


THE CAUCHY PROBLEM FOR HYPERBOLIC CONSERVATION LAWS WITH RELAXATION TERMS: A CONFERENCE REPORT

Christian Klingenberg  Yun-guang Lu *

Abstract

Consider a system of conservation laws with relaxation terms. From applications one knows these terms may be very stiff. This raises the question if their solutions converge to the solutions of the related system where relaxation has gone to zero. A method to investigate this question involves approximating the original stiff system by either adding small viscous terms or by a numerical scheme, and then letting both the approximation and the relaxation parameter tend to zero. Below we announce two such results.

1. Two equations

In this section we consider a relaxation model:

$$\left. \begin{aligned} (u + v)_t + f(u)_x &= 0 \\ v_t &= \frac{A(u) - v}{g(\delta, u, v)} \end{aligned} \right\} \quad (1.1)$$

with initial data

$$(u, v)|_{t=0} = (u_0(x), v_0(x)) \quad (1.2)$$

where the positive constant δ is referred to as relaxation time. Later on we state the assumptions that we make on (1.1), (1.2) in detail.

System (1.1) consists of a conservation law and an equation with a relaxation term. In [12], [14], [16] a similar model arising in combustion

$$\left. \begin{aligned} (u + qz)_t + f(u)_x &= 0 \\ z_t &= k\phi(u)z \end{aligned} \right\} \quad (1.3)$$

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was studied. In [3], [17], [20] other models with relaxation terms have been considered. For $g(\delta, u, v) = \delta$ the system (1.1) arises in chromatography, see [17].

In [20] the question of vanishing relaxation-discretisation for (1.1), (1.2) is considered with $g = \delta$ in the framework of BV solutions under the assumption of monotonicity of f . Our proof needs no such restrictions by using a different technique, as described next.

We study the zero relaxation limit $g(\delta, u, v) \rightarrow 0$ by using compensated compactness. This method has been used on models for combustion [12], [14] for viscoelasticity and phase transitions [3]. We show that the solutions of the viscous equations

$$\left. \begin{aligned} (u+v)_t + f(u)_x &= \epsilon(u+v)_{xx} \\ v_t &= \epsilon v_{xx} + \frac{A(u) - v}{g(\delta, u, v)} \end{aligned} \right\} \quad (1.4)$$

converge to the solutions of the equilibrium equation

$$(u + A(u))_t + f(u)_x = 0 \quad (1.5)$$

where both δ and ϵ tend to zero related by $\delta = O(\epsilon)$.

The method of proof is as follows: we consider the existence of viscous solutions of the system (1.4) with initial data

$$(u^\epsilon, v^\epsilon)|_{t=0} = (u_0^\epsilon, v_0^\epsilon) \quad , \quad (1.6)$$

where $u_0^\epsilon, v_0^\epsilon$ are smooth functions obtained by smoothing $u_0(x), v_0(x)$ with a mollifier, satisfying

$$\begin{aligned} u_0^\epsilon &\rightarrow u_0(x) \quad , \quad v_0^\epsilon \rightarrow v_0(x) \quad , \quad \text{when } \epsilon \rightarrow 0, \\ |\epsilon u_{0,x}^\epsilon(x)| &\leq M \quad , \quad |\epsilon v_{0,x}^\epsilon(x)| \leq M \quad , \end{aligned}$$

where M is a positive constant depending only on the bound of $|u_0|_{L^\infty}, |v_0|_{L^\infty}$ and is independent of ϵ . We first prove local existence using the local contraction mapping principle. The next step is to show an a priori estimate in the L^∞ norm

of solutions, which is obtained using the maximum principle, as in [13]. Then, in the heart of our analysis, it is proven that for a function pair $(\eta(u^{\epsilon,\delta}), q(u^{\epsilon,\delta}))$ which satisfies $q'(u) = \frac{\eta'(u)f'(u)}{1+A'(u)}$ we have

$$\eta(u^{\epsilon,\delta})_t + q(u^{\epsilon,\delta})_x \quad \text{is compact in} \quad H_{loc}^{-1}(R \times R^+) \quad . \quad (1.7)$$

This is proven mainly through energy estimates. Finally the method of compensated compactness as given in [2], [12] is used to study the convergence of viscous solutions $(u^{\epsilon,\delta}, v^{\epsilon,\delta})$. First the convergence of $u^{\epsilon,\delta}$ is shown, and then using a lemma, the convergence of $v^{\epsilon,\delta}$. When taking $\delta = O(\epsilon)$, the global weak solution of the equilibrium (1.5) is obtained as $\delta \rightarrow 0$. For details, see [8].

Finally we list the assumptions on (1.1), (1.2) needed in our proof in detail.

- (A1) $f(u), A(u) \in C^2$ satisfying $A'(u) \geq c_1 > 0$,
 $\text{meas} \left\{ u : \left(\frac{f'(u)}{1+A'(u)} \right)' = 0 \right\} = 0$
- (A2) $g(\delta, u, v) \in C^1(R^2)$ for any fixed δ ,
 $c_1(u, v)\delta \leq g(\delta, u, v) \leq c_2(u, v)\delta$,
 $|g_u| \leq c_3(u, v)\delta, |g_v| \leq c_3(u, v)\delta$,
 where $c_i(u, v)$ are positive, continuous functions, $i = 1, 2, 3$
- (A3) $u_0(x), v_0(x)$ are bounded in L^∞ and
 $\int_G |A(u_0(x)) - v_0(x)| dx \leq \delta M(G)$ for any compact set G in \mathbf{R} .

2. Three equations

In the second part of this report, we consider the Cauchy problem for the following nonlinear system:

$$\left. \begin{aligned} v_t - u_x &= 0 \\ u_t - \sigma(v, s)_x + \alpha u &= 0 \\ s_t + \frac{\beta(s - f(v))}{\tau} &= 0 \end{aligned} \right\} \quad (2.1)$$

with bounded L^2 measurable initial data

$$(v, u, s)|_{t=0} = (v_0(x), u_0(x), s_0(x)) \quad , \quad (2.2)$$

where α, β, τ are nonnegative constants. When $\beta = 0$, system (2.1) can be used to model the adiabatic gas flow through porous media [7], where v is the specific volume, u denotes velocity, s stands for entropy and σ denotes pressure. Its form in Euler coordinates is also a model of isothermal unsteady two phase flow in pipelines [1]. We study the global generalized solution for this case. Next we consider the case $\beta \neq 0$. Here again v is specific volume, u denotes velocity but s is the mass fraction of one of the modes of a two mode gas and $f(v)$ is a given equilibrium distribution in v . In this case, τ denotes a reaction time. When written in Euler coordinates, it can be used to model chemically reacting flow [9].

We show that the solution of the equilibrium system

$$\left. \begin{aligned} v_t - u_x &= 0 \\ u_t - \sigma(v, s)_x + \alpha u &= 0 \end{aligned} \right\} \quad (2.3)$$

is given by the limit of the solutions of the viscous approximation

$$\left. \begin{aligned} v_t - u_x &= \epsilon v_{xx} \\ u_t - \sigma(v, s)_x + \alpha u &= \epsilon u_{xx} \\ s_t + \frac{\beta(s - f(v))}{\tau} &= \epsilon s_{xx} \end{aligned} \right\} \quad (2.4)$$

as the dissipation ϵ and the reaction time τ both go to zero related by $\tau = O(\epsilon)$. Similar results about zero relaxation systems of two equations and solutions in L^∞ space are discussed in papers [3], [4], [9], [14]; in L^p space see the recent paper [5].

In dealing with the Cauchy problem (2.1), (2.2), one basic difficulty is the a priori estimate of the viscous solutions of (2.4), independent of ϵ in a suitable L^p space ($p > 1$). Since system (2.1) in general can not be diagonalized by using Riemann invariants, it is not to be expected that viscous solutions $(v^\epsilon, u^\epsilon, s^\epsilon)$ of the Cauchy problem (2.4) will be bounded in L^∞ , uniformly in ϵ , by using the invariant region principle [19]. We have to search for solutions of system (2.1) in L^p space. In some sense, the a priori estimate of the solutions of (2.4) in L^2 is easy to get, if we can find a strictly convex entropy for the system (2.1). However, a new difficulty arises by considering the compactness of the

viscous solutions in L^p space by trying to use compensated compactness. To the author's knowledge compactness of the viscous solutions in L^p space has been shown for scalar equations in [11], for a simple model of combustion close to a scalar equation in [12], for a system of two equations by M. Santos, H. Frid [6], P.X. Lin [10] and J.W. Shearer [18]. Since for the parabolic viscous system (2.4) both the L^∞ estimate and the one sided L^∞ estimates using the Riemann invariants (as given in (1.6) of [11]) are not easy to get, here we take the entropy - entropy flux pair as considered in [18] as the base of our paper. For details of the proof, see [15].

We make the following assumptions about $\sigma(v, s)$ in (2.1) and the initial data (2.2):

Case I: $\beta > 0$

A1 $f(v) = cv$

A2 $\sigma(v, s) = \sigma(v) - cs$, where $\sigma(v)$ satisfies the following conditions:

- a) $\sigma \in C^3(R)$, $\sigma(0) = 0$, $\sigma' \geq d > c^2$, d is a positive constant
- b) $\sigma'' \neq 0$, $\sigma'' \in L^1 \cap L^\infty(R)$
- c) $\sigma''' \in L^\infty(R)$, $|\sigma'''|_{L^1} \leq M$

A3 $(v_0(x), u_0(x), s_0(x))$ are bounded in L^2 and tend to zero as $|x| \rightarrow \infty$ sufficiently fast such that

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} (v_0^\epsilon(x), u_0^\epsilon(x), s_0^\epsilon(x)) &= (0, 0, 0) \\ \lim_{x \rightarrow \pm\infty} \left(\frac{dv_0^\epsilon(x)}{dx}, \frac{du_0^\epsilon(x)}{dx}, \frac{ds_0^\epsilon(x)}{dx} \right) &= (0, 0, 0) \quad , \end{aligned}$$

where $(v_0^\epsilon(x), u_0^\epsilon(x), s_0^\epsilon(x))$ are smooth functions obtained by smoothing the initial data $(v_0(x), u_0(x), s_0(x))$ with a mollifier, satisfying

$$\begin{aligned} (v_0^\epsilon(x), u_0^\epsilon(x), s_0^\epsilon(x)) &\rightarrow (v_0(x), u_0(x), s_0(x)) \quad \text{when } \epsilon \rightarrow 0 \\ |v_0^\epsilon(x)|_{L^2(R)} &\leq M \quad , \quad |u_0^\epsilon(x)|_{L^2(R)} \leq M \quad , \quad |s_0^\epsilon(x)|_{L^2(R)} \leq M \\ |v_0^\epsilon(x)|_{H^1(R)} &\leq M(\epsilon) \quad , \quad |u_0^\epsilon(x)|_{H^1(R)} \leq M(\epsilon) \quad , \quad |s_0^\epsilon(x)|_{H^1(R)} \leq M(\epsilon) \\ \left| \frac{d^i v_0^\epsilon(x)}{dx^i} \right| &\leq M(\epsilon), \left| \frac{d^i u_0^\epsilon(x)}{dx^i} \right| \leq M(\epsilon), \left| \frac{d^i s_0^\epsilon(x)}{dx^i} \right| \leq M(\epsilon), \quad i = 0, 1, 2 \quad . \end{aligned}$$

Case II: $\beta = 0$

A1 $\sigma(v, s) = \sigma(v)g(s) - cs$, $g(s) \in C^3$ and $g(s) \geq d > 0$

A2 $\sigma(v)$ satisfies (A2) a), b), c) in case I

A3 $(v_0(x), u_0(x), s_0(x))$ satisfy the same conditions as in (A3) case I.

Moreover $|s_0(x)|_{H^1_{loc}(R)} \leq M$, where M is a positive constant and independent of ϵ , $M(\epsilon)$ is a positive constant which depends on ϵ .

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Christian Klingenberg
Applied Mathematics
Würzburg University
Am Hubland, Würzburg 97074
Germany

Also:

Applied Mathematics,
Heidelberg University
Heidelberg 69120, Germany

Yun-guang Lu
Young Scientist Laboratory of
Mathematical Physics, Wuhan Institute
of Mathematical Sciences
Academia Sinica, Wuhan, China

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