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A NOTE ON THE BLOW-UP OF A NONLINEAR EVOLUTION EQUATION WITH NONLOCAL COEFFICIENTS.

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Abstract

In this work we consider the cauchy problem associated to a nonlinear model of partial differential equations with non local coefficients. We show that for a class of initial data, the solutions blow-up in finite time.

Resumo

Consideramos neste trabalho o problema de Cauchy associado a um modelo de equações diferenciais parciais não lineares com coeficientes não locais. Mostramos que para uma classe de dados iniciais, as soluções apresentam "blow-up" em tempo finito.

1. Introduction

In this note we are interested in the study of blow-up present in solutions of the initial value problem

$$\begin{cases} \partial_t u + \sum_{i=1}^{n} (-\Delta)^{-\frac{\beta_i}{2}} u \cdot \partial_{x_i} u = 0, \; x \in \mathbb{R}^n, \; t \in \mathbb{R}, \; \beta_i \in [0, n). \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

with

$$(-\Delta)^{-\beta/2} f(x) = c_{\beta,n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}} dy. \tag{R}$$

In [P], Ponce showed that for $u_0 \in C_0^\infty(\mathbb{R}^n)$ and $\beta_i \in [1, n)$, this problem has unique global solution

$$u \in C([0, \infty) : H^s(\mathbb{R}^n)), \quad \text{for any} \quad s \in \mathbb{Z}^+.$$

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We will focus our attention on problem (1) with $\beta_i \in (0,1)$.

We first consider the Euler equation for an incompressible homogeneous inviscid fluid flow,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = \nabla p, \quad x \in I\!R^2, \ t \in I\!R. \\ \text{div } u = 0, \qquad u(x,0) = u_0(x), \end{cases} \tag{2}$$

where $u = (u_1, u_2)$ is the velocity field and p the pressure. In this case the evolution of the vorticity $\omega = \nabla \times u = \partial_x u_2 - \partial_y u_1$ is given by the problem

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = 0, \\ \omega(x,0) = \omega_0. \end{cases} \tag{3}$$

The Biot-Savart law of fluid dynamics (see for instance [M]) allows us to recover the velocity field u in terms of the vorticity w, i.e., $u = (\nabla^{\perp}(-\Delta)^{-1})\omega$. Thus the problem (3) can be written in a self-contained manner as

$$\begin{cases} \partial_t \omega + (\nabla^{\perp}(-\Delta)^{-1})\omega \nabla \omega = 0 \\ \omega(x,0) = \omega_0(x). \end{cases} \tag{4}$$

Then, we can think problem (1) as an intermediate model between the inviscid Burgers equation,

$$\partial_t u + u \partial_x u = 0, \quad x \in I\!R, \ t \in I\!R, \tag{5}$$

and model (4) above.

Note that (1) appears as a model which feature both integral differential equation and nonlocal differential equation. It has the advantage that the nonlocal operator $(-\Delta)^{-\beta/2}$ called Riesz potential (see [S]) defined in (R) is positive preserving.

For $\beta_i = 1$, the number of derivatives involved in the nonlinear term is zero. This is the extremal case for which global existence in [P] was established. For $n = 1$ and $\beta_1 = 0$ it is well known that blow-up occurs in (5) for any initial data $u_0 \in C_0^{\infty}(I\!R)$. Concerning problem (1), the case $0 < \beta_i < 1$ was left open in [P].

In this note we shall show that at least for $n = 1$ and $\beta \in (0, 1)$, the solutions of (1) with $u_0 \in C_0^\infty(I\!R)$ may blow up in finite time.

The next theorem is our main result.

Theorem. *Let $n = 1$ and $\beta \in (0, 1)$. There exist a $u_0 \in C_0^\infty(I\!R)$ and a $T^* > 0$ such that the solution $u(x, t)$ of (1) with initial data u_0 satisfies*

$$\lim_{t \to T^*} ||\partial_x u(\cdot, t)||_{L^\infty} = \infty.$$

Remarks.

1) In our proof we will use the characteristics, so it is useful to notice the following: for any $\beta_i \in (0, n)$ it is clear from the characteristic method that as far as the solution $u(x, t)$ of (1) exists, then

$$||u(\cdot, t)||_{L^\infty} = ||u_0||_{L^\infty}.$$

2) This result shows that the solution of (1) with appropriate data behaves like the solution of (5) with same initial data. In fact, our proof below can be adapted to obtain blow up result for any data $u_0 \in C_0^\infty(I\!R)$, with $u_0 \geq 0$. At this point we do not know whether or not the blow up occurs in any solution corresponding to arbitrary data in $C_0^\infty(I\!R)$.

3) In the case $n > 1$, our argument below, although the computations seem take very involved, suggests that for particular kind of data in $C_0^\infty(I\!R^n)$ the blow-up of the classical solutions of (1) should occur for $\beta_i \in (0, 1)$ for some $i \in \{1, ..., n\}$.

4) We observe that (4) can be seen as the extremal case in two dimensions. Notice also that the best bound known for the global solution $\omega(x, t)$ (see Kato [K]) is

$$||\partial_x \omega(t)||_{L^\infty} \leq c||\partial_x \omega(0)||_{L^\infty} e^{ct} .$$

Proof of the Theorem. We consider a function u_0 as in the Figure 1

that is,

$$H_1 \begin{cases} u_0 \text{ is nonincreasing in } (\alpha_2(0), \alpha_1(0)) \text{ and nondecreasing in } (\alpha_4(0), \alpha_3(0)) \\ \text{satisfying} \\ u_0(x) = 1 \quad \text{for} \quad \alpha_3(0) \le x \le \alpha_2(0)\ , \\ 0 < u_0(x) < 1 \quad \text{for} \quad \alpha_4(0) < x < \alpha_3(0) \ \text{ and } \ \alpha_2(0) < x < \alpha_1(0), \\ u_0(x) = 0 \quad \text{for} \quad x \le \alpha_4(0) \ \text{ and } \ x \ge \alpha_1(0). \end{cases}$$

Now we use characteristics to compute the evolution on time of the difference $\alpha_1(t) - \alpha_2(t)$. In what follows we may write α_i, $i = 1, 2, 3, 4$, instead of $\alpha_i(t)$ for simplicity of the formulas. Then,

$$\frac{d}{dt}(\alpha_1 - \alpha_2)(t) = [(-\Delta)^{-\beta/2}u](\alpha_1(t), t) - [(-\Delta)^{-\beta/2}u](\alpha_2(t), t).$$

The definition of Riesz transform implies

$$\begin{aligned} \frac{d}{dt}(\alpha_1(t) - \alpha_2(t)) = {} & C\Big(\int_{\mathbb{R}} \frac{u(y,t)}{|\alpha_1(t) - y|^{(1-\beta)}}dy - \int_{\mathbb{R}} \frac{u(y,t)}{|\alpha_2(t) - y|^{(1-\beta)}}dy\Big) \\ = {} & C\int_{\alpha_4(t)}^{\alpha_3(t)} \Big(\frac{u(y,t)}{|\alpha_1(t) - y|^{1-\beta}} - \frac{u(y,t)}{|\alpha_2(t) - y|^{1-\beta}}\Big)dy \\ & + C\int_{\alpha_3(t)}^{\alpha_2(t)} \Big(\frac{u(y,t)}{|\alpha_1(t) - y|^{1-\beta}} - \frac{u(y,t)}{|\alpha_2(t) - y|^{1-\beta}}\Big)dy \\ & + C\int_{\alpha_2(t)}^{\alpha_1(t)} \Big(\frac{u(y,t)}{|\alpha_1(t) - y|^{1-\beta}} - \frac{u(y,t)}{|\alpha_2(t) - y|^{1-\beta}}\Big)dy \\ = {} & I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

Using the characteristics, the fact that the function $u(y,t)$ is monotone increasing (resp. decreasing) in y if the initial data $u_0(y)$ is increasing (resp. decreasing) between the points where the characteristics intersect the t-axis, and the hypothesis H_1 on u_0 we have,

$$\begin{aligned} I_1 &= C\int_{\alpha_4}^{\alpha_3}\Big(\frac{u_0(s(y,t))}{|\alpha_1-y|^{1-\beta}} - \frac{u_0(s(y,t))}{|\alpha_2-y|^{1-\beta}}\Big)dy \le 0 \\ I_2 &= C\int_{\alpha_3}^{\alpha_2}\Big(\frac{1}{|\alpha_1-y|^{1-\beta}} - \frac{1}{|\alpha_2-y|^{1-\beta}}\Big)dy \le 0 \\ I_3 &= C\int_{\alpha_2}^{\alpha_1}\Big(\frac{u_0(s(y,t))}{|\alpha_1-y|^{1-\beta}} - \frac{u_0(s(y,t))}{|\alpha_2-y|^{1-\beta}}\Big)dy \le 0, \end{aligned}$$

where $I_i = I_i(t)$, $i = 1,2,3$, and $s(y,t)$ is the point in the t-axis which lies on the characteristic passing through (y,t).
We claim

$$C_2 \left\{ \begin{array}{l} \text{Given } \; T>0 \text{ there exist } \alpha_1(0),\alpha_2(0),\alpha_3(0),\alpha_4(0) \\ \text{such that } \; \alpha_1(t)-\alpha_2(t)\le\alpha_2(t)-\alpha_3(t) \; \text{ for any } t\in[0,T] \;. \end{array}\right.$$

Assuming this for the moment, we find that

$$I_2 \le C\int_{\alpha_2-(\alpha_1-\alpha_2)}^{\alpha_2}\Big(\frac{1}{|\alpha_1-y|^{1-\beta}} - \frac{1}{|\alpha_2-y|^{1-\beta}}\Big)dy \;.$$

Therefore,

$$\frac{d}{dt}(\alpha_1(t)-\alpha_2(t)) \le \frac{(\alpha_1(t)-\alpha_2(t))^\beta}{\beta}(2^\beta-2). \tag{6}$$

The explicit solution of

$$\frac{d}{dt}(\tilde\alpha_1(t)-\tilde\alpha_2(t)) = \frac{(\tilde\alpha_1(t)-\tilde\alpha_2(t))^\beta}{\beta}(2^\beta-2).$$

is such that there exists a

$$t^* = \frac{\beta(\tilde\alpha_1(0)-\tilde\alpha_2(0))^{(1-\beta)}}{(1-\beta)(2-2^\beta)} \;, \tag{7}$$

with $\tilde\alpha_1(t^*) = \tilde\alpha_2(t^*)$.

Hence we have proved the existence of a time $\bar{t} \leq t^*$ where the characteristics crossed each other provided C_2 holds.

Next we look for conditions on u_0 such that C_2 satisfies.

First we prove the following inequalities,

$$\frac{d}{dt}(\alpha_2(t) - \alpha_3(t)) \geq -\frac{1}{\beta}(\alpha_3(t) - \alpha_4(t))^{\beta}, \qquad (A)$$

and

$$\frac{d}{dt}(\alpha_3(t) - \alpha_4(t)) \leq ((\alpha_3 - \alpha_4)^{1-\beta}(0) + \frac{2}{\beta}(1-\beta)t)^{\frac{1}{1-\beta}}. \qquad (B)$$

Using the definition in (R) and the characteristics we can write

$$\frac{d}{dt}(\alpha_2(t) - \alpha_3(t)) = \int_{\alpha_4(t)}^{\alpha_3(t)} \Big(\frac{u(y,t)}{|\alpha_2(t) - y|^{1-\beta}} - \frac{u(y,t)}{|\alpha_3(t) - y|^{1-\beta}}\Big) dy$$

$$+ \int_{\alpha_3(t)}^{\alpha_2(t)} \Big(\frac{u(y,t)}{|\alpha_2(t) - y|^{1-\beta}} - \frac{u(y,t)}{|\alpha_3(t) - y|^{1-\beta}}\Big) dy$$

$$+ \int_{\alpha_2(t)}^{\alpha_1(t)} \Big(\frac{u(y,t)}{|\alpha_2(t) - y|^{1-\beta}} - \frac{u(y,t)}{|\alpha_3(t) - y|^{1-\beta}}\Big) dy.$$

Hypothesis H_1 on u_0 implies that the second integral on the r.h.s. of the previous inequality is zero and the third integral is positive, therefore inequality (A) follows from

$$\frac{d}{dt}(\alpha_2(t) - \alpha_3(t)) \geq -\int_{\alpha_4(t)}^{\alpha_3(t)} \frac{dy}{|\alpha_3(t) - y|^{1-\beta}} = -\int_0^{(\alpha_3-\alpha_4)(t)} \frac{dv}{|v|^{1-\beta}}. \qquad (8)$$

To prove inequality (B), notice that

$$\begin{aligned} \frac{d}{dt}(\alpha_3 - \alpha_4)(t) &= \int_{\alpha_4}^{\alpha_3+(\alpha_3-\alpha_4)} \frac{u_0(s(y,t))}{|\alpha_3(t) - y|^{1-\beta}} dy + \int_{\alpha_3+(\alpha_3-\alpha_4)}^{\alpha_1} \frac{u_0(s(y,t))}{|\alpha_3 - y|^{1-\beta}} dy \\ &- \int_{\alpha_4}^{\alpha_3} \frac{u_0(s(y,t))}{|\alpha_4(t) - y|^{1-\beta}} dy - \int_{\alpha_3}^{\alpha_1} \frac{u_0(s(y,t))}{|\alpha_4(t) - y|^{1-\beta}} dy = \\ &= J_1(t) + J_2(t) - J_3(t) - J_4(t), \end{aligned}$$

where $s(y,t)$ is as before.

We claim that $J_2(t) - J_4(t) \leq 0$, for $t \in [0, t^*]$. In fact, consider the change of variables $y = z - (\alpha_4 - \alpha_3)$ in $J_2(t)$. Then,

$$J_2(t) = \int_{\alpha_3}^{\alpha_1+\alpha_4-\alpha_3} \frac{u_0(s(z+\alpha_3-\alpha_4,t))}{(z-\alpha_4(t))^{1-\beta}} dz \leq \int_{\alpha_3}^{\alpha_1} \frac{u_0(s(z+\alpha_3-\alpha_4,t))}{(z-\alpha_4(t))^{1-\beta}} dz.$$

Therefore

$$J_2(t) - J_4(t) \leq \int_{\alpha_3}^{\alpha_1} \frac{u_0(s(z+\alpha_3-\alpha_4,t)) - u_0(s(z,t))}{(z-\alpha_4(t))^{1-\beta}} dz \leq 0.$$

The inequality above is a consequence of the hypothesis H_1 on u_0 and $(\alpha_3 - \alpha_4)(t) \geq 0$.
Using also the fact that $-J_3(t) < 0$, we can write

$$\frac{d}{dt}(\alpha_3 - \alpha_4)(t) \leq \int_{\alpha_4}^{\alpha_3+(\alpha_3-\alpha_4)} \frac{u_0(s(y,t))}{|\alpha_3(t) - y|^{1-\beta}} dz \leq 2 \int_{\alpha_4}^{\alpha_3} \frac{dy}{|\alpha_3(t) - y|^{1-\beta}}.$$

Computing the last integral with the simple change of variables $z = \alpha_3(t) - y$, we obtain

$$\frac{d}{dt}(\alpha_3 - \alpha_4)(t) \leq \frac{2}{\beta}(\alpha_3 - \alpha_4)^{\beta}(t).$$

The inequality (B) follows then by solving the above differential inequality.

Now, combining the inequality (A) and (B) we have,

$$\frac{d}{dt}(\alpha_2 - \alpha_3)(t) \geq -\frac{1}{\beta}((\alpha_3 - \alpha_4)^{(1-\beta)}(0) + \frac{2}{\beta}(1-\beta)t)^{\frac{\beta}{1-\beta}}. \tag{9}$$

Integrating (9) in $[0,t]$ we obtain

$$\begin{aligned} (\alpha_2 - \alpha_3)(t) &\geq (\alpha_2 - \alpha_3)(0) - \frac{1}{2}((\alpha_3 - \alpha_4)^{(1-\beta)}(0) + \frac{2(1-\beta)}{\beta}t)^{\frac{1}{1-\beta}} \quad (10) \\ &+ \frac{1}{2}(\alpha_3 - \alpha_4)^{(1-\beta)}(0). \end{aligned}$$

Note that the second term of the inequality is a decreasing function of t, then substituting t by t^* as in (7) on the right hand side of (10), we obtain

$$(\alpha_2 - \alpha_3)(t) \geq (\alpha_2 - \alpha_3)(0) - \frac{1}{2}((\alpha_3 - \alpha_4)^{(1-\beta)}(0) \tag{11}$$

$$+ \quad \frac{2}{2^\beta - 2}(\alpha_1 - \alpha_2)^{(1-\beta)}(0))^{\frac{1}{1-\beta}} + \frac{1}{2}(\alpha_3 - \alpha_4)^{(1-\beta)}(0) \quad \text{for all} \quad t \in [0, t^*],$$

To finish the proof it is enough to choose u_0 as in the figure 1 satisfying hypothesis H_1 and $\alpha_1(0), \alpha_2(0), \alpha_3(0)$ and $\alpha_4(0)$ such that

$$(\alpha_2 - \alpha_3)(0) - \frac{1}{2}((\alpha_3 - \alpha_4)^{(1-\beta)}(0)\frac{2}{2^\beta - 2}(\alpha_1 - \alpha_2)^{(1-\beta)}(0))^{\frac{1}{1-\beta}} \tag{12}$$

$$+\frac{1}{2}(\alpha_3 - \alpha_4)^{(1-\beta)}(0) \ \geq (\alpha_1 - \alpha_2)(0).$$

This combined with (11) implies

$$(\alpha_2 - \alpha_3)(t) \geq (\alpha_1 - \alpha_2)(t) \quad \text{for all} \quad t \in [0, t^*],$$

since $(\alpha_1 - \alpha_2)(t)$ is a decreasing function. Thus, we have that C_2 is verified for t in $[0, t^*]$ and the proof of the theorem is completed.

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References

[K] Kato, T., *Remarks on the Euler and Navier-Stokes equations in* $I\!R^2$, Proceeding of Simposia in Pure Mathematics, Vol 45, (1986), 1–7.

[M] Majda, A., *Vorticity and mathematical theory of incompressible fluid flow*, Commm. Pure Appl. Math. Vol 39, (1986), 187–220.

[P] Ponce, G., *Global Existence of Solutions to a nonlinear Evolution Equation with Nonlocal Coefficients*, J. Math. Anal. and Appl., Vol 123, 1, (1987), 104–113.

[S] Stein, E., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, (1970).

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