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SIMULTANEOUS STABILIZATION OF SOLUTIONS OF MIXED PROBLEMS FOR EVOLUTION SYSTEMS

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1. Introduction

Let Ω be a bounded domain in R^n with sufficiently smooth boundary S. In $\Omega \times (0,T)$ we consider the initial boundary value problems for a pair of evolution systems. We assume that one of them is the damped hyperbolic system with locally distributed damping or with boundary dissipation. Thus, the energy $E_1(t)$ of solutions is a non increasing function of the time variable t. The second system has principal part equal to the principal part of the first system and for every solution the energy $E_2(t)$ is a non decreasing function of time t.

Our purpose is to connect these systems (in a part of domain Ω or on the boundary S) and to give sufficient conditions on this connection and damping term ensuring the uniform exponential decay of the total energy, i.e. the existence of some constants $C > 0$, $\gamma > 0$ such that

$$E(t) \equiv E_1(t) + E_2(t) \leq C\, e^{-\gamma t}\, E(0), \quad t \geq 0.$$

The case of a single damped wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + Q(x)\frac{\partial u}{\partial t} = 0, \quad Q(x) \geq Q_0 \quad \text{in} \quad \mathcal{D} \subset \Omega$$

$$u|_S = 0$$

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is by now well understood. Results by C. Bardos, G. Lebeau and I. Rauch [1] show that, when Ω and $Q(x)$ are of class C^∞, the estimate

$$\int_\Omega (|u_t|^2 + |\nabla u|^2)\, dt \leq C\, e^{-\gamma t} \int_\Omega (|u_t|^2 + |\nabla u|^2)|_{t=0}\, dx$$

holds if and only if some "geometric control condition" is satisfied. The canonical example of open subset $\mathcal{D}$ verifying this condition is when $\mathcal{D}$ is a neighbourhood of the boundary.

The coupled system of linear thermoelasticity has beeen investigated by D.C. Pereira and G.P. Menzala [12], J. Rivera [14], J. Rivera and Y. Shibata [15].

Exponential decay for a pair of hyperbolic systems of second order with boundary damping only for one of them has been obtained by author [2].

2. Hyperbolic systems of second order. Locally distributed damping.

In the cylinder $\Omega \times (0, T)$ we consider the following initial boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(A \frac{\partial u}{\partial x_i}\right) + Q(x)\frac{\partial u}{\partial t} + B(x)\frac{\partial v}{\partial t} = 0,$$

$$\frac{\partial^2 v}{\partial t^2} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(A \frac{\partial v}{\partial x_i}\right) - B(x)\frac{\partial u}{\partial t} = 0, \tag{2.1}$$

$$u|_{t=0} = f_1(x), v|_{t=0} = f_2(x), \frac{\partial u}{\partial t}|_{t=0} = f_3(x), \frac{\partial v}{\partial t}|_{t=0} = f_4(x),$$

$$u|_S = 0, \;\; v|_S = 0,$$

where $u = (u^1(x,t), \ldots, u^m(x,t))$, $v = (v^1(x,t), \ldots, v^m(x,t))$, $x = (x_1, \ldots, x_n)$; $A = A^*$, $B(x) = B^*(x)$, $Q(x) = Q^*(x)$ are square matrices of order m, $B(x)$ and $Q(x)$ are of class $L_\infty(\Omega)$.

Assume that

$$\sum_{i=1}^n A\xi_i \cdot \xi_i \geq a_0 \sum_{i=1}^n |\xi_i|^2, \quad a_0 > 0,$$

where $\xi_i = (\xi_i^1, \xi_i^2, \ldots, \xi_i^m)$ is an arbitrary vector and

$$\eta_i \cdot \xi_i = \eta_i^1 \xi_i^1 + \cdots + \eta_i^m \xi_i^m\,, \quad |\xi_i|^2 = (\xi_i^1)^2 + \cdots + (\xi_i^m)^2\,.$$

Let us consider the energy

$$E(t) = \int_\Omega \left(\sum_{i=1}^n A \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} + \sum_{i=1}^n A \frac{\partial v}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} + |u_t|^2 + |v_t|^2 \right) dx.$$

The formal computations show that for every solution of (2.1) the following identity holds

$$E(t_2) - E(t_1) = -2 \int_{t_1}^{t_2} \int_\Omega Q(x) u_t \cdot u_t \, dxdt, \quad t_2 \geq t_1 \geq 0.$$

We assume that

$$Q(x)\xi \cdot \xi \geq 0 \text{ a.e. in } \Omega \quad \text{and} \quad Q(x)\xi \cdot \xi \geq \alpha |\xi|^2 \text{ a.e. in } \mathcal{D} \ \ \forall \xi \in R^m$$

for some non empty subset $\mathcal{D}$ of Ω and some positive constant α.

The aim of this section is to give sufficient conditions on the subset $\mathcal{D}$ (where the damping term is effective) ensuring the uniform stabilization as $t \to \infty$ of the energy $E(t)$.

Throughout this paper $H^k(\Omega)$ is the usual Sobolev space. Denote by $\mathcal{H}$ the real Hilbert space of quadruples $w = \{w_1, w_2, w_3, w_4\}$ of m-component vector-functions w_i such that

$$w_1, w_2 \in H^1(\Omega), \quad w_1|_S = w_2|_S = 0, \quad w_3, w_4 \in L_2(\Omega).$$

The inner product in $\mathcal{H}$ is given by

$$\langle w, f \rangle_0 = \int_\Omega \left(\sum_{i=1}^n A \frac{\partial w_1}{\partial x_i} \cdot \frac{\partial f_1}{\partial x_i} + \sum_{i=1}^n A \frac{\partial w_2}{\partial x_i} \cdot \frac{\partial f_2}{\partial x_i} + w_3 \cdot f_3 + w_4 \cdot f_4 \right) dx,$$

where $w = \{w_1, w_2, w_3, w_4\}, \quad f = \{f_1, f_2, f_3, f_4\}$.

In $\mathcal{H}$ we define the unbounded operator $\mathcal{A}\colon \mathcal{D}(\mathcal{A})$ consists of the elements $w = \{w_1, w_2, w_3, w_4\} \in \mathcal{H}$ such that

$$w_1, w_2 \in H^2(\Omega), \quad w_3, w_4 \in H^1(\Omega), \quad w_1|_S = w_2|_S = w_3|_S = w_4|_S = 0,$$

$$\mathcal{A}\{w_1, w_2, w_3, w_4\} = \{w_3, w_4, \tilde{A} w_1 - Q w_3 - B w_4, \tilde{A} w_2 + B w_3\}$$

for $\{w_1, w_2, w_3, w_4\} \in \mathcal{D}(\mathcal{A})$. Here $\tilde{A}u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(A\frac{\partial u}{\partial x_i}\right)$.

It can be shown that $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative. The operator $\mathcal{A}$ thus generates a strongly continuous semigroup of contractions $U(t)$, $t > 0$.

As is well known, $U(t)f$ is strongly differentiable with respect to t for $f \in \mathcal{D}(\mathcal{A})$ and

$$\frac{d}{du}U(t)f = \mathcal{A}U(t)f.$$

It follows that if $\{u, v, u_1, v_1\} = U(t)f$ then u, v is a solution to problem (2.1), moreover, $u_t = u_1$, $v_t = v_1$.

Let $f \in \mathcal{H}$. Then $U(t)f$ satisfies the following identity:

$$\int_0^T \left\{\langle U(t)f, \frac{d\Psi}{dt}\rangle_0 + \langle U(t)f, \mathcal{A}^*\Psi\rangle_0\right\}dt = -\langle f, \Psi(0)\rangle_0 \,,$$

where $\Psi \in L_2(0,T;\mathcal{D}(\mathcal{A}^*))$, $\Psi_t \in L_2(0,T;\mathcal{H})$, $\Psi(T) = 0$.

Thus, $U(t)f$ is the weak solution in $\mathcal{H}$ to the abstract Cauchy problem

$$\frac{dw}{dt} = \mathcal{A}w, \quad w|_{t=0} = f.$$

We introduce the notation

$$\Phi(u) = \sum_{i=1}^{n} A\frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_i}\,.$$

The proof of the stabilization is based on the following identity

$$
\begin{aligned}
&(T+t_0)E(T)-t_0\,E(0)+\textstyle\int_\Omega[2(\nabla\varphi,\nabla)u\cdot u_t+2(\nabla\varphi,\nabla)v\cdot v_t+\\
&+\frac{1}{2}\,CQu\cdot u+Cu_t\cdot u+Cv_t\cdot v]dx|_{t=0}^{t=T}=\int_0^T\int_S\frac{\partial\varphi}{\partial\nu}(\Phi(u)+\Phi(v))dSdt+\\
&+\textstyle\int_0^T\int_\Omega[-2(t+t_0)Q(x)u_t\cdot u_t+2(\nabla\varphi,\nabla)v\cdot Bu_t-2(\nabla\varphi,\nabla)u\cdot Bv_t-\\
&-2(\nabla\varphi,\nabla)u\cdot Qu_t-Cv\cdot Bu_t-CBv_t\cdot u-\\
&-\sum_{i=j}^n\frac{\partial C}{\partial x_j}u\cdot A\frac{\partial u}{\partial x_j}-\sum_{j=1}^n\frac{\partial C}{\partial x_j}v\cdot A\frac{\partial v}{\partial x_j}+\mathcal{K}(u)+\mathcal{K}(v)]dxdt.
\end{aligned}
\tag{2.2}
$$

Here $\varphi=\varphi(x)$ and $C=C(x)$ are scalar funcitons in Ω, ν is the unit outer normal,

$$\mathcal{K}(u)=(\Delta\varphi+1-C(x))\Phi(u)+(1+C(x)-\Delta\varphi)|u_t|^2-2\sum_{i,j=1}^n\frac{\partial^2\varphi}{\partial x_i\partial x_j}A\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i}.$$

We also use the estimate

$$\int_0^T\int_\Omega Bv_t\cdot v_t\,dxdt\le 2E(0)+\int_0^T\int_\Omega(2q+4p)Qu_t\cdot u_t\,dxdt, \tag{2.3}$$

where $p,q>0$ are constant such that $(pB(x)-Q(x))\xi\cdot\xi\ge 0$, $(qQ(x)-B(x))\xi\cdot\xi\ge 0$ in Ω, $p>0$, $q>0$, $\xi\in R^m$.

Henceforth we assume that Ω and $\mathcal{D}$ (where the damping term is effective) satisfy the following conditions:

There exists a function $\varphi(x)\in C^4(\Omega)$ such that

(i) $\dfrac{\partial\varphi}{\partial\nu}\le 0$ on S,

(ii) $\varphi_0(x)\ge\mu>0$ in $\Omega\backslash\mathcal{D}$, where $\varphi_0(x)=\inf\limits_{|\eta|=1}\sum\limits_{i,j=1}^n\dfrac{\partial^2\varphi}{\partial x_i\partial x_j}\eta_i\eta_j$, $\eta=(\eta_1,\dots,\eta_n)$,

(iii) $\Delta^2\varphi-2\Delta\varphi_0\le 0$ in Ω.

Example 2.1. $\Omega=\mathcal{P}\backslash G$, where $\mathcal{P},G\subset R^n$ are bounded domains, $\overline{G}\subset\mathcal{P}$, $S_0=\partial\mathcal{P}$, $S_1=\partial G$, $\partial\Omega=S_0\cup S_1\equiv S$.

Assume that $\mathcal{P}$, G are star-shaped with respect to some point x^0:

$$(x-x^0,\nu)\geq 0,\ \ x\in S_0\,;\quad (x-x^0,\nu)\leq 0,\ \ x\in S_1\,.$$

Assume there is $R>0$ such that

$$G\subset\{x:|x-x^0|<R\},\quad \{x:|x-x^0|<R\}\subset\mathcal{P}.$$

We set (for $n\geq 3$)

$$\varphi(x)=\frac{1}{2}R^k|x-x^0|^2-\frac{1}{2+k}|x-x^0|^{2+k}\,,\quad 0<k\leq 1.$$

Then the subset

$$\mathcal{D}=\{x\in\Omega:|x-x^0|>R_1\},\quad \text{where}\quad R_1=\left(\frac{R^k-\mu}{1+k}\right)^{1/k},\ 0<\mu\leq 1,\mu<R^k$$

has the required properties.

Example 2.2. Let Ω be a bounded domain in R^n with boundary S. For $x^0\in R^n$ we set

$$S_0(x^0)=\{x\in S:(x-x^0,\nu)\leq 0\},\quad S_1(x^0)=S\backslash S_0(x^0).$$

Assume that for some $x^0\in R^n$ and $R>0$, $x\in S_0(x^0)$ if and only if $|x-x^0|\leq R$. We use the same function $\varphi(x)$.

In this case we can set

$$\mathcal{D}=\left\{x\in\Omega:|x-x^0|^k>\frac{R^k-\mu}{1+k}\right\},\ \text{where}\ 0<k,\mu\leq 1,\ \mu<R^k.$$

Remark 2.1. If the damping term is effective in Ω ($\mathcal{D}=\Omega$), the conditions (ii) and (iii) are not required. In this case for an arbitrary domain Ω we define a function $\varphi(x)$ as a solution of the problem

$$\Delta\varphi=-1\quad\text{in}\quad\Omega,\qquad \frac{\partial\varphi}{\partial\nu}=-\frac{\operatorname{mes}\Omega}{\operatorname{mes}S}\quad\text{on}\quad S.$$

We set

$$C(x)=\Delta\varphi-2\varphi_0(x)+\mu$$
$$(\text{if}\quad \mathcal{D}=\Omega, C(x)\equiv C=\Delta\varphi-2\inf_{\Omega}\varphi_0(x)+\mu,\ 0<\mu\leq 1).$$

From (2.2), (2.3), the properties of $\varphi(x)$, $C(x)$ we arrive at the following assertion ([5]).

Theorem 2.1. *Suppose that $Q(x)$ and $B(x)$ satisfy the assumptions:*

1) $Q(x)\xi\cdot\xi \geq 0$, $(qQ(x)-B(x))\xi\cdot\xi \geq 0$, $(pB(x)-Q(x))\xi\cdot\xi \geq 0$ *in* Ω, $p>0$, $q>0$, $\xi \in R^m$;

2) $Q(x)\xi\cdot\xi \geq \alpha|\xi|^2$ *in* $\mathcal{D}$, $\alpha>0$, $\forall\,\xi\in R^m$.

Assume that there exists a function $\varphi(x)$ with the properties (i)-(iii). *Then for all* $f=\{f_1,f_2,f_3,f_4\}\in\mathcal{H}$, $t>0$

$$||U(t)f||_0^2 \leq (2C_1+t_0)\frac{1}{t_0^{1-\beta}(t+t_0)^{\beta}}||f||_0^2$$

with $0<\beta<\mu\leq 1$, $t_0\geq t^*=t^*(\Omega,A,Q,B)$, $C_1=C_1(\Omega,A,Q,B)$.

Corollary 2.1. *$U(t)$ takes the space $\mathcal{H}$ into itself and*

$$||U(t)||_{\mathcal{H}\to\mathcal{H}}<1$$

for

$$t>t_1=t_0\Big[\Big(1+\frac{2C_1}{t_0}\Big)^{1/\beta}-1\Big].$$

Using Pazy's theorem [11], we obtain

Corollary 2.2. *Suppose $f(x)\in\mathcal{H}$. There exist C, $\gamma>0$ such that*

$$||U(t)f||_0^2 \leq C\ \exp(-\gamma t)||f||_0^2\,.$$

Using the estimate of Theorem 2.1 we study the following exact controllability problem:

Given the initial distribution $f(x)=\{f_1(x),f_2(x),f_3(x),f_4(x)\}$, time $T>0$, and a desired terminal state $g(x)=\{g_1(x),g_2(x),g_3(x),g_4(x)\}$ with $f(x)$, $g(x)$ in $\mathcal{H}$, find a vector-valued function $p(x,t)$ such that the solution of the problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(A\frac{\partial u}{\partial x_i}\right) + B(x)\frac{\partial v}{\partial t} = \chi_{\mathcal{D}}(x)p(x,t),\\
&\frac{\partial^2 v}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(A\frac{\partial u}{\partial x_i}\right) - B(x)\frac{\partial u}{\partial t} = 0,\\
&u|_{t=0} = f_1(x),\ v|_{t=0} = f_2(x),\ \frac{\partial u}{\partial t}|_{t=0} = f_3(x),\ \frac{\partial v}{\partial t}|_{t=0} = f_4(x),\\
&u|_S = 0,\ \ v|_S = 0,
\end{aligned} \tag{2.4}$$

satisfies

$$u|_{t=T} = g_1(x),\ v|_{t=T} = g_2(x),\ \frac{\partial u}{\partial t}|_{t=T} = g_3(x),\ \frac{\partial v}{\partial t}|_{t=T} = g_4(x). \tag{2.5}$$

Here we denote by $\chi_{\mathcal{D}}(x)$ the characteristic function of $\mathcal{D}$.

Theorem 2.2. *Assume that $B(x)$ satisfies the following assumptions:*

1) $B(x)\xi\cdot\xi \geq 0$, $x\in\Omega$, $B(x)\xi\cdot\xi \geq \beta_0|\xi|^2$, $x\in\mathcal{D}$, $\xi\in R^m$, $\beta_0>0$;

2) $B(x)\equiv 0$, $x\in\Omega\backslash\overline{\mathcal{D}}$.

Let Ω and $\mathcal{D}$ be such that there exists function $\varphi(x)$ with properties (i)-(iii).

Then for any $T>t_1$, given any initial data $f\in\mathcal{H}$ and any $g\in\mathcal{H}$ there exists a control $p(x,t)\in L_2(D\times(0,T))$ such that the corresponding solution of (2.4) *satisfies* (2.5). *Moreover,*

$$||p||^2_{L^2(\mathcal{D}\times(0,T))} \leq C(||f||_0^2 + ||g||_0^2).$$

3. Hyperbolic systems of second order. Boundary damping.

Let Ω be a bounded domain in R^n with boundary S which consists of two disjoint closed surfaces S_0 and S_1 (the case $S_1=\phi$ is not excluded).

Consider the following mixed problem in the cylinder $\Omega\times(0,T)$:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right)=0,\ \frac{\partial^2 v}{\partial t^2}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial v}{\partial x_j}\right)=0,\\
&u|_{t=0}=f_1(x),\ v|_{t=0}=f_2(x),\ \frac{\partial u}{\partial t}|_{t=0}=f_3(x),\ \frac{\partial v}{\partial t}|_{t=0}=f_4(x),\\
&\sum_{i,j=1}^{n}A_{ij}\frac{\partial u}{\partial x_j}\nu_i+au+bu_t+\xi v_t|_{S_0}=0,\ u|_{S_1}=0,\\
&\sum_{i,j=1}^{n}A_{ij}\frac{\partial v}{\partial x_j}\nu_i+cv-\xi u_t|_{S_0}=0,\ v|_{S_1}=0.
\end{aligned}\tag{3.1}$$

Here $u=(u^1(x,t),\dots,u^m(x,t)),\ v=(v^1(x,t),\dots,v^m(x,t)), x=(x_1,x_2,\dots,x_n)$, $A_{ij}=A^*_{ji}$ are square matrices of order m, $\nu=(\nu_1,\dots,\nu_n)$ is the unit outer normal, and a, b, c, ξ are positive constants.

Assume that

$$\sum_{i,j}A_{ij}\xi_j\xi_i\ge a_0\sum_i|\xi_i|^2,\quad a_0>0,\quad \xi_i=(\xi_i^1,\dots,\xi_i^m)\in R^m.$$

Denote by $\mathcal{H}$ the real Hilbert space of quadruples $\{u,v,u_1,v_1\}$ of m-component vector-functions such that

$$u,v\in H^1(\Omega),\quad u|_{S_1}=v|_{S_1}=0,\quad u_1,v_1\in L_2(\Omega).$$

We define the inner product in $\mathcal{H}$ by

$$\langle\{u,v,u_1,v_1\},\{f,g,f_1,g_1\}\rangle_0=\int_{S_0}(au\cdot f+cv\cdot g)dS+$$

$$+\int_\Omega\left(A_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial f}{\partial x_i}+A_{ij}\frac{\partial v}{\partial x_j}\cdot\frac{\partial g}{\partial x_i}+u_1\cdot f_1+v_1\cdot g_1\right)dx.$$

In $\mathcal{H}$ we define the unbounded operator $\mathcal{A}\colon\mathcal{D}(\mathcal{A})$ consists of the elements $\{u,v,u_1,v_1\}\in\mathcal{H}$ such that

$$u, v \in H^2(\Omega), \quad u_1, v_1 \in H^1(\Omega),$$

$$\sum_{i,j=1}^{n} A_{ij} \frac{\partial u}{\partial x_j} \nu_i + au + bu_1 + \xi v_1|_{S_0} = 0,\ u_1|_{S_1} = u|_{S_1} = 0,$$

$$\sum_{i,j=1}^{n} A_{ij} \frac{\partial v}{\partial x_j} \nu_i + cv - \xi u_1|_{S_0} = 0,\ v_1|_{S_1} = v|_{S_1} = 0.$$

For $\{u, v, u_1, v_1\} \in \mathcal{D}(\mathcal{A})$

$$\mathcal{A}\{u, v, u_1, v_1\} = \{u_1, v_1, \tilde{A}u, \tilde{A}v\}, \text{ where } \tilde{A}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(A_{ij} \frac{\partial u}{\partial x_j} \right).$$

In a standard way, we can check that $\mathcal{A}$ and the adjoint operator $\mathcal{A}^*$ are dissipative. Thus, $\mathcal{A}$ generates a strongly continuous semigroup of contractions $U(t)$, $t > 0$. It follows that if $\{u, v, u_1, v_1\} = U(t)f$ $(f \in \mathcal{D}(\mathcal{A}))$ then u, v is a solution of (3.1) and $u_t = u_1$, $v_t = v_1$. Moreover,

$$\frac{d}{dt} E(t) \equiv \frac{d}{dt} ||U(t)f||_0^2 = -2 \int_{S_0} b|u_1|^2 \, dS.$$

Observe that, for $f \in \mathcal{H}$, $U(t)f$ is a weak solution in $\mathcal{H}$ to the abstract Cauchy problem

$$\frac{d}{dw} w = \mathcal{A}w, \quad w|_{t=0} = f$$

in the following sense:

$$\int_0^T \left(\langle U(t)f, \frac{d\Psi}{dt} \rangle_0 + \langle U(t)f, \mathcal{A}^*\Psi \rangle_0 \right) dt = -\langle f, \Psi(0) \rangle$$

for every $\Psi \in L_2(0, T; \mathcal{D}(\mathcal{A}^*))$, $\Psi_t \in L_2(0, T; \mathcal{H})$, $\Psi(T) = 0$.

The following formulas give grounds for proving stabilization of a solution to problem (3.1):

$$
\begin{aligned}
&(T+t_0)E(t)-t_0\,E(0)+\\
&+\textstyle\int_\Omega[2(\nabla\varphi,\nabla)u\cdot u_t+2(\nabla\varphi,\nabla)v\cdot v_t+(n-1-\gamma)(u\cdot u_t+v\cdot v_t)dx|_{t=0}^{t=T}+\\
&+\textstyle\int_{S_0}\dfrac{n-1-\gamma}{2}b|u|^2\,dS|_{t=0}^{t=T}=\displaystyle\int_0^T\int_\Omega\{(\Delta\varphi-n+2+\gamma)(\Phi(u)+\Phi(v))-\\
&-(\Delta\varphi-n+\gamma)(|u_t|^2+|v_t|^2)-2\sum_{i,j,p=1}^n\frac{\partial^2\varphi}{\partial x_p\partial x_i}(A_{ij}u_{x_j}\cdot u_{x_p}+A_{ij}v_{x_j}\cdot v_{x_p})\}dxdt+\\
&+\textstyle\int_0^T\int_{S_1}\dfrac{\partial\varphi}{\partial\nu}(\Phi(u)+\Phi(v))dSdt+\displaystyle\int_0^T\int_{S_0}\{-2(t+t_0)b|u_t|^2+\frac{\partial\varphi}{\partial\nu}(|u_t|^2+|u_t|^2)-\\
&-\frac{\partial\varphi}{\partial\nu}(\Phi(u)+\Phi(v))-(n-2-\gamma)(a|u|^2+c|v|^2)-(n-1-\gamma)\xi u\cdot v_t+\\
&+(n-1-\gamma)\xi u_t\cdot v-\sum_{p=1}^n 2b\varphi_{x_p}u_{x_p}\cdot u_t-\sum_{p=1}^n 2\xi\varphi_{x_p}u_{x_p}\cdot v_t+\sum_{p=1}^n 2\xi\varphi_{x_p}v_{x_p}\cdot u_t-\\
&-\sum_{p=1}^n 2a\varphi_{x_p}u_{x_p}\cdot u-\sum_{p=1}^n 2c\varphi_{x_p}v_{x_p}\cdot v\}dSdt,
\end{aligned}
\tag{3.2}
$$

$$
\textstyle\int_\Omega\left(u_t\cdot v_t+\displaystyle\sum_{i,j=1}^n A_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial v}{\partial x_i}\right)dx|_{t=0}^{t=T}=
$$

$$
=\textstyle\int_0^T\int_{S_0}(\xi|u_t|^2-\xi|v_t|^2-bu_t\cdot v_t-(c-a)u_t\cdot v-\dfrac{\partial}{\partial t}(au\cdot v))dSdt.
\tag{3.3}
$$

Here $\Phi(u)=\sum_{i,j=1}^n A_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i}$, $\varphi=\varphi(x)$ is an arbitrary smooth function, γ is positive constant.

Henceforth we assume that Ω satisfies the geometric conditions that are listed below.

Let $\Psi(x)$ be a solution to the problem

$$
\Delta\Psi=\frac{a_1}{a_0},\quad x\in\Omega,\quad \left.\frac{\partial\Psi}{\partial\nu}\right|_{S_0}=\frac{a_1}{a_0}\frac{\operatorname{mes}\Omega}{\operatorname{mes}S_0},\quad \left.\frac{\partial\Psi}{\partial\nu}\right|_{S_1}=0,
$$

where $a_1=\max|a_{ij}^{pq}|$, a_{ij}^{pq} are the entries of the matrix A_{ij}, and the constant a_0 is defined as above (observe that $A_{ij}=\delta_{ij}I$ and $a_0=a_1=1$ for the wave operator). Assign

$$
w=\max_{i,j}\sup_{x\in\overline{\Omega}}|\Psi_{x_ix_j}(x)|.
$$

Suppose that the domain Ω satisfies the following conditions: there is a point $x^0\in R^m$ such that

(a) S_1 is starlike with respect to $x^0 : (x - x^0, \nu) \leq 0$ for $x \in S_1$;

(b) for some $0 < \varepsilon \leq 1$,

$$(x - x^0, \nu) > -\frac{1}{\varepsilon + nw} \frac{\operatorname{mes} \Omega}{\operatorname{mes} S_0}, \quad x \in S_0 .$$

Clearly, condition (b) holds if the surface S_0 is starlike with respect to the point x^0.

Choose the function $\varphi(x)$ in (3.2) as follows

$$\varphi(x) = \frac{a_0}{a_1} \Psi(x) + \frac{1}{2\theta} |x - x^0|^2 , \quad \theta > 0.$$

We now can deduce from (3.2), (3.3) the following assertion ([2]).

Theorem 3.1. *Suppose that Ω satisfies the above-listed conditions with a parameter $0 < \varepsilon \leq 1$ and*

$$0 < a < \frac{\delta a_0 n^2 w}{3r}, 0 < c < \frac{\delta a_0 n^2 w}{3r} \left(\frac{\partial \varphi}{\partial \nu} \geq |\nabla \varphi| \delta, \delta > 0, x \in S_0, r = \sup_{x \in \overline{\Omega}} |\nabla \varphi| \right).$$

Then there are $t^* > 0$ and $C^* > 0$ such that for $t_0 \geq t^*$

$$||U(t)f||_0^2 \leq (2C^* + t_0) \frac{1}{t_0^{1-\varepsilon} (t + t_0)^\varepsilon} ||f||_0^2$$

for every $f \in \mathcal{H}$.

We use the estimate of Theorem 3.1 to prove exact boundary controllability for the system

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij} \frac{\partial u}{\partial x_j} \right) = 0, \quad \frac{\partial^2 v}{\partial t^2} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij} \frac{\partial v}{\partial x_j} \right) = 0, \\
&u|_{t=0} = g_1(x), \ v|_{t=0} = h_1(x), \ \frac{\partial u}{\partial t}|_{t=0} = g_2(x), \ \frac{\partial v}{\partial t}|_{t=0} = h_2(x), \\
&\sum_{i,j=1}^n A_{ij} \frac{\partial u}{\partial x_j} \nu_i + au + \xi v_t|_{S_0} = p(x,t), \ u|_{S_1} = 0, \\
&\sum_{i,j=1}^n A_{ij} \frac{\partial v}{\partial x_j} \nu_i + cv - \xi u_t|_{S_0} = 0, \ v|_{S_1} = 0
\end{aligned} \tag{3.4}$$

where $\{g_1, h_1, g_2, h_2\} \in \mathcal{H}$. For every element $\{G_1, H_1, G_2, H_2\} \in \mathcal{H}$, we have to find vector-function $p(x,t)$ such that the solution of problem (3.4) satisfies

$$u|_{t=T} = G_1(x), \quad v|_{t=T} = H_1(x), \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = G_2(x), \quad \frac{\partial v}{\partial t}\bigg|_{t=0} = H_2(x) \tag{3.5}$$

for $T > t_1$. Here t_1 is such that $||U(t)||_{\mathcal{H}\to\mathcal{H}} < 1$ for $t > t_1$.

Theorem 3.2. *Assume that the coefficients a and c in boundary conditions of* (3.4) *satisfy the assumptions of Theorem* 3.1.

Suppose that Ω satisfies the above-listed conditions. Then for any $T > t_1$, given initial data $\{g_1, h_1, g_2, h_2\} \in \mathcal{H}$ and any $\{G_1, H_1, G_2, H_2\}$ $\left\{u, v, \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right\}\bigg|_{t=T} = \{G_1, H_2, G_2, H_2\}$ at time T. Moreover,

$$||p||^2_{L_2(S_0\times(0,T))} \leq C(||\{g_1, h_1, g_2, h_2\}||_0^2 + ||\{G_1, H_1, G_2, H_2\}||_0^2).$$

Theorem 3.2 implies a simultaneous exact boundary control. Consider the following two mixed problems in $\Omega \times (0,T)$:

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right) = 0, \;\; u|_{t=0} = g_1(x), \;\; \frac{\partial u}{\partial t}|_{t=0} = g_2(x), \\ &u|_{S_0} = p(x,t), \quad u|_{S_1} = 0; \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\frac{\partial^2 v}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial v}{\partial x_j}\right) = 0, \;\; v|_{t=0} = h_1(x), \;\; \frac{\partial vu}{\partial t}|_{t=0} = h_2(x), \\ &\sum_{i,j=1}^{n} A_{ij}\frac{\partial v}{\partial x_j}\nu_i + cv|_{S_0} = q(x,t), \quad v|_{S_1} = 0. \end{aligned} \tag{3.7}$$

Theorem 3.3. *Suppose that Ω satisfies the above-listed conditions and the coefficient c satisfies the assumptions of Theorem* 3.2. *Then for any $T > t_1$, given initial data $\{g_1, h_1, g_2, h_2\} \in \mathcal{H}$ and any $\{G_1, H_1, G_2, H_2\} \in \mathcal{H}$ there exists a vector-function $p(x,t) \in L_2(S_0 \times (0,T))$ transferring system* (3.6) *to the state $\{G_1, G_2\}$ at time T:*

$$u|_{t=T} = G_1(x), \quad \frac{\partial u}{\partial t}\bigg|_{t=T} = G_2(x).$$

Moreover, $p_t(x,t)$ belongs to $L_2(S_0 \times (0,T))$ and $q(x,t) = p_t(x,t)$ transfers system (3.7) to the state $\{H_1, H_2\}$ at the same time T:

$$v\big|_{t=T} = H_1(x), \quad \frac{\partial v}{\partial t}\Big|_{t=T} = H_2(x).$$

We remark that simultaneous boundary control for the wave equation has been established by Lions [10] using Hilbert Uniqueness Method ([9], [10]). Controllability of the coupled system of thermoelasticity has been proved by Zuazua [16].

Remark 3.1. All assertions of this section are valid for the pair of systems of elasticity theory.

4. Maxwell's equations. Boundary damping.

Let $\Omega \subset R^3$ be a bounded domain with sufficiently smooth boundary S. In $\Omega \times (0,T)$ we consider the initial boundary value problem for a pair of the Maxwell systems

$$\begin{cases} u^1_t = \operatorname{curl}(\mu u^2), u^2_t = -\operatorname{curl}(\lambda u^1), \operatorname{div} u^1 = \operatorname{div} u^2 = 0, \\ u^3_t = \operatorname{curl}(\mu u^4), u^4_t = -\operatorname{curl}(\lambda u^3), \operatorname{div} u^3 = \operatorname{div} u^4 = 0, \end{cases} \tag{4.1}$$

$$u^1(x,0) = f^1(x), u^2(x,0) = f^2(x), u^3(x,0) = f^3(x), u^4(x,0) = f^4(x), \tag{4.2}$$

$$\begin{cases} [u^2, \nu] - \alpha[\nu, [u^1, \nu]] + \xi[\nu, [\nu, u^3]] = 0, \\ [u^4, \nu] - \beta[\nu, [u^3, \nu]] - \xi[\nu, [\nu, u^1]] = 0, \end{cases} \quad (x,t) \in S \times (0,T) \tag{4.3}$$

where u^i are three-dimensional vector-valued functions of t, $x = (x_1, x_2, x_3)$, ν is the unit outer normal, $[\,\cdot\,,\,\cdot\,]$ is the vector product, $\mu = \mu(x)$ and $\lambda = \lambda(x)$ are scalar functions in Ω, and $\alpha = \alpha(x)$, $\beta = \beta(x)$, $\xi = \xi(x)$ are continuously differentiable functions on S with $\operatorname{Re}\alpha \geq 0$, $\operatorname{Re}\beta \geq 0$.

For $\xi \equiv 0$ boundary conditions (4.3) (the Leontovich conditions) mean that surface S is a conductor and complex-valued function $\alpha(x)$ (and $\beta(x)$) is a surface impedance.

If $\xi \equiv 0$ the mixed problem (4.1)-(4.3) disintegrates into two initial boundary value problems for Maxwell's equations. In the case of damping boundary conditions ($\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$) exponential decay of the total energy ($\xi \equiv 0$)

$$E(t) = \int_\Omega [\lambda(|u^1|^2 + |u^3|^2) + \mu(|u^2|^2 + |u^4|^2)]dx$$

has been proved by the author [3].

Our purpose is to prove the uniform simultaneous stabilization as $t \to \infty$ of the total energy $E(t)$. It means that (4.3) are damping boundary conditions only for the first half of system (4.1): $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta \equiv 0$,

$$\frac{d}{dt}E(t) = -2\int_S \operatorname{Re}\alpha|[u^1,\nu]|^2\,dS.$$

Using this result we study the exact controllability problem for (4.1) with boundary control

$$\begin{cases} [u^2,\nu] - i\operatorname{Im}\alpha[\nu,[u^1,\nu]] + \xi[\nu,[\nu,u^3]] = p(x,t), \\ [u^4,\nu] - \beta[\nu,[u^3,\nu]] - \xi[\nu,[\nu,u^1]] = 0. \end{cases} \tag{4.4}$$

The exact controllability problem for the Maxwell system ($\lambda(x) \equiv \lambda_0$, $\mu(x) \equiv \mu_0$ for $x \in \Omega$) with boundary control by means of currents flowing tangentially in the boundary of the region

$$[\nu,u^1] = q(x,t) \quad (x,t) \in S \times (0,T)$$

has been studied by Russell [13] for a circular cylindrical region, by Kime [7] for a spherical region, and by Lagnese [8] for an arbitrary domain. In [8] control problem is solved by the Hilbert Uniqueness Method introduced by Lions [9], [10].

Let $\lambda(x)$ and $\mu(x)$ be continuously differentiable functions in Ω satisfying the conditions

$$0 < \lambda_0 \le \lambda(x) \le \lambda_1\,, \quad 0 < \mu_0 \le \mu(x) \le \mu_1\,.$$

We denote by $\mathcal{H}$ the Hilbert space of quadruples $u = \{u^1, u^2, u^3, u^4\}$ of three-component complex-valued functions $u^i \in L_2(\Omega)$ with the inner product

$$\langle u, v\rangle_0 = \int_\Omega [\lambda(u^1, \overline{v}^1) + \mu(u^2, \overline{v}^2) + \lambda(u^3, \overline{v}^3) + \mu(u^4, \overline{v}^4)]dx.$$

In $\mathcal{H}$ we can define the unbounded operator $\mathcal{A}$ in the same way as in [3], [4]. It can be shown that $\mathcal{A}$ and the adjoint operator $\mathcal{A}^*$ are dissipative for $\operatorname{Re}\alpha(x) \geq 0$, $\operatorname{Re}\beta(x) \geq 0$. From it follows that $\mathcal{A}$ generates a strongly continuous semigroup of contractions $U(t)$, $t > 0$.

Let M be the orthogonal complement of the kernel of $\mathcal{A}^*$ in $\mathcal{H}$. We note that $U(t)f$ for $f \in M$ is a weak solution of (4.1)-(4.3).

We now define a special class of domains which includes starlike domains.

Consider the problem

$$\Delta\Phi = 1 \,\text{in}\, \Omega, \qquad \frac{\partial\Phi}{\partial\nu}\Big|_S = \frac{\operatorname{mes}\Omega}{\operatorname{mes} S}\,.$$

We define the following quantity:

$$k(\Omega) = \sup_{x\in\Omega_{|\xi|=1}} 2\operatorname{Re}\Phi_{x_i x_j}\,\xi^i\overline{\xi}^j\,, \qquad \xi = (\xi^1, \xi^2, \xi^3) \in C^3\,.$$

We shall say that Ω is substarlike if:

(i) $k(\Omega) < 1$, or

(ii) $k(\Omega) \geq 1$; there exists a point $x^0 \in \Omega$ such that for some $0 < \varepsilon \leq 1$

$$(x - x^0, \nu) > -\frac{1}{k + \varepsilon - 1}\frac{\operatorname{mes}\Omega}{\operatorname{mes} S}\,.$$

The following proposition is proved in [3].

Lemma 4.1. *Assume that Ω is substarlike domain. Then there exists a function $\varphi(x) \in C^2(\Omega) \cap C^1(\Omega)$ such that*

(i) $(\nabla\varphi, \nu) > 0$ *on* S_0,

(ii) $2\operatorname{Re}\varphi_{x_i x_j}\,\xi^i\xi^j - (\Delta\varphi - 1)|\xi|^2 \leq (1 - w)|\xi|^2$ *in* Ω, *where* $0 < w \leq 1$.

We now assume that $\operatorname{Re}\alpha(x) \geq \alpha_0 > 0$, $\operatorname{Re}\beta(x) \equiv 0$, $\operatorname{Re}\xi(x) \geq \xi_0 > 0$.

The proof of the next theorem is based on the invariance of the Maxwell system in vacuum relative to the group of dilations in all variables ([6]).

Theorem 4.1. *Assume that* Ω *is substarlike,* $(\nabla\varphi, \nabla\lambda) \leq \lambda(w-\gamma)$, $(\nabla\varphi, \nabla\mu) \leq \mu(w-\gamma)$, *where* $\varphi(x)$, w *are defined in Lemma* 4.1, $0 < \gamma \leq 1$. *Then for all* $f \in M$, $t \geq 0$

$$||U(t)f||_0^2 \leq \left(\frac{2d}{\sqrt{\lambda_0\mu_0}} + 2C + t_0\right)\frac{1}{t_0^{1-\gamma}} \cdot \frac{1}{(t+t_0)^\gamma}||f||_0^2 .$$

Here $d = \sup\limits_{\overline{\Omega}} |\nabla\varphi|$, $C = C(\varphi, \lambda, \mu, \xi)$, $t_0 = t_0(\varphi, \lambda, \mu, \xi)$ are constants.

Corollary 4.1. *Suppose* $f(x) \in M$. *There exist* C_1 , $\beta > 0$ *such that*

$$E(t) \leq C_1 \exp(-\beta t)E(0).$$

We use the estimate of Theorem 4.1 to prove exact controllability to an arbitrary state of solutions of (4.1), (4.2), (4.4) ([6]).

Theorem 4.2. *Assume that* Ω *is substarlike,* $\operatorname{Im}\alpha, \beta, \xi \in C^1(S)$, $\operatorname{Re}\beta \equiv 0$, $\operatorname{Re}\xi \geq \xi_0 > 0$. *Suppose that* $\lambda(x)$ *and* $\mu(x)$ *satisfy the conditions of Theorem* 4.1. *There is a* $T_0 > 0$ *such that, for any* $T > T_0$, *given* $f = \{f_1, f_2, f_3, f_4\}$, $g = \{g_1, g_2, g_3, g_4\} \in M$ *there exists a boundary control* $p(x,t) \in L_2(S \times (0,T))$ *such that the corresponding solution of* (4.1), (4.2), (4.4) *satisfies*

$$\{u^1(x,T), u^2(x,T), u^3(x,T), u^4(x,T)\} = \{g^1(x), g^2(x), g^3(x), g^4(x)\}.$$

Moreover,

$$||p||_{L_2(S\times(0,T))}^2 \leq C(||f||_0^2 + ||g||_0^2).$$

Using this result we obtain simultaneous exact control for the following problems ($\operatorname{Re}\beta \equiv 0$):

$$\begin{cases} e_t = \operatorname{curl}(\mu h), \quad h_t = -\operatorname{curl}(\lambda e), \operatorname{div} e = \operatorname{div} h = 0, \\ e(x,0) = f^1(x), \quad h(x,0) = f^2(x), \\ [\nu, e] = p(x,t), \quad (x,t) \in S \times (0,T); \end{cases} \tag{4.5}$$

$$\begin{cases} E_t = \operatorname{curl}(\lambda H), \quad H_t = -\operatorname{curl}(\mu E), \quad \operatorname{div} E = \operatorname{div} H = 0, \\ E(x,0) = f^3(x), \quad H(x,0) = f^4(x), \\ [H,\nu] - \beta[\nu,[E,\nu]] = q(x,t), \quad (x,t) \in S \times (0,T) \end{cases} \tag{4.6}$$

For any $T > T_0$, given $f = \{f^1, f^2, f^3, f^4\}$, $g = \{g^1, g^2, g^3, g^4\} \in M$ there exists a control $p(x,t)$ driving the system (4.5) to the state g^1, g^2 at time T:

$$e(x,T) = g^1(x), \quad h(x,T) = g^2(x),$$

while the function $q = [\nu, p]$ drives the system (4.6) to the state g^3, g^4 at the same time T:

$$E(x,T) = g^3(x), \quad H(x,T) = g^4(x).$$

References

[**1**] C. Bardos, G. Lebeau and J. Rauch, *Contrôle et Stabilisation dans les Problèmes Hyperboliques*, Appendix II in J.L. Lions [7].

[**2**] B. V. Kapitonov, *Uniform stabilization and simultaneous exact boundary controllability for a pair of hyperbolic systems,* Siberian Mathematical Journal, 35 (4), (1994), pp. 722–734.

[**3**] B. V. Kapitonov, *Stabilization and exact boundary controllability for Maxwell's equations,* SIAM J. Control Optim., 32 (1994), pp. 408–421.

[**4**] B. V. Kapitonov, *On exponential decay as $t \to \infty$ of solutions of an exterior boundary value problem for the Maxwell systems,* Math. USSR Sb. 66 (1990), pp. 475–498.

[**5**] B. V. Kapitonov, *Uniform stabilization and exact controllability for a class of coupled hyperbolic system*, Computational and Applied Mathematics, to appear.

[6] B. V. Kapitonov, *Stabilization and simultaneous boundary controllability for a pair of Maxwell's equations,* Computational and Applied Mathematics, to appear.

[7] K. A. Kime, *Boundary controllability of Maxwell's equations in a spherical region,* SIAM J. Control Optim., 18 (1990), pp. 294–319.

[8] J. E. Lagnese, *Exact boundary controllability of Maxwell's equations in a general region,* SIAM J. Control Optim., 27 (1989), pp. 374–388.

[9] J. L. Lions, *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués,* Tome 1. Contrôlabilité Exacte, Masson, Paris, Collection R.M.A., 1988.

[10] J. L. Lions, *Exact controllability, stabilization and perturbations for distributed systems,* SIAM Review, 30 (1988), pp. 1–68.

[11] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Applied Mathematical Sciences, Springer, Berlin-New York, 1983.

[12] D. C. Pereira and G.P. Menzala, *Exponential decay of solutions to a coupled system of equations of linear thermoelasticity,* Mat. Aplic. Comp., 8 (3) (1989), pp. 193–204.

[13] D. L. Russell, *The Dirichlet-Neumann boundary control problem associated with Maxwell's equations in a cylindrical region,* SIAM J. Control Optim., 24 (1986), pp. 199–229.

[14] J. M. Rivera, *Energy decay rates in linear thermoelasticity,* Funkcialaj Ekvacioj, 35 (1) (1992), pp. 19–30.

[15] J. M. Rivera and Y. Shibata, *A linear thermoelastic plate equation with Dirichlet boundary condition,* Preprint.

[16] E. Zuazua, *Contrôlabilité de système de la thermo élasticité sans restriction sur les paramètres de couplage,* C.R. Acad. Sci., 318, Série I (1994), pp. 643–648.

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