.\nabla\varphi^n, \xi) = (f, \xi), \quad \forall \xi \in W_k \ , \\ u^n = P_n u_0 \quad ; \quad \varphi^n = P_n \varphi_0 \ . \end{cases} \tag{2.3}$$

By using these approximations, the authors proved in [7] a local in time existence theorem for (2.2). The result is (with slight modifications in presentation).

Proposition 2.1. *Let Ω be a bounded domain in $I\!R^n$ ($N = 2$ or 3) with $C^{1,1}$ boundary; we suppose ν, k satisfying (2.1), $g \in L^\infty(0,T;(L^2(\Omega))^N$, $f \in L^2(0,T;L^2(\Omega))$, $\partial_t f \in L^2(0,T;L^2(\Omega))$; $h \in L^2(0,T;(L^2(\Omega))^N)$; $u_0 \in V$, $\varphi_0 \in$*

$H_0^1(\Omega) \cap H^2(\Omega)$. *Then there exists a positive number* $T^* \leq T$ *such that the problem (2.2) has a unique solution* (u, φ) *satisfying*

$$u \in L^\infty(0,T^*;V) \cap L^2(0,T^*;(H^2(\Omega))^N);\ \partial_t u \in L^2(0,T^*;(L^2(\Omega))^N) \quad (2.4)$$

$$\varphi \in L^\infty(0,T^*;H^2(\Omega));\ \partial_t \varphi \in L^\infty(0,T^*;L^2(\Omega)) \quad (2.5)$$

$$u(t) \to u_0 \text{ strongly in } V \text{ and } \varphi(t) \to \varphi_0 \text{ weakly in } H^2(\Omega) \text{ as } t \to 0+ \quad (2.6)$$

The proof of Proposition 2.1 follows by proving estimates for the approxima-

tions (u^n, φ^n) that are uniform in n. They are then carried to (u, φ) in the limit.

In the case in which $u_0 \in V \cap (H^2(\Omega)^N)$ and g and h are more regular, we can obtain estimates for $|\tilde{\Delta} u^n(t)|$, uniformly in n which are shown in the same way as it was done for $|\Delta \varphi^n(t)|$ in [7], Theorem 2.2. These estimates imply the following result.

Proposition 2.2. *Under the conditions of Proposition 2.1, if* $\partial_t g \in (L^2(0,T; L^2(\Omega))^N)$, $\partial_t h \in (L^2(0,T;L^2(\Omega))^N)$ *and* $u_0 \in V \cap (H^2(\Omega)^N)$, *then there is* $T^* > 0$ *such that* (u, φ) *satisfies (2.5); (2.6) and*

$$u \in L^\infty(0,T^*;(H^2(\Omega))^N);\ \partial_t u \in L^\infty(0,T^*;(L^2(\Omega))^N) \quad (2.7)$$

$$u(t) \to u_0 \quad \text{weakly in } (H^2(\Omega))^N \text{ as } t \to 0+\,. \quad (2.8)$$

Now, we state the first new result in this paper.

Theorem 2.3. *Let* Ω *be a bounded domain in* $I\!R^N$ $(N = 2$ *or* $3)$ *with* C^∞ *boundary. Assume* ν, k *of class* C^∞ *satisfying (2.1),* $g, h \in (C^\infty(\overline{\Omega} \times [0,T]))^N$, $f \in C^\infty(\overline{\Omega} \times [0,T])$, $\varphi_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $u_0 \in V \cap (H^2(\Omega))^N$. *Then, the solution* (u, φ) *obtained in Proposition 2.1 is a classic solution, that is,* $u \in (C^\infty(\overline{\Omega} \times (0,T^*])^N \cap (C(\overline{\Omega} \times [0,T^*])^N$ *and* $\varphi \in \mathcal{C}^\infty(\overline{\Omega} \times [0,T^*]) \cap \mathcal{C}(\overline{\Omega} \times [0,T^*])$.

We first state some lemmas which are necessary for proving this theorem. Below, $\partial_t^k \varphi$ represents the $\text{k}^{\underline{\text{th}}}$ derivative of φ with respect to t, and $\partial_x^\ell \varphi$ is an arbitrary $\ell^{\underline{\text{th}}}$ order derivative of φ with respect to the spatial variables. Similar notation are used with u.

Lemma 2.4. *Let h be any function of class C^k, $k \geq 1$ such that* $\sup\left\{|\dfrac{d^i h}{dt^i}(s)|,\ s \in I\!R\right\} \leq C < +\infty; i = 0, 1, \ldots, k.$

Let φ be a function in $\mathcal{C}^k([0,T]; H^2(\Omega))$. Define

$$M_\ell(\varphi(t)) = \sum_{i=0}^{\ell} ||\partial_t^i \varphi(t)||_2 \quad , \quad t \in [0,T] \quad , \quad \ell = 0, 1, \ldots, k \ .$$

Then, we have the following estimate

$$|\partial_t^k h(\varphi(t))|_\infty \leq J_k(M_{k-1}(\varphi(t))) + C_k |\partial_t^k \varphi(t)|_\infty$$

for all $t \in [0,T]$. Where J_k is a continuous increasing function and C_k is a positive constant which depends on h.

Proof. We proceed by induction on k. If $k = 1$, we can take $J_1 \equiv 0$ and $C_1 \geq \sup_{I\!R} |h'(s)|$. So, suppose the result is true for any $j \in I\!N$ such that $0 < j < k$. Then, we have

$$\begin{aligned} \partial_t^k h(\varphi) &= \partial_t^{k-1}(h'(\varphi)\partial_t \varphi) = \sum_{j=0}^{k-1} C(j) \partial_t^j h'(\varphi) \partial_t^{k-j} \varphi \\ &= \sum_{j=1}^{k-1} C(j) \partial_t^j h'(\varphi) \partial_t^{k-j} \varphi + h'(\varphi) \partial_t^k \varphi \ . \end{aligned}$$

Thus, by the inductive hypothesis and Sobolev imbedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we have

$$\begin{aligned} |\partial_t^k h(\varphi)|_\infty &\leq \sum_{j=1}^{k-1} C(j) |\partial_t^j h'(\varphi)|_\infty |\partial_t^{k-j} \varphi|_\infty + C_k |\partial_t^k \varphi|_\infty \\ &\leq \sum_{j=1}^{k-1} C(j)(J_j(M_{j-1}(\varphi)) + C_j |\partial_t^j \varphi|_\infty) |\partial_t^{k-1} \varphi|_\infty + C_k |\partial_t^k \varphi|_\infty \\ &\leq \sum_{j=1}^{k-1} C(j)(J_j(M_{k-1}(\varphi)) + C_j K M_{k-1}(\varphi)) K M_{k-1}(\varphi) + C_k |\partial_t^k \varphi|_\infty \\ &= J_k(M_{k-1}(\varphi)) + C_k |\partial_t^k \varphi|_\infty, \end{aligned}$$

and the lemma is proved. □

Similarly, we can show:

Lemma 2.5. *If h satisfies the conditions of Lemma 2.4, then for all $\varphi \in \mathcal{C}^k([0,T];H^2(\Omega))$ we have*

$$||\partial_t^k h(\varphi(t))||_{1,4} \leq L_k(M_{k-1}(\varphi(t))) + C_k|\nabla \partial_t^k \varphi(t)|_4 + C_k|\partial_t^k \varphi(t)|_\infty |\nabla \varphi(t)|_4$$

for all $t \in [0,T]$. Where L_k is a continuous increasing function.

To obtain a classical solution, we need estimates of the solution's higher order derivatives. We work first to establish the regularity of u and φ with respect to t. Our main task is to prove further estimates for the Galerkin approximations.

Lemma 2.6. *Under the conditions of Proposition 2.2. For every $j = 0, 1, 2, \ldots$ and every $0 < \varepsilon < T^*$; there exist continuous functions $F_j(t,\varepsilon), G_j(t,\varepsilon)$ and $H_j(t,\varepsilon)$ of $t \in [\varepsilon, T^*]$, such that*

$$\begin{aligned} |\nabla \partial_t^j u^n(t)|^2 + |\nabla \partial_t^j \varphi^n(t)|^2 &+ \int_\varepsilon^t (|\tilde{\Delta} \partial_t^j u^n|^2 + |\Delta \partial_t^j \varphi^n|) ds \qquad (2.9) \\ &\leq F_j(t,\varepsilon) \\ |\partial_t^{j+1} u^n(t)|^2 + |\partial_t^{j+1} \varphi^n(t)|^2 &+ \int_\varepsilon^t (|\nabla \partial_t^{j+1} u^n|^2 + |\nabla \partial_t^{j+1} \varphi^n|^2) ds \qquad (2.10) \\ &\leq G_j(t,\varepsilon) \end{aligned}$$

and

$$|\tilde{\Delta} \partial_t^j u^n(t)|^2 + |\Delta \partial_t^j \varphi^n(t)|^2 \leq H_j(t,\varepsilon) \qquad (2.11)$$

for $t \in [\varepsilon, T^*]$. The functions $F_j(\cdot,\varepsilon)$, $G_j(\cdot,\varepsilon)$ and $H_j(\cdot,\varepsilon)$ do not depend on n. The right endpoint T^*, of the interval on which the estimates (2.9)–(2.11) hold, is the same as for the Proposition 2.2.

Proof. The estimates (2.9)–(2.11) will be proved by induction on j. For $j = 0$; estimates (2.9)–(2.11) can be deduced from the proof of Proposition 2.2, see [7];

p. 14 (in fact, the proof also follows by taking $k = 0$ in the next estimates). Moreover, in this case we can take $\varepsilon = 0$.

Let $k \geq 1$ be a natural number; we have to prove that (2.9)–(2.11) are true for $j = k$.

As induction hypothesis, let us suppose that these estimates are true for any $j \in I\!N$ such that $0 \leq j < k$ (independently of n).

To obtain the estimates for $j = k$, we proceed as follows: by differentiating (2.3) k-times with respect to t and then taking $v = -\tilde{\Delta}\partial_t^k u(t)$ (to easy the notation, on what follows we will suppress the superscript n), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|\nabla\partial_t^k u|^2 + (\mathrm{div}\ \partial_t^k(\nu(\varphi)\nabla u), \tilde{\Delta}\partial_t^k u) \\ &= (\partial_t^k(u\nabla u), \tilde{\Delta}\partial_t^k u) - (\partial_t^k(\alpha\varphi g) + \partial_t^k h, \tilde{\Delta}\partial_t^k u)\ .\end{aligned} \tag{2.12}$$

Most of the right-hand side terms are essentially the terms found by differentiating k-times with respect to t the Galerkin approximation for the usual Navier-Stokes equation. We can use the estimates of Heywood [4]

$$\begin{aligned}|(\partial_t^i u\nabla\partial_t^{k-i}u, \tilde{\Delta}\partial_t^k u)| &\leq |\partial_t^i u|_6|\nabla\partial_t^{k-i}u|_3|\tilde{\Delta}\partial_t^k u| \\ &\leq C_\delta|\nabla\partial_t^i u|^2|\tilde{\Delta}\partial_t^{k-i}u|^2 + \delta|\tilde{\Delta}\partial_t^k u|^2\end{aligned} \tag{2.13}$$

for $1 \leq i \leq k$, and

$$\begin{aligned}|(u\nabla\partial_t^k u, \tilde{\Delta}\partial_t^k u)| &\leq |u|_\infty|\nabla\partial_t^k u||\tilde{\Delta}\partial_t^k u| \\ &\leq C_\delta|\tilde{\Delta}u|^2|\nabla\partial_t^k u|^2 + \delta|\tilde{\Delta}\partial_t^k u|^2\ .\end{aligned} \tag{2.14}$$

Next, we estimate the new terms on the right-hand side of (2.12) as follows

$$\begin{aligned}|(\partial_t^{k-i}\varphi\ \partial_t^i g, \tilde{\Delta}\partial_t^k u)| &\leq |\partial_t^i g|\ |\partial_t^{k-i}\varphi|_\infty|\tilde{\Delta}\partial_t^k u|^2 \\ &\leq C_\delta|\partial_t^i g|^2\ |\Delta\partial_t^{k-i}\varphi|^2 + \delta|\tilde{\Delta}\partial_t^k u|^2\end{aligned} \tag{2.15}$$

for $0 < i \leq k$, and

$$\begin{aligned}&|(\partial_t^k\varphi g, \tilde{\Delta}\partial_t^k u)| \leq |g|\ |\partial_t^k\varphi|_\infty|\tilde{\Delta}\partial_t^k u| \\ &\leq C|g|\ |\nabla\partial_t^k\varphi|_4|\tilde{\Delta}\partial_t^k u| \leq C|g|\ |\nabla\partial_t^k\varphi|^{1/4}|\Delta\partial_t^k\varphi|^{3/4}|\tilde{\Delta}\partial_t^k u| \\ &\leq C_\delta|g|^8|\nabla\partial_t^k\varphi|^2 + \delta|\tilde{\Delta}\partial_t^k u|^2 + \delta|\Delta\partial_t^k\varphi|^2\ .\end{aligned} \tag{2.16}$$

To estimate the new term on the left side of (2.12), we observe that

$$\text{div } \partial_t^k(\nu(\varphi)\nabla u) = \text{div } (\nu(\varphi)\nabla\partial_t^k u) + \sum_{i=1}^{k} C(i)\text{div } (\partial_t^i\nu(\varphi)\nabla\partial_t^{k-i}u)$$

and

$$\text{div } (\partial_t^i\nu(\varphi)\nabla\partial_t^{k-i}u) = \partial_t^i\nu(\varphi)\Delta\partial_t^{k-i}u + \nabla\partial_t^i\nu(\varphi)\nabla\partial_t^{k-i}u \tag{2.17}$$

where $\nabla\partial_t^i\nu(\varphi)\nabla\partial_t^{k-i}u$ denotes the vector fields which $\text{j}^{\underline{\text{th}}}$ component is given by $[\nabla\partial_t^i\nu(\varphi)\nabla\partial_t^{k-i}u]_j = (\nabla\partial_t^i\nu(\varphi), \nabla\partial_t^{k-i}u)_{I\!R^N}$.

From the above, we obtain the following estimates

$$\begin{aligned} |(\text{div } (\partial_t^i\nu(\varphi)\nabla\partial_t^{k-i}u), \tilde{\Delta}\partial_t^k u)| &\leq C_\delta|\partial_t^i\nu(\varphi)|_\infty^2|\Delta\partial_t^{k-i}u|^2 \\ &+C_\delta|\nabla\partial_t^i\nu(\varphi)|_4^2|\nabla\partial_t^{k-i}u|_4^2 + \delta|\tilde{\Delta}\partial_t^k u|^2 \end{aligned} \tag{2.18}$$

for $1 \leq i \leq k$.

In the following, we use Lemmas 2.4 and 2.5 and Sobolev embeddings to conclude that for $i = 1, \ldots, k-1$ there hold

$$\begin{cases} |\partial_t^i\nu(\varphi)|_\infty^2|\Delta\partial_t^{k-i}u|^2 \leq \tilde{J}_i(M_{k-1}(\varphi))|\tilde{\Delta}\partial_t^{k-i}u|^2, \\ |\nabla\partial_t^i\nu(\varphi)|_4^2|\nabla\partial_t^{k-i}u|_4^2 \leq \tilde{L}_i(M_{k-1}(\varphi))|\tilde{\Delta}\partial_t^{k-i}u|^2, \end{cases} \tag{2.19}$$

where $\tilde{J}_i(\tau) = (J_i(\tau) + C_i\tau)^2$ and $\tilde{L}_i(\tau) = (L_i(\tau) + C_k\tau + C_k\tau^2)^2$.

When $i = k$, we have the estimates

$$\begin{aligned} |\partial_t^k\nu(\varphi)|_\infty^2|\Delta u|^2 &\leq (J_k(M_{k-1}(\varphi)) + C|\partial_t^k\varphi|_\infty^2)|\tilde{\Delta}u|^2 \\ &\leq J_k(M_{k-1}(\varphi))|\tilde{\Delta}u|^2 + C|\nabla\partial_t^k\varphi|^{1/2}|\Delta\partial_t^k\varphi|^{3/2}|\tilde{\Delta}u|^2 \\ &\leq J_k(M_{k-1}(\varphi))|\tilde{\Delta}u|^2 + C_\delta|\nabla\partial_t^k\varphi|^2|\tilde{\Delta}u|^8 + \delta|\Delta\partial_t^k\varphi|^2 \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &|\nabla\partial_t^k\nu(\varphi)|_4^2|\nabla u|_4^2 \leq (L_k(M_{k-1}(\varphi)) + C|\nabla\partial_t^k\varphi|_4^2 + C|\partial_t^k\varphi|_\infty^2|\nabla\varphi|_4^2)|\tilde{\Delta}u|^2 \\ &\leq L_k(M_{k-1}(\varphi))|\tilde{\Delta}u|^2 + C|\nabla\partial_t^k\varphi|_4^2(|\tilde{\Delta}u|^2 + |\nabla\varphi|_4^2|\tilde{\Delta}u|^2) \\ &\leq L_k(M_{k-1}(\varphi))|\tilde{\Delta}u|^2 + C_\delta|\nabla\partial_t^k\varphi|^2(|\nabla u|^2 + |\nabla\varphi|_4^2|\tilde{\Delta}u|^2)^4 + \delta|\Delta\partial_t^k\varphi|^2. \end{aligned} \tag{2.21}$$

Combining estimates (2.13)–(2.21) with (2.12), we conclude

$$
\begin{aligned}
&\frac{1}{2}\ \frac{d}{dt}|\nabla\partial_t^k u|^2 - (\mathrm{div}(\nu(\varphi)\nabla\partial_t^k u), \tilde{\Delta}\partial_t^k u) \qquad (2.22)\\
&\le C_\delta(\tilde{J}(M_{k-1}(\varphi)\sum_{\ell=0}^{k-1}|\tilde{\Delta}\partial_t^\ell u|^2 + \sum_{\ell=0}^{k-1}|\nabla\partial_t^{k-\ell}u|\ |\tilde{\Delta}\partial_t^\ell u|^2)\\
&+C_\delta(\sum_{\ell=0}^{k-1}|\partial_t^{k-\ell}g|^2\ |\Delta\partial_t^\ell\varphi|^2 + |\partial_t^k h|^2 + |\tilde{\Delta}u|^2|\nabla\partial_t^k u|^2)\\
&+C_\delta(|g|^2 + |\tilde{\Delta}u|^2 + |\tilde{\Delta}u|\ |\Delta\varphi|^2)^4|\nabla\partial_t^k\varphi|^2) + \delta|\tilde{\Delta}\partial_t^k u|^2 + \delta|\Delta\partial_t^k\varphi|^2\ .
\end{aligned}
$$

In the following, we use the decomposition $\tilde{\Delta}\partial_t^k u + \nabla\partial_t^k q = -\Delta\partial_t^k u$ (see Temam [12]) for the second term on the left side of (2.22).

$$
\begin{aligned}
&(\mathrm{div}(\nu(\varphi)\nabla\partial_t^k u), \tilde{\Delta}\partial_t^k u)(\nu(\varphi)\Delta\partial_t^k u, \tilde{\Delta}\partial_t^k u) + (\nu'(\theta)\nabla\varphi\nabla\partial_t^k u, \tilde{\Delta}\partial_t^k u)\\
&\quad -(\nu(\varphi)\tilde{\Delta}\partial_t^k u, \tilde{\Delta}\partial_t^k u) - (\nu(\varphi)\nabla\partial_t^k q, \tilde{\Delta}\partial_t^k u) + (\nu'(\theta)\nabla\varphi\nabla\partial_t^k u, \tilde{\Delta}\partial_t^k u).
\end{aligned}
$$

By using estimates similar to the ones used in Theorem 2.2 of [7], we arrive at the following estimates

$$
\begin{aligned}
|(\nu'(\varphi)\nabla\varphi\nabla\partial_t^k u, \tilde{\Delta}\partial_t^k u)| &\le C_\delta|\nabla\varphi|_4^2|\nabla\partial_t^k u|_4^2 + \delta/2|\tilde{\Delta}\partial_t^k u|^2\\
&\le C_\delta|\nabla\varphi|_4^8|\nabla\partial_t^k u|^2 + \delta|\tilde{\Delta}\partial_t^k u|^2,
\end{aligned}
$$

$$
\begin{aligned}
|(\nu(\varphi)\nabla\partial_t^k q, \tilde{\Delta}\partial_t^k u)| &= |(\partial_t^k q, \mathrm{div}(\nu(\varphi)\tilde{\Delta}\partial_t^k u)|\\
&= |(\partial_t^k q, \nu'(\varphi)\nabla\varphi\tilde{\Delta}\partial_t^k u)|\\
&\le C|\partial_t^k q|_4|\nabla\varphi|_4|\tilde{\Delta}\partial_t^k u|\\
&\le C_\varepsilon|\nabla\varphi|_4|\nabla\partial_t^k u|^{1/4}|\tilde{\Delta}\partial_t^k u|^{7/4} + \varepsilon|\nabla\varphi|_4|\tilde{\Delta}\partial_t^k u|^2\\
&\le C_\delta|\nabla\varphi|_4^8|\nabla\partial_t^k u|^2 + \delta|\tilde{\Delta}\partial_t^k u|^2\ .
\end{aligned}
$$

Theses estimates together (2.22) imply

$$
\begin{aligned}
&\frac{1}{2}\ \frac{d}{dt}|\nabla\partial_t^k u|^2 + \nu_0|\tilde{\Delta}\partial_t^k u|^2 \le C_\delta\hat{J}(M_{k-1}(\varphi), M_{k-1}(u), \sum_{\ell=0}^{k-1}|\partial_t^{k-\ell}g|^2)\\
&+C_\delta|\partial_t^k h|^2 + C_\delta\hat{L}(|g|, |\tilde{\Delta}u|, |\Delta\varphi|)(|\nabla\partial_t^k\varphi|^2 + |\nabla\partial_t^k u|^2) + \delta|\tilde{\Delta}\partial_t^k u|^2 +\\
&+\delta|\Delta\partial_t^k\varphi|^2
\end{aligned}
$$

where $\hat{J}$ and $\hat{L}$ are continuous increasing functions. $M_{k-1}(\varphi)$ and $M_{k-1}(u)$ are as in Lemma 2.4.

The proof of the following inequality is completely analogous to the previous one

$$\frac{1}{2}\frac{d}{dt}|\nabla\partial_t^k\varphi|^2+k_0|\Delta\partial_t^k\varphi|^2\le C_\delta\hat{J}_1(M_{k-1}(\varphi),M_{k-1}(u))+C_\delta|\partial^k f|^2$$

$$+\hat{L}_1(|\tilde{\Delta}u|,|\Delta\varphi|)(|\nabla\partial_t^k u|^2+|\nabla\partial_t^k\varphi|^2)+\delta|\Delta\partial_t^k\varphi|^2\ .$$

Finally, by adding the above inequalitie and by choosing a suitable small δ, we deduce the following differential inequality (we put back the superscrip n, in order to be clear that the estimates to be obtained in the following are independent of n):

$$\begin{aligned}\frac{d}{dt}\mathcal{N}^n(t)+\tau^n(t)\qquad &\le \hat{J}_2(M_{k-1}(\varphi^n),M_{k-1}(u^n))+(C|\partial_t^k h|^2+|\partial_t^k f|^2)\\ &\quad+\hat{L}_2(|g|,|\tilde{\Delta}u^n|,|\Delta\varphi^n|)\mathcal{N}^n(t),\end{aligned}\tag{2.23}$$

where

$$\mathcal{N}^n(t)=|\nabla\partial_t^k u^n(t)|^2+|\nabla\partial_t^k\varphi^n(t)|^2\ \ ;\ \ \tau^n(t)=|\tilde{\Delta}\partial_t^k u^n(t)|^2+|\Delta\partial_t^k\varphi^n(t)|^2\ .$$

and $\hat{J}_2$, $\hat{L}_2$ can be taken as continuous increasing functions (which are independent of n).

Next, we must find estimates, holding independently of n, for the "initial values" of $\mathcal{N}^n(t)$. We recall that our induction hypothesis is that the estimates (2.9)–(2.11) are true for any j such that $0\le j<k$, independently of n. Thus, by using the induction assumption (2.10), for every $\varepsilon>0$, one obtains

$$\int_{\varepsilon/2}^{t}(|\nabla\partial_t^k u^n|^2+|\nabla\partial_t^k\varphi^n|^2)ds\le G_{k-1}(t,\varepsilon/2)$$

for $t\in[\varepsilon/2,T^*]$, independently of n. From this, we conclude that for each Galerkin approximation (u^n,φ^n) there exists a number γ_n, with $\varepsilon/2<\gamma_n<\varepsilon$, such that

$$\mathcal{N}^n(\gamma_n)=|\nabla\partial_t^k u^n(\gamma_n)|^2+|\nabla\partial_t^k\varphi^n(\gamma_n)|\le(\varepsilon/2)^{-1}G_{k-1}(\varepsilon,\varepsilon/2)\ .\tag{2.24}$$

One can integrate (2.23) from γ_n to t and apply Gronwall's Lemma to find

$$\begin{aligned}&\mathcal{N}^n(t) + \int_{\gamma_n}^t \tau^n(s)ds \\ &\leq A^n(t,\gamma_n)\ \exp\ [\int_{\gamma_n}^t \hat{L}_2(|g(s)|, |\tilde{\Delta} u^n(s)|, |\Delta\varphi^n(s)|)ds].\end{aligned}$$

where

$$\begin{aligned}A^n(t,\gamma_n) \ &= \mathcal{N}^n(\gamma_n) + \int_{\gamma_n}^t (\hat{J}_2(M_{k-1}(\varphi^n(s)), M_{k-1}(u^n(s))) + (C|\partial_t^k h(s)|^2 \\ &+|\partial_t^k f(s)|^2)ds.\end{aligned}$$

From (2.24) and the induction hypothesis (we observe that the functions M_{k-1} depend only on the the norms of $|\partial_t^j \varphi^n|$ and $|\partial_t^j u^n|$, for $0 \leq j \leq k-1$, which are uniformly bounded by the induction hypothesis), the terms in the right-hand side of the above inequality are bounded independently of n. Thus, there is a continuous increasing function $F_k(t,\varepsilon)$, independent of n such that

$$\mathcal{N}^n(t) + \int_{\gamma_n}^t \tau^n(t)(s)ds \leq F_k(t,\varepsilon)$$

for $t \in [\gamma_n, T^*]$. Since $\gamma_n < \varepsilon$, we obtain (2.9).

In the following, we prove (2.10). Differentiating (2.3) $k+1$-times with respect to t and then taking $v = \partial_t^{k+1}u(t)$, one obtains

$$\frac{1}{2}\frac{d}{dt}|\partial_t^{k+1}u|^2 + (\partial_t^{k+1}(\nu(\varphi)\nabla u), \nabla\partial_t^{k+1}u) = (\partial_t^{k+1}(u\nabla u + \varphi g + h), \partial_t^{k+1}u)\ . \quad (2.25)$$

Using estimates similar to the ones used by Heywood [4] for the Navier-Stokes equations, we can obtain a bounds for the terms on the right side of (2.25). The other terms on the left of (2.25) can be estimated in the same way as we shown (2.23), for example we get

$$\begin{aligned}&|(\partial_t^k \nu(\varphi)\nabla\partial_t u, \nabla\partial_t^{k+1}u)| \leq |\partial_t^k\nu(\varphi)|_\infty |\nabla\partial_t u||\nabla\partial_t^{k+1}u| \\ &\quad \leq C_\delta(\tilde{J}_{k-1}(M_{k-1}(\varphi)) + |\partial_t^k\varphi|_\infty^2)|\nabla\partial_t u|^2 + \delta|\nabla\partial_t^{k+1}u|^2\end{aligned} \quad (2.26)$$

$$\begin{aligned}&|(\partial_t^{k+1}\nu(\varphi)\nabla u, \nabla\partial_t^{k+1}u)| = |(\partial_t^k(\nu'(\varphi)\partial_t\varphi)\nabla u, \nabla\partial_t^{k+1}u)| \\ &\quad = |\sum_{i=0}^k C(i)(\partial_t^i\nu'(\varphi)\partial_t^{k+1-i}\varphi\nabla u, \nabla\partial_t^{k+1}u)| \\ &\quad \leq C_\delta(|\nabla u|_6^4|\partial_t^{k+1}\varphi|^2 + \sum_{i=1}^k \tilde{J}(M_{k-1}(\varphi))|\nabla u|_4^2|\partial_t^{k+1-i}\varphi|_4^2) \\ &\quad +C_\delta|\partial_t^k\varphi|_\infty^2|\partial_t\varphi|_4^2|\nabla u|_4^2 + \delta|\nabla\partial_t^{k+1}u|^2 + \delta|\nabla\partial_t^{k+1}\varphi|^2\ .\end{aligned} \quad (2.27)$$

Thus, we can obtain the following inequality.

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}|\partial_t^{k+1}u|^2+\nu_0|\nabla\partial_t^{k+1}u|^2\le C_\delta\hat{J}(M_{k-1}(\varphi))\sum_{\ell=1}^{k}(|\nabla\partial_t^\ell u|^2+|\nabla u|_4^2|\nabla\partial_t^\ell\varphi|^2)\\
&+C_\delta(\sum_{\ell=1}^{k}|\nabla\partial_t^\ell u|^2|\nabla\partial_t^{k+1-\ell}u|^2+\sum_{\ell=1}^{k+1}|\partial_t^\ell g|^2|\nabla\partial_t^{k+1-\ell}\varphi|^2\\
&+C_\delta|\Delta\partial_t^k\varphi|^2(|\nabla\partial_t u|^2+|\nabla\partial_t\varphi|^2|\nabla u|_4^2)+C_\delta|\partial_t^{k+1}h|^2\\
&+C_\delta(|g|^4\tilde{\Delta}u|^4)|+|\partial_t^{k+1}\varphi|^2+C_\delta|\nabla u|^4|\partial_t^{k+1}u|^2+\delta|\nabla\partial_t^{k+1}u|^2+\delta|\nabla\partial_t^{k+1}\varphi|^2\ .
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}|\partial_t^{k+1}\varphi|^2+k_0|\nabla\partial_t^{k+1}\varphi|^2\le C_\delta\hat{J}(M_{k-1}(\varphi))\sum_{\ell=1}^{k}(\nabla\partial_t^\ell\varphi|^2+|\nabla\varphi|_4^2|\nabla\partial_t^\ell\varphi|^2)\\
&+C_\delta(\sum_{\ell=1}^{k}|\nabla\partial_t^\ell u|^2|\nabla\partial_t^{k+1-\ell}u|^2+|\Delta\partial_t^k\varphi|^2|\nabla\partial_t\varphi|^2(1+|\nabla\varphi|_4^2))\\
&+C_\delta|\Delta\varphi|^4(|\nabla\partial_t^{k+1}\varphi|^2+|\nabla\partial_t^{k+1}u|^2)+C_\delta|\partial_t^{k+1}f|^2+\delta|\nabla\partial_t^{k+1}u|^2\\
&+\delta|\nabla\partial_t^{k+1}\varphi|^2\ .
\end{aligned}$$

Then, for appropriate $\delta>0$ we find

$$\begin{aligned}
&\frac{d}{dt}\mathcal{N}_1^n(t)+\tau_1^n(t)\le\hat{J}_3(M_{k-1}(\varphi^n),\sum_{\ell=1}^{n}|\nabla\partial_t^\ell u^n|^2)\\
&\quad+C(|\partial_t^{k+1}f|^2+|\partial_t^{k+1}h|^2)+C\hat{L}_3(|g|,|\tilde{\Delta}u^n|,|\Delta\varphi^n|)\mathcal{N}_1^n(t)\\
&\quad+C|\Delta\partial_t^k\varphi^n|^2(|\nabla\partial_t u^n|^2+|\nabla\partial_{tt}\varphi^n|^2)(1+\Delta u^n|^2+|\Delta\varphi^n|^2)
\end{aligned}\tag{2.28}$$

where

$$\mathcal{N}_1^n(t)=|\partial_t^{k+1}u^n(t)|^2+|\partial_t^{k+1}\varphi^n(t)|^2\quad;\quad\tau_1^n(t)=|\nabla\partial_t^{k+1}u^n(t)|^2+|\nabla\partial_t^{k+1}\varphi^n(t)|^2$$

and $\hat{J}_3,\ \hat{L}_3$ are continuous functions.

Now, we observe the equality

$$|\partial_t^{k+1}u^n|^2=(\operatorname{div}(\partial_t^k\nu(\varphi^n)\nabla u^n),\partial_t^{k+1}u^n)+(\partial_t^k(g\varphi^n)+\partial_t^k h-\partial_t^k(u^n\nabla u^n),\partial_t^{k+1}u^n)$$

which is obtained by differentiating (2.3) k-times with respect to t and taking $v=\partial_t^{k+1}u^n(t)$. We use the estimates (2.13)–(2.21) together (2.23) in order to conclude that

$$\int_\varepsilon^t|\partial_t^{k+1}u^n|^2ds\le I_k(t,\varepsilon)$$

for $t \in [\varepsilon, T^*]$, $\varepsilon > 0$. Analogously, we get

$$\int_{\varepsilon}^{t} |\partial_t^{k+1}\varphi^n|^2 ds \le I_k(t,\varepsilon)\ .$$

Thus, proceeding as before, we conclude that for each Galerkin approximation (u^n, φ^n), there exists a number γ_n, with $\varepsilon/2 < \gamma_n < \varepsilon$, such that

$$|\partial_t^{k+1}u^n(\gamma_n)|^2 + |\partial_t^{k+1}\varphi^n(\gamma_n)|^2 \le (\varepsilon/2)^{-1} I_k(t,\varepsilon/2)\ .$$

Then, since ε is arbitrary, by integrating (2.28) from γ_n to t we obtain (2.10).

Finally, we observe that inequality (2.23) implies

$$\begin{aligned} \tau^n(t) &\le \hat{J}_2(t) + C(|\partial_t^k h(t)|^2 + |\partial_t^k f(t)|^2 + \hat{L}_2(t)\mathcal{N}^n(t) \\ &+ 2(\partial_t^{k+1}u(t), \tilde{\Delta}\partial_t^k u(t)) + 2(\partial_t^{k+1}\varphi(t), \Delta\partial_t^k\varphi(t)) \\ &\le \hat{J}_2(t) + C(|\partial_t^k h(t)|^2 + |\partial_t^k f(t)|^2) + \hat{L}_2(t)\mathcal{N}^n(t) \\ &+ C\sqrt{\mathcal{N}_1^n(t)}\ \sqrt{\tau^n(t)}\ . \end{aligned}$$

Since $\mathcal{N}_1^n(t)$ is bounded, we obtain estimates (2.11). This completes the proof of lemma. □

As an immediate consequence of Lemma 2.6 we obtain

Corollary 2.7. *The solution (u,φ) obtained in Proposition 2.2 satisfies $(u,\varphi) \in \mathcal{C}^\infty((0,T^*];(H^2(\Omega))^{N+1}) \cap \mathcal{C}([0,T^*];(W^{1,4}(\Omega))^{N+1})$.*

Proof. We follows Rautmann [11] p. 433 in applying the theorem of Ascoli and Arzelá, in order to conclude that

$$(u,\varphi) \in \mathcal{C}^n((0,T^*]; V \times (H_0^1(\Omega))^N) \quad \text{for any} \quad n = 0,1,2. \tag{2.29}$$

Next, we observe that estimates (2.9) and (2.10), for the solution, imply $\partial_t^k u, \partial_t^k \varphi$ are almost everywhere equal to functions of $C((0,T^*];H^2(\Omega))$, for any $k = 0,1,2,\ldots$. This fact together (2.29) imply $(u,\varphi) \in \mathcal{C}^\infty((0,T^*];(H^2(\Omega))^{N+1})$. By Proposition 2.2 we have $(u(t),\varphi(t)) \to (u_0,\varphi_0)$ weakly in $(H^2(\Omega))^{N+1})$ as $t \to 0^+$, then by Sobolev's embedding $(u,\varphi) \in C([0,T^*];(W^{1,4}(\Omega))^{N+1})$. □

Now, we are in a position to prove the full classical regularity of the solution.

Proof of Theorem 2.3. By the usual Sobolev imbeddings we note that it is sufficient to show

$$u \in \mathcal{C}^\infty((0,T^*];(H^{\ell+2}(\Omega))^N)\ ;\ \varphi \in \mathcal{C}^\infty((0,T^*];W^{\ell+2,4}(\Omega)) \qquad (2.30.\ell)$$

for any $\ell = 0,1,2,\ldots$.

The proof of (2.30.ℓ) is done by induction on ℓ. We begin by observing that for any fixed t, and for any $k = 0,1,2,\ldots$, $\partial_t^k u$ and $\partial_t^k \varphi$ are solution of

$$\begin{cases} -\mathrm{div}(\nu(\varphi)\nabla\partial_t^k u) + \nabla\partial_t^k\ p = \tilde{k}_k, \\ \quad \mathrm{div}\ \partial_t^k u = 0, \\ -\mathrm{div}(k(\varphi)\nabla\partial_t^k \varphi) = \tilde{f}_k \quad \text{in}\ \ \Omega\ , \\ \partial_t^k u = 0, \\ \partial_t^k \varphi = 0 \quad \text{on}\ \ \partial\Omega\ , \end{cases}$$

where

$$\begin{aligned} \tilde{k}_k &= \sum_{i=0}^{k} C(i)(\partial_t^i u\nabla\partial_t^{k-i}u + \partial_t^{k-i}\varphi\partial_t^i g) \\ &+ \sum_{i=1}^{k} C(i)\mathrm{div}(\partial_t^i\nu(\varphi)\nabla\partial_t^{k-i}u) + \partial_t^k h - \partial_t^{k+1}u \\ \tilde{f}_k &= \sum_{i=0}^{k} C(i)(\partial_t^i u\nabla\partial_t^{k-i}\varphi) + \sum_{i=1}^{k} C(i)\mathrm{div}(\partial_t^i k(\varphi)\nabla\partial_t^{k-i}\varphi) + \partial_t^k f - \partial_t^{k+1}\varphi\ . \end{aligned}$$

Because $\tilde{f}_0 \in \mathcal{C}((0,T^*];L^4(\Omega))$ and by regularity properties in the stationary case (see [6], Theorem 2.3), we can conclude that $\varphi \in \mathcal{C}((0,T^*];W^{2,4}(\Omega))$. This fact together with the known regularity of (u,φ) imply that $\tilde{f}_1 \in C((0,T^*];L^4(\Omega))$; thus, we can apply the L^p-regularity on the stationary problem once again, to obtain $\partial_t\varphi \in \mathcal{C}((0,T^*];W^{2,4}(\Omega))$. By induction one see $\partial_t^k\varphi \in \mathcal{C}((0,T^*];W^{2,4}(\Omega))$ for $k = 0,1,2,\ldots$. It follows (2.30.0).

Now, we suppose (2.30.$\ell-1$) holds (that is, $u \in \mathcal{C}^\infty((0,T^*];H^{\ell+1}(\Omega))$ and $\varphi \in \mathcal{C}^\infty((0,T^*];W^{\ell+1,4}(\Omega))$. Then a slight modification of Theorem 2.3 in [6] allows us to conclude $\varphi \in \mathcal{C}((0,T^*];W^{\ell+2,4}(\Omega))$ and $u \in \mathcal{C}((0,T^*];H^{\ell+2}(\Omega))$. Next, by induction on k (combining arguments of the proof of [6] Theorem

2.3 and of Lemma 2.6 in order to prove $\tilde{f}_k \in \mathcal{C}((0,T^*];W^{k,4}(\Omega))$ and $\tilde{h}_k \in \mathcal{C}((0,T^*];H^k(\Omega)))$, we obtain $\partial_t^k\varphi \in \mathcal{C}((0,T^*];W^{\ell+2,4}(\Omega))$ and $\partial_t^k u \in \mathcal{C}((0,T^*];H^{\ell+2}(\Omega))$ for any $k=0,1,\dots$. This proves Theorem 2.3. $\square$

Remark. In a standard way we can obtain information on the associated pressure. In fact, under the hypothesis of Theorem 2.3, if the classical (u,φ) is the solution of (2.2), there exist $p \in C^\infty(\overline{\Omega}\times(0,T^*])$, $\int_\Omega p(t,x)dx = 0$ for any $t \in (0,T^*]$ such that (u,φ,p) is solution of (1.1), (1.2). This is a direct consequence of regularity results (see Amrouche and Girault [2]).

3. Global Existence

In this section we present a sequence of estimates for the (strong) solutions of (2.2) and their spectral approximations. These estimates are important to obtain a global solution, an also they are used in a essencial way to get uniform in time error bounds for the spectral approximation (see [8])

We have the folling result

Theorem 3.1 *Let Ω be a bounded domain in $I\!R^3$ with $C^{1,1}$ boundary, we suppose ν,k satisfying (2.1). Assume also that $g \in L^\infty(0,\infty;L^2(\Omega))$, $h \in L^\infty(0,\infty;L^2(\Omega))$, $f \in L^\infty(0,\infty;L^2(\Omega))$, $\partial_t f \in L^\infty(0,\infty;L^2(\Omega))$, $u_0 \in V$ and $\varphi_0 \in H_0^1(\Omega)\cap H^2(\Omega)$. If $|\nabla u_0|$, $|\Delta\varphi_0|$, $||f||_{L^\infty(0,\infty;L^2(\Omega))}$, $||\partial_t f||_{L^\infty(0,\infty;L^2(\Omega))}$, $||h||_{L^\infty(0,\infty;L^2(\Omega))}$ and $\alpha||g||_{L^\infty(0,\infty;L^2(\Omega))}$ (where α is the parameter associated to the volume expansion) are sufficient small. Then the solution (u,φ) described in Proposition 2.1 exists globally in time. Moreover, there exist some finite positive constants β and $M = M(u_0,\varphi_0,f,h,\alpha g)$ such that*

$$\sup_{t\geq 0}|\Delta\varphi(t)| \leq \beta \quad , \sup_{t\geq 0}\{|\nabla u(t)| + |\nabla\varphi(t)|\} \leq M \tag{3.1}$$

Also, the same kind of estimate hold uniformly in n for the spectral Galerkin approximations.

Proof. The estimates will be proved for the approximation (u^n,φ^n); as usual they can be carried to (u,φ) in the limit. Also, when one obtains the above estimates (specifically the fact that $|\nabla u(t)|$ does not blow up in finite time), the existence of global solutions in time is immediate. To prove the estimates, we work as in the proof of local existence theorem (Proposition 2.1, see [8]; p. 14), to find

$$\begin{aligned}
&\frac{d}{dt}|\nabla u^n|^2 + |\tilde{\Delta} u^n|^2 \leq C|\nabla u^n|^6 + C|\Delta\varphi^n|^2|\tilde{\Delta} u^n|^2 \\
&+C\alpha|g|^2|\Delta\varphi^n|^2 + C|h|^2, && (3.2)\\
&|\partial_t u^n|^2 \leq C(|\Delta u^n|^6 + C|\nabla\varphi^n|^2|\tilde{\Delta} u^n|^2 + C|g|^2|\nabla\varphi^n|^2 \\
&+C|\tilde{\Delta} u^n|^2 + C|h|^2, && (3.3)\\
&\frac{d}{dt}|\partial_t\varphi^n|^2 + |\nabla\partial_t\varphi^n|^2 \leq C|\nabla\varphi^n|^2|\partial_t u^n|^2 + C|\nabla\partial_t\varphi^n|^2|\Delta\varphi^n|^2 \\
&+C|\partial_t f|^2, && (3.4)\\
&|\partial_t\varphi^n|^2 \leq C(|\nabla u^n|^4 + |\Delta\varphi^n|^2)|\Delta\varphi^n|^2 + C|f|^2 + C|\Delta\varphi^n|^2, && (3.5)\\
&|\Delta\varphi^n|^2 \leq C(|\nabla u^n|^2 + |\Delta\varphi^n|^2)|\Delta\varphi^n|^2 + C|f|^2 + C|\partial_t\varphi^n|^2. && (3.6)
\end{aligned}$$

We observe that, since the parameter α is one of the parameters that must be controled, we have explicited it in (3.2). Also, the constants C in the above estimates are independent of n. These estimates can be obtained in standard way; for example, estimate (3.6) is obtained by taking $\xi = -\Delta\varphi^n$ in (2.3).

Now, by combining (3.2)–(3.4), we obtain

$$\begin{aligned}
&\frac{d}{dt}(|\nabla u^n|^2 + |\partial_t\varphi^n|^2) + |\tilde{\Delta} u^n|^2 + |\nabla\partial_t\varphi^n|^2 \\
&\leq C|\Delta\varphi^n|^2|\nabla\partial_t\varphi^n|^2 + C|\partial_t f|^2 + C|\Delta\varphi^n|^2|\tilde{\Delta} u^n|^2 \\
&+C(1 + C|\Delta\varphi^n|^2)(|\nabla u^n|^6 + |\Delta\varphi^n|^2|\tilde{\Delta} u^n|^2 + \alpha|\Delta\varphi^n|^2|g|^2 + |h|^2),
\end{aligned}$$

which, by adding suitable positive terms to the right-hand side of the last inequality and rearranging, it can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}(|\nabla u^n|^2 + |\partial_t\varphi^n|^2) + |\tilde{\Delta} u^n|^2 + |\nabla\partial_t\varphi^n|^2 \\
&\leq (C|\Delta\varphi^n|^2)[2 + (C|\Delta\varphi^n|^2)][|\tilde{\Delta} u^n|^2 + |\nabla\partial_t\varphi^n|^2] && (3.7)\\
&+C[1 + (C|\Delta\varphi^n|^2)][|\nabla u^n|^2 + |\partial_t\varphi^n|^2]^3 \\
&+(C|\Delta\varphi^n|^2)[1 + (C|\Delta\varphi^n|^2)]\alpha|g|^2 + C[1 + (|\Delta\varphi^n|^2)]|h|^2 + C|\partial_t f|^2,
\end{aligned}$$

where the positive constant C is independente of n and of the initial conditions. Also, by taking the larger one, we can consider the constant C appearing in (3.7) the same as the constant C appearing in (3.6). We stress this point because this constant will play a special role in what follows.

Now, we will choose the constant $\beta > 0$ for which the stated result is true. Fix β as a positive constant satisfying

$$C\beta < \frac{1}{8}, \tag{3.8}$$

where C is the positive constant appearing in (3.7) (and (3.6)).

In the following, we will show that, when the initial data and external fields are suitably small, there hold

$$\sup_{t\geq 0} |\Delta\varphi^n(t)|^2 < \beta \tag{3.9}$$

and $|\nabla u^n(t)| + |\nabla\varphi^n(t)|\} \leq M$ for all $t \geq 0$ and some $M > 0$.

In fact, let the initial condition $\varphi(0)$ satisfy $|\Delta\varphi(0)|^2 < \beta$. Then, by our choice of basis function, it is true that $|\Delta\varphi^n(0)|^2 < \beta$ for any n.

Observe that (3.9) means that, for small initial data and external force fields, $\varphi(t)$ exists globally in time and satisfies the stated estimate.

To prove this, we start by remarking that

$$\varphi^n(t) = \sum_{i=1}^{n} c_i^n(t)\psi^i(x)$$

where the $c_i^n(t)$ are found by solving a system of ordinary differential equations, and therefore continuous in t in the interval of existence $[0, t_2^n)$, where $0 < t_2^n \leq \infty$. Thus,

$$\Delta\varphi^n(t) = \sum_{i=1}^{n} c_i^n(t)\lambda_i\psi^i(x)$$

is also continuous on $[0, t_2^n)$ with values in $L^2(\Omega)$; since $|\Delta\varphi^n(0)|^2 < \beta$, either $\sup_{t\geq 0} |\Delta\varphi^n(t)|^2 < \beta$ or all $t \in [0, t_2^n))$, which means that there is no blow-up and in fact $t_2^n = \infty$ and (3.9) is true, or there is t_1^n, with $0 < t_1^n < t_2^n$, a time for which we have $|\Delta\varphi^n(t)|^2 < \beta$ for $t \in [0, t_1^n)$ and $|\Delta\varphi^n(t_1^n)|^2 = \beta$.

Let us show, arguing by contradiction, that this late possibility does not hold for small initial data and external force fields. For this, assume that $\varphi(t)$ satisfies the second possibility.

Introduce the auxiliary variable

$$\eta_n(t) = |\nabla u^n(t)|^2 + |\partial_t \varphi^n(t)|^2,$$

and observe that due to the assumption that $|\Delta\varphi^n(t)| \leq \beta$ for $t \in [0, t_1^n]$, on this interval we have $C|\Delta\varphi(t)|^2 \leq C\beta < 1/8$ (according to our choice of β in (3.8). Thus, on the interval $t \in [0, t_1^n]$, we can estimate the right-hand side of (3.7) to obtain

$$\frac{d}{dt}\eta_n + \frac{1}{2}(|\tilde{\Delta} u^n|^2 + |\nabla \partial_t \varphi^n|^2) \leq 2C\eta_n^3 + 2C(\alpha|g|^2\beta + |h|^2 + |\partial_t f|^2) \tag{3.10}$$

Now we observe that there exists $C_1 > 0$, which is independent of n such that

$$C_1(|\nabla u^n|^2 + |\partial_t \varphi^n|^2) \leq \frac{1}{2}(|\tilde{\Delta} u^n|^2 + |\nabla \partial_t \varphi^n|^2).$$

Then, (3.10) implies that on the interval $[0, t_1^n]$ it holds

$$\begin{aligned} \frac{d}{dt}\eta_n &\leq 2C\eta_n^3 - C_1\eta_n + C_2, \\ \eta_n(0) &= |\nabla u_0^n|^2 + |\partial_t \varphi^n(0)|^2, \end{aligned}$$

where

$$C_2 = 2C \sup_{t \geq 0}\{\alpha|g(t)|^2\beta + |h(t)|^2 + |\partial_t f(t)|^2\}.$$

The comparison theorem for differential inequalities implies $\eta_n(t) \leq \phi(t)$ for all t in the interval $[0, t_1]$ where ϕ satisfies

$$\begin{aligned} \frac{d}{dt}\phi &= 2C\phi^3 - C_1\phi + C_2 = R(\phi, C_2), \\ \phi(0) &= |\nabla u_0|^2 + |\partial_t \varphi(0)|^2. \end{aligned}$$

(Observe that, again by our choice of basis functions, $|\nabla u_0^n|^2 + |\partial_t \varphi^n(0)|^2 \leq |\nabla u_0|^2 + |\partial_t \varphi(0)|^2$; also, observe that C and C_1 do not depend on the initial data and the external force field.)

Next, we observe that $R(\phi, 0) = 2C\phi^3 - C_1\phi$ has three simple roots: $r_1(0) = -\left(\frac{C_1}{2C}\right)^{1/2}$, which is unstable, $r_2(0) = 0$, which is stable, and $r_3(0) = \left(\frac{C_1}{2C}\right)^{1/2}$, which is unstable. For $C_2 > 0$ small, $R(\phi, C_2)$ has one unstable simple root $r_1(C_2)$, a stable simple root $r_2(C_2)$ and a unstable simple root $r_3(C_2)$, satisfying $r_1(C_2) < 0 < r_2(C_2) < r_1(C_2)$, and, moreover, $r_2(C_2) \to 0+$ (in a monotonically decreasing way) as $C_2 \to 0+$. Also, for such small values of $C_2 > 0$, $R(\phi, C_2) > 0$ for $\phi \in (0, r_2(C_2))$.

Thus, there exists $\delta > 0$ such that, if the following assumption holds

$$C_2 < \delta \quad \text{and} \quad \eta_n(0) \le \phi(0) < r_2(C_2), \tag{3.11}$$

then we have

$$0 \le \eta_n(t) = |\nabla u^n(t)|^2 + |\partial_t \varphi^n(t)|^2 \le \phi(t) < r_2(C_2) \tag{3.12}$$

for all t in the interval $[0, t_1^n]$.

Now, the assumption that $|\Delta\varphi(t)|^2 \le \beta$ for $t \in [0, t_1^n]$ permit us to estimate inequality (3.6), on $[0, t_1^n]$, as

$$\begin{aligned} |\Delta\varphi^n(t)|^2 &\le (C|\Delta\varphi^n(t)|^2)|\nabla u^n(t)|^2 + (C|\Delta\varphi^n(t)|^2)|\Delta\varphi^n(t)|^2 \\ &\quad + C|f(t)|^2 + C|\partial_t\varphi^n(t)|^2 \\ &\le (1/8)|\nabla u^n(t)|^2 + (1/8)|\Delta\varphi^n(t)|^2 + C|\partial_t\varphi^n(t)|^2 + C|f(t)|^2, \end{aligned}$$

which implies for $t \in [0, t_1^n]$ that

$$\begin{aligned} |\Delta\varphi^n(t)|^2 &\le (1/7)|\nabla u^n(t)|^2 + (8C/7)|\partial_t\varphi^n(t)|^2 + (8C/7)|f(t)|^2 \\ &\le \max\{1/7, 8C/7\}\eta_n(t) + (8C/7)|f(t)|^2 \\ &\le \max\{1/7, 8C/7\}r_2(C_2) + (8C/7)|f(t)|^2 \\ &\le (1/2)\beta, \end{aligned} \tag{3.13}$$

if we take f, h and $\alpha||g||^2_{L^\infty(0,\infty;L^2(\Omega))}$ sufficiently small such way that there hold the following:

$$\begin{aligned} &C_2 = 2C\sup_{t\ge 0}\{\alpha|g(t)|^2\beta + |h(t)|^2 + |\partial_t f(t)|^2\} \\ &= 2C[\beta\alpha||g||^2_{L^\infty(0,\infty;L^2(\Omega))} + ||h||^2_{L^\infty(0,\infty;L^2(\Omega))} + ||\partial_t f||^2_{L^\infty(0,\infty;L^2(\Omega))}] < \delta, \end{aligned}$$

(where δ is given by (3.11)) and

$$(8C/7)||f||^2_{L^\infty(0,\infty;L^2(\Omega))} \le \beta/4$$

and

$$\max\{1/7, 8C/7\}r_2(C_2) + (8C/7)||f||^2_{L^\infty(0,\infty;L^2(\Omega))} \leq \beta/4.$$

In particular, (3.130 implies that $|\Delta\varphi^n(t_1^n)| \leq \beta/2$, which is in contradiction with our choice of t_1^n.

Thus, for such small initial data and external fields, we conclude that the approximate solutions exist for all time ($t_2^n = \infty$) and also that $|\Delta\varphi^n(t)|^2 < \beta$ for all $t \geq 0$. Therefore, we can proceed as above to obtain $\eta_n(t) < r_2(C_2)$ for all $t \geq 0$, and now, the definition of $\eta_n(t)$, implies that $|\nabla u^n(t)|^2 \leq \eta_n(t) < r_2(C_2)$ again for all $t \geq 0$. These two estimates implies the stated one for the approximated solutions.

As we said in the beginning of the proof, these estimates can be carried over to the solution in the limit, and the theorem is proved. □

Also, by arguing as above, it is possible to prove a result on global existence in time for the solutions obtained in the case $u_0 \in V \cap H^2(\Omega)$ (Proposition 2.2, see [8]; p. 7). In fact, it holds

Theorem 3.2 *Under the conditions of the Theorem 3.1, assume that* $\partial_t g \in L^\infty(0,\infty; L^2(\Omega))$, $\partial_t h \in L^\infty(0,\infty; L^2(\Omega))$ *and* $u_0 \in V \cap H^2$.

Then, if $||u_0||_{H^2(\Omega)}$ *and* $||\partial_t h||_{L^\infty(0,\infty;L^2(\Omega))}$ *are small enough, the solution described in Theorem 3.1 satisfies the aditional estimates*

$$\sup_{t\geq 0} |\tilde{\Delta} u(t)| + |\partial_t u(t)| < +\infty$$

for any $\gamma > 0$. *Also, the same kind of estimates hold uniformly in* n *for the spectral Galerkin approximations.*

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