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INITIAL BOUNDARY VALUE PROBLEM FOR A CLASS OF QUADRATIC SYSTEMS OF CONSERVATION LAWS

Pui Tak Kan * **Marcelo M. Santos** † **Zhouping Xin** ‡

Abstract

This paper concerns the Initial Boundary Value Problem for a class of systems of conservation laws with a quadratic flux and a characteristic boundary condition. We prove the existence of solution in L^∞, with the boundary condition strongly satisfied in L^2. We use the Godunov's numerical method to obtain approximate solutions, and we obtain the solution as limit of these approximate solutions. The limit in the interior is given by the theory of compensated compactness and by a priory uniform estimate of the approximate solutions in L^∞. The limit at the boundary is obtained by solving analytically the lateral Riemann problem with the given boundary condition.

Resumo

Este artigo consiste do Problema de Valor Inicial e de Fronteira para uma classe de sistemas de leis conservação quadráticos com uma condição de fronteira característica. Provamos a existência de solução em L^∞, com a condição de fronteira sendo satisfeita fortemente em L^2. Usamos o método numérico de Godunov para obter soluções aproximadas e obtemos a solução como limite destas soluções aproximadas. O limite no interior é dado pela Teoria de Compacidade Compensada, e por estimativas uniformes das soluções aproximadas no L^∞. O limite na fronteira é obtido resolvendo-se analiticamente o Problema de Riemann lateral com a condição de fronteira dada.

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1. Introduction

Consider the symmetric quadratic systems of conservation laws

$$\begin{cases} u_t + \frac{1}{2}(au^2 + v^2)_x &= 0 \\ \qquad v_t + (uv)_x &= 0, \end{cases} \tag{1.1}$$

where $x > 0$, $t > 0$, $U = (u, v) \in \mathbf{R}^2_+ \overset{\text{def}}{=} \{(u, v) \in \mathbf{R}^2;\ v \geq 0\}$, and $1 < a < 2$. These systems are the case III of symmetric quadratic system introduced in [14]. The quadratic systems arise from 2×2 systems of nonstrictly hyperbolic conservation laws by neglecting high order terms in the Taylor series of the flux functions, and they can be used as model to oil recovery [14]. The solution of their Cauchy or Riemann problem presents complexities that distinguish its own theory, see e.g. [1, 2, 5, 9, 10, 11, 13, 14]. In this paper we consider a boundary condition besides an initial condition for the systems (1.1).

We prescribe the initial data for the systems (1.1)

$$U(x, 0) = U_0(x), \tag{1.2}$$

where $U_0(x) \in \mathbf{R}^2_+$ for all $x > 0$ and $U_0 \in L^2 \cap L^\infty$, and the following boundary condition

$$\sqrt{a}\, u(0, t) - v(0, t) = 0. \tag{1.3}$$

We prove the existence of a solution (u, v) to the Initial Boundary Value Problem (IBVP) (1.1)–(1.3). We use the Godunov numerical scheme to obtain an approximate solution $U^\varepsilon(x, t)$ and then we take the limit as ε goes to zero to obtain an exact solution.

Once the Godunov scheme is defined, we have an approximate solution $U^\varepsilon(x, t)$ for (1.1)–(1.3). We prove that U^ε is uniformly bounded in $L^\infty(\mathbf{R}^2_+)$ with respect to ε. Then [1, 2] prove that U^ε converges pointwise a.e. in $x > 0$, $t > 0$ to an *interior* solution U for (1.1)–(1.3), that is, to a solution U for the Cauchy problem (1.1)–(1.2). The boundary condition (1.3), as we shall see, is satisfied in the following sense

$$\int_0^\infty \int_0^\infty 2(u - \sqrt{a}v)\phi_t + (\sqrt{a}u - v)^2 \phi_x dx dt + \int_0^\infty \left(2(u - \sqrt{a}v)\phi\right)(x, 0) dx = 0$$

for all $\phi \in C_0^1(\mathbf{R}^2)$, which implies that

$$* - \lim_{\delta\to 0+} \frac{1}{\delta} \int_0^\delta \left(\sqrt{a}u(x,\cdot) - v(x,\cdot)\right)^2 dx = 0, \tag{1.4}$$

where $* - \lim$ stands for the weak-$*$ limit in $L^\infty(t > 0)$ (see section 4).

In the forthcoming paper [12] we extend the boundary condition (1.3) to a class of boundary conditions for which the Godunov scheme is well defined, and show the existence of trace γq at the boundary $x = 0$ of entropy fluxes q composed with an interior solution. The existence of this trace γq holds for general $n \times n$ systems of conservation laws

$$U_t + F(U)_x = 0,\ t > 0,\ x > 0, \tag{1.5}$$

and it satisfies the inequality

$$\gamma q \le q(\overline{U}_0) + \eta(\overline{U}_0) \cdot \{\gamma F - F(\overline{U}_0)\} \tag{1.6}$$

for any entropy–entropy flux pair (η, q) of (1.5). Here we assume that (1.5) has a solution U that is the limit a.e. in $x > 0,\ t > 0$ of approximate solutions U^ε generated either by Godunov scheme or an appropriate vanishing viscosity method as in [7], $q(\overline{U}_0)$ and $\eta(\overline{U}_0)$ are respectively the weak-$*$ limit of $q(U^\varepsilon(0,\cdot)$ and $\eta(U^\varepsilon(0,\cdot))$ at the boundary $x = 0$, and γF is the trace of the flux F at the boundary $x = 0$. The trace γq can be computed by the formula

$$\gamma q = * - \lim_{\delta\to 0} \frac{1}{\delta} \int_0^\delta q\left(U(x,\cdot)\right) dx. \tag{1.7}$$

Besides, if the interior solution U is generated by Godunov scheme then we have

$$\gamma F = F(\overline{U}_0),$$

so (1.6) reduces to

$$\gamma q \le q(\overline{U}_0)$$

in this case. These facts will be proved in [12].

From (1.4) and (1.7) we see that the boundary condition is also satisfied in the sense that the trace of $(\sqrt{a}u(x,t) - v(x,t))^2$ at the boundary $x = 0$ exists and it is equal to zero. Note that $(\sqrt{a}u - v)^2$ is an entropy flux for the system (1.1) associated with the entropy $2(u - \sqrt{a}v)$.

We point out that the condition (1.4) also underlines a phenomenon of regularity of the solution: Although the functions $u(x,t)$ and $v(x,t)$ may oscillate when x goes to the boundary $x = 0$, the quantity $(\sqrt{a}u(x,t) - v(x,t))^2$ has a weak limit as x goes to zero in the sense of (1.4). Actually, the existence of the trace γq for the $n \times n$ systems of conservation laws (1.5) implies that

$$\int_0^\infty q\left(U(x,t)\right)\zeta(t)dt$$

is a function of bounded variation in $x > 0$ for any $\zeta \in C_0^1(t > 0)$ [12].

This paper is organized as follows. In section 2 we recall preliminaries results about the systems (1.1). In section 3 we give an explicit parametrization of the Hugoniot loci of (1.1) that helps to provide a rigorous analytical solution of the the Riemann problem for (1.1). In the last section, 4, we give the solution of the *lateral* Riemann problem for (1.1), show the Godunov scheme for (1.1)–(1.3), and prove that the approximate solutions for (1.1)–(1.3) obtained by the Godunov scheme are uniformily bounded in L^∞. This uses Hoff's theorem [8] on invariant regions for Riemann problems. To apply this theorem we give a formula for the third derivatives of the difference between rarefaction and shock curves at the initial point: see Proposition 4.4 below. We also prove in section 4 the existence of solution of the IBVP (1.1)–(1.3).

2. Preliminaries

In this section we recall some basic facts on the system (1.1). For the details we refer the reader to [10], section 2 of [11], and [1, 2].

The eigenvalues of (1.1) are

$$\lambda_k = \frac{1}{2}\left\{(a+1)u + (-1)^k\sqrt{(a-1)^2u^2 + 4v^2}\right\}, \tag{2.1}$$

$k = 1, 2$. According to their signs, the upper half plane is divided in three regions: $K_1 \stackrel{\text{def}}{=} \{U \in \mathbf{R}^2_+;\ \lambda_1(U) < \lambda_2(U) < 0\}$, $K_2 \stackrel{\text{def}}{=} \{U \in \mathbf{R}^2_+;\ \lambda_1(U) <$

$0 \leq \lambda_2(U)\}$, and $K_3 \stackrel{\text{def}}{=} \{U \in \mathbf{R}^2_+; 0 \leq \lambda_1(U) \leq \lambda_2(U)\}$. Corresponding eigeinvectores are

$$r_{1,2} = (v, \lambda_{1,2} - au). \tag{2.2}$$

It is easy to check that $r_j \cdot \nabla\lambda_j \neq 0$, $j = 1, 2$, for all (u, v) such that $v > 0$, that is, the systems (1.1) are genuinely nonlinear for $v > 0$. Then we normalize $r_{1,2}$ by the condition

$$r_j \cdot \nabla\lambda_j \equiv 1, \;\; j = 1, 2,$$

in $\mathbf{R}^2/\{(0, 0)\}$. Integrating these fields on the plane, we get the *rarefaction curves* of (1.1). See [10].

Associates to the rarefaction curves, we have a pair of *Riemann invariants* w_1, w_2, that is, a pair of real functions on $\mathbf{R}^2$ such that

$$\nabla w_i \cdot r_j = 0, \;\; i \neq j, \, i, j = 1, 2. \tag{2.3}$$

Besides we can choose (w_1, w_2) such that $\nabla w_i \cdot r_i > 0$, $i = 1, 2$. We note that the conditions (2.3) mean that the rarefaction curves are the level curves of w_1 and w_2.

3. Hugoniot loci

Let $U_0 = (u_0, v_0)$ be a point in $\mathbf{R}^2_+ \stackrel{\text{def}}{=} \{(u, v) \in \mathbf{R}^2; v \geq 0\}$. The *Hugoniot locus* $\mathcal{H}(U_0)$ for the systems (1.1) is the set of points $U = (u, v)$ in the plane satisfying the *Rankine–Hugoniot relation* based at U_0, that is, the equations

$$2s(u - u_0) = au^2 + v^2 - au_0^2 - v_0^2, \tag{3.1}$$

$$s(v - v_0) = uv - u_0v_0, \tag{3.2}$$

for some real number $s = s(U_0; U)$, the *shock speed* between the states U and U_0. If $v_0 = 0$ then

$$\begin{aligned} \mathcal{H}(U_0) = \;\; & \{(u, v) \in \mathbf{R}^2\,;\; v = 0\} \\ & \cup\{(u, v) \in \mathbf{R}^2\,;\; (2 - a)u = u_0 \pm \sqrt{(a - 1)^2u_0^2 + (2 - a)v^2}\}. \end{aligned} \tag{3.3}$$

In particular,

$$\mathcal{H}\left((0,0)\right) = \{(u,v)\in \mathbf{R}^2\,;\, v=0\} \cup \{(u,v)\in \mathbf{R}^2\,;\, v = \pm\sqrt{(2-a)}u\}. \tag{3.4}$$

Now let $v_0 > 0$ and $0 \le v \ne v_0$. Canceling s in (3.1)–(3.2), we obtain the quadratic equation in u,

$$(av_0 + (2-a)v)u^2 - 2u_0(v+v_0)u + 2u_0^2 v_0 - (v-v_0)(v^2 - au_0^2 - v_0^2) = 0, \tag{3.5}$$

which has discriminant Δ given by

$$\begin{aligned}\Delta = {} & 4u_0^2(v+v_0)^2 - 4(av_0+(2-a)v)\left[2u_0^2v_0 - (v-v_0)(v^2-au_0^2-v_0^2)\right] \\ = {} & 4\left\{(2-a)v^4 + 2(a-1)v_0v^3 + (a-1)^2u_0^2(v-v_0)^2 \right. \\ & \left. -2v_0^2v^2 - 2(a-1)v_0^3 v + av_0^4\right\}.\end{aligned} \tag{3.6}$$

From (3.6) we can easily check that v_0 is a root of second order of Δ. So we can divide (3.6) by $(v-v_0)^2$ to obtain

$$\Delta = 4(v-v_0)^2\left\{(a-1)^2u_0^2 + (v+v_0)(av_0+(2-a)v)\right\}.$$

Note that $\Delta \ge 0$ since v, $v_0 \ge 0$, and $1 < a < 2$. Then the roots of (3.5) are given by the formula

$$\begin{aligned} u = {} & \frac{1}{av_0+(2-a)v}\left\{u_0(v+v_0)\right. \\ & \left. \pm(v-v_0)\sqrt{(a-1)^2u_0^2 + (v+v_0)(av_0+(2-a)v)}\right\}.\end{aligned} \tag{3.7}$$

The formula (3.7) expresses the Hugoniot locus $\mathcal{H}(U_0)$ in terms of two functions of v. We rewrite it as

$$u - u_0 = \frac{v-v_0}{av_0+(2-a)v}\left\{(a-1)u_0 + (-1)^k\sqrt{}\right\}, \tag{3.7_k}$$

where $k = 1, 2$, and

$$\sqrt{} \stackrel{\text{def}}{=} \sqrt{(a-1)^2u_0^2 + (v+v_0)(av_0+(2-a)v)}. \tag{3.8}$$

We note that in the case $v_0^2 - (2-a)u_0^2 = 0$, we have

$$\sqrt{} = \sqrt{\frac{1}{2-a}\left(v_0+(2-a)v\right)^2} = \frac{1}{\sqrt{2-a}}\left(v_0+(2-a)v\right), \tag{3.9}$$

where we used that $(v_0 + (2-a)v) \geq 0$, so

$$u - u_0 = \frac{v - v_0}{av_0 + (2-a)v}\{(a-1)u_0 + (-1)^k \frac{1}{\sqrt{2-a}}(v_0 + (2-a)v)\}, \quad (3.9)_k$$

$k = 1, 2$. From $(3.9)_k$, if $k = 1$ and $u_0 > 0$ we have

$$u = \frac{(2-a)v_0 v - (2-a)v^2 + 2v_0^2}{\sqrt{2-a}\,(av_0 + (2-a)v)}, \tag{3.10}$$

equivalently,

$$\frac{u - u_0}{v - v_0} = -\sqrt{2-a}\frac{v + v_0}{av_0 + (2-a)v}; \tag{3.11}$$

if $k = 1$ and $u_0 < 0$ we have

$$\frac{u - u_0}{v - v_0} = -\frac{1}{\sqrt{2-a}}; \tag{3.12}$$

if $k = 2$ and $u_0 > 0$ we have

$$\frac{u - u_0}{v - v_0} = \frac{1}{\sqrt{2-a}}; \tag{3.13}$$

and if $k = 2$ and $u_0 < 0$ we have

$$u = \frac{(2-a)v_0 v - (2-a)v^2 + 2(a-1)v_0^2}{\sqrt{2-a}\,(av_0 + (2-a)v)}, \tag{3.14}$$

equivalently,

$$\frac{u - u_0}{v - v_0} = \sqrt{2-a}\frac{v + v_0}{av_0 + (2-a)v}. \tag{3.15}$$

Definition 3.1. *Let $U_0 = (u_0, v_0)$ with $v_0 \geq 0$. We define $\mathcal{T}^1(U_0)$ to be the set of points $U = (u, v) \in \mathbf{R}^2$ satisfying (3.7)$_1$ and $v > v_0$, and $\mathcal{T}^2(U_0)$ to be the set of points $U = (u, v) \in \mathbf{R}^2$ satisfying (3.7)$_2$ and $0 \leq v < v_0$ ($\mathcal{T}^2(U_0) = \emptyset$ if $v_0 = 0$).*

In this section we prove that $\mathcal{T}^k$ is a curve of $k-$shocks in the sense of Lax, that is,

$$\lambda_1(U) < s < \lambda_2(U), \tag{3.16}$$

$$s < \lambda_1(U_0), \tag{3.17}$$

for all $U \in \mathcal{T}^1(U_0)$, and

$$\lambda_2(U) < s, \tag{3.18}$$

$$\lambda_1(U_0) < s < \lambda_2(U_0), \tag{3.19}$$

for all $U \in \mathcal{T}^2(U_0)$, where λ_1 and λ_2 are the eigenvalues of (1.1). We also prove other results that will be useful in the next sections.

The eigenvalues λ_k, $k = 1, 2$, are the roots of the polynomial in λ,

$$\begin{aligned} P(\lambda; U) \stackrel{\text{def}}{=} & \begin{vmatrix} au - \lambda & v \\ v & u - \lambda \end{vmatrix} \\ = & \lambda^2 - [(a+1)u]\lambda + [au^2 - v^2] = (\lambda - \lambda_1)(\lambda - \lambda_2). \end{aligned}$$

Note that (3.16) is equivalent to

$$P(s; U) < 0, \ \ \forall U \in \mathcal{T}^1(U_0), \tag{3.16$'$}$$

and (3.19) is equivalent to

$$P(s; U_0) < 0, \ \ \forall U \in \mathcal{T}^2(U_0). \tag{3.19$'$}$$

Lemma 3.2.

$$P(s; U) < 0, \ \ \forall U \in \mathcal{T}^1(U_0).$$

Proof:

$$\begin{aligned} P(\lambda; U) = \ & \lambda^2 - (a+1)(u - u_0)\lambda - (a+1)u_0\lambda + a(u - u_0)^2 \\ & +2auu_0 - au_0^2 - v^2 \\ = \ & \lambda^2 - (a+1)(u - u_0)\lambda - (a+1)u_0\lambda + a(u - u_0)^2 \\ & +2au_0(u - u_0) + au_0^2 - v^2. \end{aligned} \tag{3.20}$$

From (3.2) we have

$$s = \frac{uv - u_0 v_0}{v - v_0} = v\left(\frac{u - u_0}{v - v_0}\right) + u_0. \tag{3.21}$$

From (3.20) and (3.21) we have

$$\begin{aligned}
P(s;U) = \;& v^2\left(\frac{u-u_0}{v-v_0}\right)^2 + 2u_0v\left(\frac{u-u_0}{v-v_0}\right) + u_0^2 \\
& -(a+1)v(v-v_0)\left(\frac{u-u_0}{v-v_0}\right)^2 - (a+1)u_0(v-v_0)\left(\frac{u-u_0}{v-v_0}\right) \\
& -(a+1)u_0v\left(\frac{u-u_0}{v-v_0}\right) - (a+1)u_0^2 + a(v-v_0)^2\left(\frac{u-u_0}{v-v_0}\right)^2 \\
& +2au_0(v-v_0)\left(\frac{u-u_0}{v-v_0}\right) + au_0^2 - v^2 \\
= \;& \left[v^2 - (a+1)v(v-v_0) + a(v-v_0)^2\right]\left(\frac{u-u_0}{v-v_0}\right)^2 \\
& + [2u_0v - (a+1)u_0(v-v_0) - (a+1)u_0v \\
& +2au_0(v-v_0)]\left(\frac{u-u_0}{v-v_0}\right) - v^2.
\end{aligned}$$

Thus

$$\begin{aligned}
P(s(U;U_0);U) \equiv \;& P(s;U) \\
= \;& v_0(av_0 + (1-a)v)\left(\frac{u-u_0}{v-v_0}\right)^2 \\
& +(1-a)u_0v_0\left(\frac{u-u_0}{v-v_0}\right) - v^2, \quad \forall U,\ U_0.
\end{aligned} \tag{3.22}$$

From $(3.7)_k$ we have

$$\frac{u-u_0}{v-v_0} = \frac{1}{av_0 + (2-a)v}\left\{(a-1)u_0 + (-1)^k\sqrt{\quad}\right\}, \quad \forall\,(u,v)\in\mathcal{T}^k. \qquad (3.23)_k$$

where $\sqrt{\quad}$ is defined in (3.8). So

$$\begin{aligned}
\left(\frac{u-u_0}{v-v_0}\right)^2 = \;& \frac{1}{(av_0+(2-a)v)^2}\Big\{2(a-1)^2u_0^2 \\
& +(v+v_0)(av_0+(2-a)v) + (-1)^k 2(a-1)u_0\sqrt{\quad}\Big\}
\end{aligned} \qquad (3.24)_k$$

for all $(u,v)\in\mathcal{T}^k$. Substituting $(3.23)_1$ and $(3.24)_1$ into (3.22), we obtain

$$P(s;U) = \frac{1}{(av_0+(2-a)v)^2}\{A\sqrt{\quad} + B\},$$

where

$$\begin{aligned}
A \stackrel{\text{def}}{=} \;& -2(a-1)u_0v_0(av_0+(1-a)v) - (1-a)u_0v_0(av_0+(2-a)v) \\
= \;& (a-1)u_0v_0(-2av_0 - 2(1-a)v + av_0 + (2-a)v) \\
= \;& a(a-1)u_0v_0(v-v_0).
\end{aligned}$$

and

$$\begin{aligned}
B \stackrel{\text{def}}{=} & \ 2(a-1)^2u_0^2v_0(av_0+(1-a)v) \\
& +v_0(v+v_0)(av_0+(2-a)v)(av_0+(1-a)v) \\
& -(a-1)^2u_0^2v_0(av_0+(2-a)v)-v^2(av_0+(2-a)v)^2 \\
= & \ (a-1)^2u_0^2v_0(2av_0+2(1-a)v-av_0-(2-a)v) \\
& +(av_0+(2-a)v)(av_0^2v+(1-a)v_0v^2+av_0^3 \\
& +(1-a)v_0^2v-av_0v^2-(2-a)v^3) \\
= & \ -a(a-1)^2u_0^2v_0(v-v_0) \\
& +(av_0+(2-a)v)((a-2)v^3+(1-2a)v_0v^2+v_0^2v+av_0^3) \\
= & \ (v-v_0)\left[-a(a-1)^2u_0^2v_0\right. \\
& \left.+(av_0+(2-a)v)((a-2)v^2-(a+1)v_0v-av_0^2)\right],
\end{aligned}$$

Hence,

$$P(s;U)=\frac{v-v_0}{(av_0+(2-a)v)^2}\{\widetilde{A}\sqrt{}-\widetilde{B}-C\}, \tag{3.25}$$

where

$$\widetilde{A}\stackrel{\text{def}}{=}a(a-1)u_0v_0,\quad \widetilde{B}\stackrel{\text{def}}{=}a(a-1)^2u_0^2v_0,$$

and

$$C\stackrel{\text{def}}{=}(av_0+(2-a)v)((2-a)v^2+(a+1)v_0v+av_0^2)$$

Since $v>v_0$ on $\mathcal{T}^1$, we finish the proof if

$$(\widetilde{A}\sqrt{})^2<\left(\widetilde{B}+C\right)^2$$

(note that $\widetilde{B}>0$ and $C>0$). That is, if

$$\begin{aligned}
& a^2(a-1)^4u_0^4v_0^2+a^2(a-1)^2u_0^2v_0^2(v+v_0)(av_0+(2-a)v) \\
< \ & a^2(a-1)^4u_0^4v_0^2 \\
& +2a(a-1)^2u_0^2v_0(av_0+(2-a)v)((2-a)v^2+(a+1)v_0v+av_0^2) \\
& +(av_0+(2-a)v)^2((2-a)v^2+(a+1)v_0v+av_0^2) \\
\Longleftrightarrow \ & a(a-1)^2u_0^2v_0(-av_0v-av_0^2+2(2-a)v^2+2(a+1)v_0v+2av_0^2) \\
& +(av_0+(2-a)v)((2-a)v^2+(a+1)v_0v+av_0^2)>0 \\
\Longleftrightarrow \ & a(a-1)^2u_0^2v_0(2(2-a)v^2+(2+a)v_0v+av_0^2) \\
& +(av_0+(2-a)v)((2-a)v^2+(a+1)v_0v+av_0^2)>0.
\end{aligned}$$

□

Lemma 3.3.

$$P(s;U_0)<0,\ \ \forall U\in\mathcal{T}^2(U_0).$$

Proof: From the definition of P we have

$$P(s;U_0) = s^2 - [(a+1)u_0]s + [au_0^2 - v_0^2]. \tag{3.26}$$

From (3.21) and (3.26) we have

$$\begin{aligned} P(s(U;U_0);U_0) &\equiv P(s;U_0) \\ &= v^2\left(\frac{u-u_0}{v-v_0}\right)^2 + (1-a)u_0 v\left(\frac{u-u_0}{v-v_0}\right) - v_0^2 \end{aligned} \tag{3.27}$$

for all $U,\ U_0$. Substituting $(3.23)_2$ and $(3.24)_2$ into (3.27), we obtain

$$P(s;U_0) = \frac{1}{(av_0 + (2-a)v)^2}\left\{A + B\sqrt{\quad}\right\},$$

where

$$\begin{aligned} A &\stackrel{\text{def}}{=} 2(a-1)^2u_0^2v_0^2 + (v+v_0)(av_0+(2-a)v)v^2 \\ &\quad -(a-1)^2u_0^2v(av_0+(2-a)v) - v_0^2(av_0+(2-a)v)^2 \\ &= (a-1)^2u_0^2v(2v - av_0 - (2-a)v) \\ &\quad +(av_0+(2-a)v)(v^3 + v_0v^2 - av_0^3 - (2-a)v_0^2v) \\ &= a(a-1)^2u_0^2v(v-v_0) + (av_0+(2-a)v)(v^2+2v_0v+av_0^2)(v-v_0) \\ &= (v-v_0)[(a-1)^2u_0^2v + (av_0+(2-a)v)(v^2+2v_0v+av_0^2)], \end{aligned}$$

and

$$\begin{aligned} B &\stackrel{\text{def}}{=} 2(a-1)u_0v^2 + (1-a)u_0v(av_0+(2-a)v) \\ &= (a-1)u_0v(2v - av_0 - (2-a)v) \\ &= a(a-1)u_0v(v-v_0). \end{aligned}$$

Hence,

$$P(s;U_0) = \frac{v-v_0}{(av_0+(2-a)v)^2}\left\{\widetilde{A}\sqrt{\quad} + \widetilde{B} + C\right\}, \tag{3.28}$$

where

$$\widetilde{A} \stackrel{\text{def}}{=} a(a-1)u_0, \quad \widetilde{B} \stackrel{\text{def}}{=} a(a-1)^2u^2v,$$

and

$$C \stackrel{\text{def}}{=} (av_0 + (2-a)v)(v^2 + 2v_0v + av_0^2).$$

Since $v < v_0$ on $\mathcal{T}^2$, we finish the proof if

$$\left(\widetilde{A}\sqrt{\quad}\right)^2 < \left(\widetilde{B} + C\right)^2$$

(note that $\widetilde{B} > 0$ and $C > 0$). That is, if

$$\begin{array}{rl} & a^2(a-1)^4u_0^4v^2 + a^2(a-1)^2u_0^2v^2(v+v_0)(av_0+(2-a)v) \\ < & a^2(a-1)^4u_0^4v^2 + 2a(a-1)^2u_0^2v(av_0+(2-a)v)(v^2+2v_0v+av_0^2) \\ & +(av_0+(2-a)v)^2(v^2+2v_0v+av_0^2)^2 \\ \Longleftrightarrow & 0 < a(a-1)^2u_0^2v(-av^2-av_0v+2v^2+4v_0v+2av_0^2) \\ & \quad +(av_0+(2-a)v)(v^2+2v_0v+av_0^2)^2 \\ \Longleftrightarrow & 0 < a(a-1)^2u_0^2v((2-a)v^2+(4-a)v_0v+av_0^2) \\ & \quad +(av_0+(2-a)v)(v^2+2v_0v+av_0^2)^2. \end{array}$$

□

Lemmas 3.2 and 3.3 prove (3.16) and (3.19). Next we deduce some additional properties of the "Lax curves" $\mathcal{T}^1$ and $\mathcal{T}^2$, from which we can infer in particular (3.17) and (3.18).

Lemma 3.4. *For fixed $k \in \{1, 2\}$, we have*

$$s(U;U_0) = \lambda_k\left(\frac{U+U_0}{2}\right) \quad \forall U \in \mathcal{T}^k(U_0).$$

Proof: We will prove the case $k = 1$. Similarly we can prove the case $k = 2$. In [9] is proved that

$$s = \lambda_p\left(\frac{U+U_0}{2}\right), \quad \forall U \in \mathcal{H}(U_0)$$

where either $p = 1$ or 2. So we have to prove that $p = 1$ if $U \in \mathcal{T}^1$. First we note that the case $U_0 = (0,0)$ can be easily verified by using the formulas for s and λ_k in this case. Thus we assume $U_0 \neq (0,0)$. Then we have

$$\lim_{\substack{U \to U_0 \\ U \in \mathcal{T}^1(U_0)}} s(U;U_0) = \lambda_1(U_0) \tag{3.29}$$

[15], since the system is strictly hyperbolic at $U_0 \neq (0,0)$. Now we suppose that $p = 2$ for some $U \in \mathcal{T}^1(U_0)$, and we shall get a contradiction. Let $U_* = (u_*, v_*)$ be the first point in $\mathcal{T}^1$ where the condition $p = 1$ fails. If $U_* = U_0$ then, by (3.29) and continuity, we have $\lambda_1(U_0) = \lambda_2(U_0)$, which is contradiction because $U_0 \neq (0,0)$. If $U_* \neq U_0$, by continuity, we have

$$\lambda_1\left(\frac{U_*+U_0}{2}\right) = \lambda_2\left(\frac{U_*+U_0}{2}\right),$$

which is also a contradiction, because $\frac{U_*+U_0}{2} \neq (0,0)$, since $U_*,\ U_0 \in \mathbf{R}^2_+$ and $U_0 \neq (0,0)$.

$\square$

Lemma 3.5.
(a) $\dfrac{du}{dv} < 0$ on $\mathcal{T}^1$;
(b) $\dfrac{du}{dv} > 0$ on $\mathcal{T}^2$.

Proof: From $(3.7)_k$ we have

$$\begin{aligned}
\frac{du}{dv} &= \frac{av_0+(2-a)v-(2-a)(v-v_0)}{(av_0+(2-a)v)^2}\left\{(a-1)u_0+(-1)^k\sqrt{}\right\}\\
&\qquad + \frac{v-v_0}{av_0+(2-a)v}\left\{(-1)^k\frac{(2-a)v+v_0}{\sqrt{}}\right\}\\
&= \frac{1}{(av_0+(2-a)v)^2\sqrt{}}\Big\{2(a-1)u_0v_0\sqrt{}\\
&\qquad + (-1)^k\Big[2(a-1)^2u_0^2v_0+2v_0(v+v_0)(av_0+(2-a)v)\\
&\qquad +(av_0+(2-a)v)(v-v_0)(v_0+(2-a)v)\Big]\Big\}\\
&= \frac{A_k}{(av_0+(2-a)v)^2\sqrt{}},
\end{aligned}$$

where

$$\begin{aligned}
A_k \stackrel{\text{def}}{=} \ & 2(a-1)u_0v_0\sqrt{}+(-1)^k2(a-1)^2u_0^2v_0\\
&+(-1)^k(av_0+(2-a)v)((2-a)v^2+(a+1)v_0v+v_0^2),
\end{aligned}$$

$k=1,2$. Then $(-1)^k\frac{du}{dv}>0$ if and only if

$$(-1)^kA_k>0. \tag{3.30}$$

Since $v\geq 0$, and $1<a<2$, (3.30) is equivalent to

$$\begin{aligned}
(-1)^{k+1}2(a-1)u_0v_0\sqrt{} \ < \ & (av_0+(2-a)v)((2-a)v^2+(a+1)v_0v+v_0^2)\\
&+2(a-1)^2u_0^2v_0;
\end{aligned}$$

so, squaring and using the definition of $\sqrt{}$, we get that $(-1)^kA_k>0$ if

$$\begin{aligned}
& 4(a-1)^4u_0^4v_0^2+4(a-1)^2u_0^2v_0^2(v+v_0)(av_0+(2-a)v)\\
< \ & 4(a-1)^4u_0^4v_0^2\\
& +4(a-1)^2u_0^2v_0(av_0+(2-a)v)((2-a)v^2+(a+1)v_0v+av_0^2)\\
& +(av_0+(2-a)v)^2((2-a)v^2+(a+1)v_0v+av_0^2)^2\\
\iff \ & 0<4(a-1)^2u_0^2v_0((2-a)v^2+av_0v)\\
& \qquad +(av_0+(2-a)v)((2-a)v^2+(a+1)v_0v+av_0^2)^2\\
\iff \ & 0<4(a-1)^2u_0^2v_0v+((2-a)v^2+(a+1)v_0v+v_0^2)^2.
\end{aligned}$$

$\square$

For future reference, we write down the formula for $\dfrac{du}{dv}$, computed in the proof above:

$$\begin{aligned}\frac{du}{dv} = & \frac{1}{(av_0+(2-a)v)^2\sqrt{\ }}\Big\{2(a-1)u_0v_0\sqrt{\ } + (-1)^k 2(a-1)^2u_0^2v_0 \\ & +(-1)^k(av_0+(2-a)v)((2-a)v^2+(a+1)v_0v+v_0^2)\Big\},\end{aligned} \tag{3.31}$$

where $\sqrt{\ }$ is defined in (3.8).

In the proof above, we got the following formula for $\dfrac{ds}{dv}$:

$$\begin{aligned}\frac{ds}{dv} = & \frac{1}{(av_0+(2-a)v)^2\sqrt{\ }}\{a(a-1)u_0v_0\sqrt{\ } + (-1)^k\Big[a(a-1)^2u_0^2v_0 \\ & +(av_0+(2-a)v)((2-a)v^2+(a+1)v_0v+av_0^2)\Big]\}.\end{aligned} \tag{3.32}$$

Lemma 3.6. *Let $U_0=(u_0,v_0)$ such that $v_0>0$ and $v_0\neq -\sqrt{2-a}u_0$ (respect. $v_0\neq \sqrt{2-a}u_0$). For any $U\in\mathcal{T}^1(U_0)$ (respect. $\mathcal{T}^2_-(U_0)$), $U-U_0$ is not parallel to $r_1(U)$ (respect. $r_2(U)$).*

Proof: We will prove only that $U-U_0$ is not parallel to $r_1(U)$, since the proof of the rest is similar. Suppose that $U-U_0$ is parallel to $r_1(U_0)$ at some point $U=(u,v)\in U\in\mathcal{T}^1(U_0)$, and we shall get a contradiction. Since $r_1(U)$ is perpendicular to $r_2(U)$ because the Jacobian matrix of the system (1.1) is symmetric, and

$$r_2(U)=\left(v,\frac{1}{2}\{(1-a)u+\sqrt{(a-1)^2u^2+4v^2}\}\right)$$

(see (2.2)) then we have

$$(U-U_0)\cdot r_2(U)=0$$

$$\Longleftrightarrow (u-u_0,v-v_0)\cdot\left(v,\frac{1}{2}\{(1-a)u+\sqrt{(a-1)^2+4v^2}\}\right)=0$$

$$\Longleftrightarrow 2v\frac{u-u_0}{v-v_0}+(1-a)u=-\sqrt{(a-1)^2+4v^2}$$

$$\iff 4v^2\left(\frac{u-u_0}{v-v_0}\right)^2+4(1-a)uv\left(\frac{u-u_0}{v-v_0}\right)+(1-a)^2u^2=(1-a)^2u^2+4v^2$$
$$\iff v\left(\frac{u-u_0}{v-v_0}\right)^2+(1-a)u\left(\frac{u-u_0}{v-v_0}\right)-v=0;$$

since

$$u=\left(\frac{u-u_0}{v-v_0}\right)(v-v_0)+u_0$$

from the last equation we have

$$(v+(1-a)(v-v_0))\left(\frac{u-u_0}{v-v_0}\right)^2+(1-a)u_0\left(\frac{u-u_0}{v-v_0}\right)-v=0. \tag{3.38}$$

Now we take $\left(\frac{u-u_0}{v-v_0}\right)$ from $(3.7)_1$ and substitute into (3.38) to get

$$\begin{array}{l}(v+(1-a)(v-v_0))\,(2(a-1)^2u_0^2\\ +(v+v_0)\,(av_0+(2-a)v)-2(a-1)u_0\sqrt{\quad}\,)\\ +(1-a)u_0\,(av_0+(2-a)v)\left((a-1)u_0-\sqrt{\quad}\right)\\ -v\,(av_0+(2-a)v)^2=0,\end{array} \tag{3.34}$$

that is,

$$\begin{array}{l}(a-1)^2u_0^2(2-a)(v-v_0)+(a-1)v_0(v_0-v)\,(av_0+(2-a)v)\\ -(a-1)u_0(2-a)(v-v_0)\sqrt{\quad}=0\end{array}$$

so

$$(a-1)u_0^2(2-a)+v_0\,(av_0+(2-a)v)=u_0(2-a)\sqrt{\quad};$$

squaring we obtain

$$\begin{array}{rl} & (2-a)^2(a-1)^2u_0^4-2(2-a)(a-1)u_0^2v_0\,(av_0+(2-a)v)\\ & +v_0^2\,(av_0+(2-a)v)^2\\ = & (2-a)^2u_0^2\left((a-1)^2u_0^2+(v+v_0)\,(av_0+(2-a)v)\right)\end{array}$$

$$\iff \quad -2(2-a)(a-1)u_0^2v_0+v_0^2\,(av_0+(2-a)v)=(2-a)^2u_0^2(v+v_0)$$
$$\iff -(2-a)u_0^2v_0\,(2(a-1)+(2-a))+(2-a)\left(v_0^2-(2-a)u_0^2\right)v+av_0^3=0$$
$$\iff a(a-2)u_0^2v_0+(2-a)\left(v_0^2-(2-a)u_0^2\right)v+av_0^3=0$$
$$\iff ((2-a)v+av_0)\left(v_0^2-(2-a)u_0^2\right)=0. \tag{3.35}$$

From (3.35) we have $v_0^2-(2-a)u_0^2=0$ or $v=-\left(a/(2-a)\right)v_0<0$. This last is a contradiction because $\mathcal{T}^1\subset\mathbf{R}_+^2$ so $v\geq 0$. Therefore, to end the proof, it just remains to consider that $v_0^2 - (2-a)u_0^2 = 0$ and also get a contradiction. In this case we have $u_0 > 0$ since, by hypothesis, $v_0 \neq -\sqrt{2-a}u_0$ and $v_0 > 0$. Substituting (3.11) into (3.33) we have

$$\begin{aligned} &(v + (1-a)(v - v_0))\,(2-a)\frac{(v+v_0)^2}{(av_0 + (2-a)v)^2} \\ &-(1-a)u_0\sqrt{2-a}\frac{v+v_0}{av_0+(2-a)v} - v = 0, \end{aligned}$$

which gives after simplifications

$$v_0\{(a-2)v^2 + (1-a)v_0 v + v_0^2\} = 0;$$

since $v_0 > 0$, we obtain

$$v = \frac{1}{a-2} < 0 \quad \text{ or } \quad v = v_0,$$

which yields the desired contradiction.

□

The next lemma shows the behavior of the shock speed $s = s(U; U_0)$ on the $\mathcal{T}^k = \mathcal{T}^k(U_0)$ curve, where $U_0 = (u_0, v_0)$.

Lemma 3.7.

(a) $\dfrac{ds}{dv} < 0$ on $\mathcal{T}^1$;
(b) $\dfrac{ds}{dv} > 0$ on $\mathcal{T}^2$.

Proof: If $v_0^2 = (2-a)u_0^2$, the result follows easily from (3.11), (3.12), (3.13), (3.15), and (3.21). Otherwise, we combine Lemma 3.6 with the Bethe-Wendroff theorem [6, 17] or *tangency rule* [9] to get the result. Here we used that at the starting point v_0 we have $ds/dv < 0$ if $k = 1$, and $ds/dv > 0$ if $k = 2$; see Corollary 17.13 of [15].

□

Corollary 3.8.

(a) $s(U, U_0) < \lambda_1(U_0), \;\; \forall\, U \in \mathcal{T}^1(U_0)$;
(b) $s(U, U_0) > \lambda_1(U_0), \;\; \forall\, U \in \mathcal{T}^2(U_0)$.

Proof: If $U_0 = (0,0)$, the proof is straightforward. Consider $U_0 \neq (0,0)$. Since the system (1.1) is strictly hyperbolic in a neighborhood of U_0, Lax's theory imply the result near U_0. Use Lemma 3.7 to complete the proof.

$\square$

Lemma 3.9.

(a) $\lambda_k(\alpha U) = \alpha\lambda_k(U)$, $k = 1, 2$, $\forall U,\ U_0 \in \mathbf{R}^2$, $\forall \alpha \geq 0$;

(b) $\lambda_k\left(\frac{U+U_0}{2}\right) > \frac{1}{2}[\lambda_i(U) + \lambda_j(U_0)]$, *either* $i = j = k = 1$ *or* $k = 2$ *and* $(i,j) \neq (2,2)$, $\forall U,\ U_0$ *such that* $u_0 v \neq u v_0$;

(c) $\lambda_k\left(\frac{U+U_0}{2}\right) < \frac{1}{2}(\lambda_i(U) + \lambda_j(U_0))$, *either* $i = j = k = 2$ *or* $k = 1$ *and* $(i,j) \neq (1,1)$, $\forall U,\ U_0$ *such that* $u_0 v \neq u v_0$

If $u_0 v = u v_0$, *we have equalities in (b) and (c).*

Proof: The eigeinvalues λ_k, $k = 1, 2$, are given by

$$\lambda_k(U) = \frac{1}{2}\left\{(a+1)u + (-1)^k\sqrt{(a-1)^2u^2 + 4v^2}\right\} \tag{3.36}$$

(see (2.1)) which yields (a) trivially. To prove the remainder, note that

$$\lambda_k\left(\frac{U+U_0}{2}\right) = \frac{1}{2}[\lambda_i(U) + \lambda_j(U_0)] - \frac{1}{4}A, \quad i,j,k = 1,2,$$

where

$$\begin{aligned} A \stackrel{\text{def}}{=} & \ (-1)^i\sqrt{(a-1)^2u^2+4v^2} + (-1)^j\sqrt{(a-1)^2u_0^2+4v_0^2} \\ & -(-1)^k\sqrt{(a-1)^2(u+u_0)^2+4(v+v_0)^2} \\ \equiv & \ (-1)^iB + (-1)^jB_l - (-1)^kC \qquad (B,\ B_l,\ C \geq 0). \end{aligned}$$

First we verify that $A = 0$ if and only if $u_0 v = u v_0$, $\forall i,j,k = 1, 2$. We have

$$\begin{aligned} C^2 = & \ (a-1)^2(u+u_0)^2 + 4(v+v_0)^2 \\ = & \ B^2 + B_l^2 + 2(a-1)^2uu_0 + 8vv_0 \end{aligned}$$

and

$$(B \pm B_l)^2 = B^2 + B_l^2 \pm 2BB_l,$$

so

$$\begin{aligned} A = 0 \iff & C^2 = (B \pm B_l)^2 \\ \iff & 2(a-1)^2uu_0 + 8vv_0 = \pm 2\sqrt{(a-1)^2u^2+4v^2}\sqrt{(a-1)^2u_0^2+4v_0^2} \\ \iff & (a-1)^4(uu_0)^2 + 16(vv_0)^2 + 8(a-1)^2uu_0vv_0 \\ & = (a-1)^4(uu_0)^2 + 16(vv_0)^2 + 4(a-1)^2\left[(uv_0)^2 + (vu_0)^2\right] \\ \iff & (uv_0 - vu_0)^2 = 0. \end{aligned}$$

Next, assuming $u_0 v \neq u v_0$, we prove the case (b) for $i = j = k = 1$. In this case, we have $A < 0$ if and only if

$$0 > C - B - B_l \iff C^2 < (B + B_L)^2$$

$$\iff (a-1)^2 u u_0 + 4 v v_0 < B B_l. \tag{3.36}$$

Since $BB_l \geq 0$, squaring both sides of (3.36) we have $A < 0$ if

$$\begin{aligned}(a-1)^4 (uu_0)^2 &+ 8(a-1) uu_0 vv_0 + 16(vv_0)^2 \\ &< (a-1)^4 (uu_0)^2 + 16(vv_0)^2 + 4(a-1)^2 \left[(uv_0)^2 + (vu_0)^2\right]\end{aligned}$$

which is equivalent to

$$0 < (uv_0 - vu_0)^2.$$

Then $A < 0$ and this proves *(b)* for $i = j = k = 1$. The proof of (c) for $i = j = k = 2$ is the same (replace A by $-A$). The proof of the other cases is similar.

□

Lemma 3.10.
(a) $\dfrac{d\lambda_1}{dv} < 0$ on $\mathcal{T}^1$;
(b) $\dfrac{d\lambda_2}{dv} > 0$ on $\mathcal{T}^2$.

Proof: From (3.36) we have

$$2\frac{d\lambda_k}{dv} = \frac{1}{\rho}\left\{A\frac{du}{dv} + (-1)^k 4v\right\}$$

where

$$\rho \stackrel{\text{def}}{=} \sqrt{(a-1)^2 u^2 + 4v^2}$$

and

$$A \stackrel{\text{def}}{=} (a+1)\rho + (-1)^k (a-1)^2 u.$$

Note that $A > 0$ because

$$\begin{aligned}\left[(-1)^k (a-1)^2 u\right]^2 &< (a+1)\rho]^2, \\ (a-1)^4 u^2 &< (a+1)^2 (a-1)^2 u^2 + 4(a+1)^2 v^2, \\ 0 &< 4a(a-1)^2 u^2 4(a+1)^2 v^2.\end{aligned}$$

Since $(-1)^k \dfrac{du}{dv} > 0$ on $\mathcal{T}^k$ (see Lemma 3.5), the proof is completed.

□

Corollary 3.11. $\lambda_k(U) < \lambda_k(U_0)$, $k = 1, 2$, $\forall\, U \in \mathcal{T}^k(U_0)$.

Proof: We have $v > v_0$ on $\mathcal{T}^1$ and $v < v_0$ on $\mathcal{T}^2$. Then, Lemma 3.10 yields the result.

□

Remark 3.12. Lemmas 3.4 and 3.9, and Corollary 3.11, give another proof of half of (3.16), and (3.19). Indeed, applying these results we have

$$s = \lambda_1\left(\frac{U + U_0}{2}\right) > \frac{1}{2}\left[\lambda_1(U) + \lambda_1(U_0)\right] > \lambda_1(U)$$

on $\mathcal{T}^1$, and

$$s = \lambda_2\left(\frac{U + U_0}{2}\right) < \frac{1}{2}\left[\lambda_2(U) + \lambda_2(U_0)\right] < \lambda_2(U_0)$$

on $\mathcal{T}^2$.

It remains to prove the inequality (3.18).

Lemma 3.13.

$$P(s; U) > 0, \ \ \forall\, U \in \mathcal{T}^2(U_0).$$

Proof: A review at the proof of Lemma 3.2 shows that

$$P(s; U) = \frac{v_0 - v}{(av_0 + (2 - a)v)^2}\{\widetilde{A}\sqrt{} + \widetilde{B} + C\},$$

Here $\widetilde{A}$, $\widetilde{B}$, and C are the same as in formula (3.25). Since $\widetilde{A}$, $\widetilde{B}$, and $C > 0$, and $0 \leq v < v_0$ on $\mathcal{T}^2$, the proof ends.

□

Corollary 3.14.

$$\lambda_2(U) < s, \quad \forall\, U \in \mathcal{T}^2(U_0).$$

Proof: Lemma 3.4 yields

$$s = \lambda_2\left(\frac{U+U_0}{2}\right), \quad \forall U \in \mathcal{T}^2(U_0).$$

From this and (2.1) we obtain

$$s = \frac{1}{4}\{(a+1)(u+u_0)+C\} \text{ and } \lambda_1(U) = \frac{1}{2}\{(a+1)u - B\}$$

where

$$C = \sqrt{(a-1)^2(u+u_0)^2+4(v+v_0)^2} \text{ and } B = \sqrt{(a-1)^2u^2+4v^2}.$$

Then

$$s > \lambda_1(U) \iff \frac{a+1}{2}(u_0-u)+\frac{1}{2}C+B>0. \tag{3.37}$$

From $(3.7)_2$ we have that $u_0 - u > 0$ on $\mathcal{T}^2$, so (3.37) implies that $s > \lambda_1(U)$ on $\mathcal{T}^2$. Now apply Lemma 3.12.

□

It will be useful to define the *reverse curves* $\mathcal{T}_-^k(U_0)$, $k = 1, 2$:

Definition 3.15.

$$\mathcal{T}_-^1(U_0) = \{(u,v) \in \mathbf{R}_+^2\,;\ (3.7)_1 \textit{ is satisfied and } 0 \le v < v_0\,\}$$

$$\mathcal{T}_-^2(U_0) = \{(u,v) \in \mathbf{R}_+^2\,;\ (3.7)_2 \textit{ is satisfied and } v > v_0\}.$$

Lemma 3.16. *For any $U_R = (u_R, v_R)$ such that $v_R > 0$, $\mathcal{T}_-^k(U_R)$, $k = 1, 2$, is a curve of Lax k–shocks.*

Proof: By the definition of $\mathcal{T}_-^1(U_R)$, if $U_L = (u_L,\, v_L) \in \mathcal{T}_-^1(U_R)$ then $0 \le v_L < v_R$ and

$$\frac{u_L - u_R}{v_L - v_R} < 0. \tag{3.38}$$

But $\mathcal{T}_-^1(U_R) \subset \mathcal{H}(U_R)$ and $U_L \in \mathcal{H}$ iff $U_R \in \mathcal{H}$. Since $v_0 > 0$, it follows that U_R satisfies $(3.7)_k$ with $U_0 = U_R$, where either $k = 1$ or $k = 2$. From (3.38) we have $k = 1$, so $U_R \in \mathcal{T}^1(U_L)$, which proves that U_L is connected to U_R by an 1–shock. Similarly we prove that $\mathcal{T}_-^2(U_R)$ is a curve of 2–shocks.

□

4. Boundary condition, Riemann problems, and Godunov scheme

The first step in defining the Godunov scheme is to solve the Riemann problem, and the *lateral* Riemann problem. The Riemann problem for (1.1) is solved in [10]. We solve below the lateral Riemann problem for (1.1) in the upper half plane $\{v \geq\}$ with the boundary condition (1.3).

We recall the construction of the solution of the Riemann problem for (1.1) in the upper half plane. It consists of *rarefaction waves, Lax shock waves* and *compressive shock waves.* Let $U_0 \in \mathbf{R}^2_+$. We denote by $\mathcal{R}^1(U_0)$, $\mathcal{S}^1(U_0)$, and $\mathcal{C}(U_0)$, the *1–rarefaction wave curve* of U_0, the *Lax 1–shock curve* of U_0, and the *compressive shock curve* of U_0, respectively. We also denote by $\mathcal{R}^2_-(U_0)$, $\mathcal{S}^2_-(U_0)$, and $\mathcal{C}_-(U_0)$, the *reverse 2–rarefaction shock curve* of U_0, the *reverse Lax 2–shock curve* of U_0, and the *reverse compressive shock curve* of U_0, respectively. These curves are defined by the following conditions, where U denote an arbitrary point in $\mathbf{R}^2_+$:

$$U \in \mathcal{R}^1(U_0) \text{ iff } w_2(U) = w_2(U_0) \text{ and } \lambda_1(U) > \lambda_1(U_0),$$

$$U \in \mathcal{S}^1(U_0) \text{ iff (3.16)–(3.17) is satisfied,}$$

$$U \in \mathcal{C}(U_0) \text{ iff } \lambda_2(U) \leq s(U_0, U) \leq \lambda_1(U_0);$$

$$U \in \mathcal{R}^2_-(U_0) \text{ iff } w_1(U) = w_1(U_0) \text{ and } \lambda_2(U) < \lambda_2(U_0),$$

$$U \in \mathcal{S}^2_-(U_0) \text{ iff (3.18)–(3.19) is satisfied,}$$

$$U \in \mathcal{C}_-(U_0) \text{ iff } \lambda_2(U_0) \leq s(U, U_0) \leq \lambda_1(U).$$

We proved in section 3 that if $U_0 = (u_0, v_0)$ and $v_0 > 0$, then $\mathcal{T}^1(U_0) = \mathcal{S}^1(U_0)$ and $\mathcal{T}^2_-(U_0) = \mathcal{S}^2_-(U_0)$. The curves $w_1 = w_1(U_0)$ and $w_2 = w_2(U_0)$ are shown in Figure 1 of [10]. Besides we can easily compute all the curves above in $\mathbf{R}^2_+$ for the case $U_0 = (u_0, v_0)$ with $v_0 = 0$. In conclusion we have the following.

Case $v_0 > 0$:

$$\mathcal{R}^1(U_0) = \{\, U = (u, v) \in \mathbf{R}^2_+\,;\, w_1(U) = w_1(U_0) \text{ and } 0 \leq v \leq v_0\},$$

$$\mathcal{R}_-^2(U_0) = \{\ U = (u,\, v) \in \mathbf{R}_+^2\,;\ w_2(U) = w_2(U_0) \text{ and } 0 \le v \le v_0\};$$
$$\mathcal{S}^1(U_0) = \mathcal{T}^1(U_0), \quad \mathcal{S}_-^2(U_0) = \mathcal{T}_-^2(U_0),$$
$$\mathcal{C}(U_0) = \mathcal{C}_-(U_0) = \emptyset;$$

Case $v_0 = 0$ and $u_0 > 0$:

$$\mathcal{R}^1(U_0) = \emptyset,$$
$$\mathcal{R}_-^2(U_0) = \{\ U = (u,\, v) \in \mathbf{R}_+^2\,;\ v = 0 \text{ and } 0 \le u \le u_0\},$$
$$\mathcal{S}^1(U_0) = \mathcal{T}^1(U_0),$$
$$\begin{aligned} \mathcal{S}_-^2(U_0) = \ & \mathcal{T}_-^2\left((\tfrac{a}{2-a}u_0,\, 0)\right) \\ & \cup\{\ U = (u,\, v) \in \mathbf{R}_+^2\,;\ v = 0 \text{ and } u_0 < u < \tfrac{a}{2-a}u_0\}, \end{aligned}$$
$$\mathcal{C}\left((u_0,\, 0)\right) = \{(u, v) \in \mathbf{R}^2\,;\ v = 0 \text{ and } u \le \frac{2-a}{a}u_0\},$$
$$\mathcal{C}_-\left((u_0,\, 0)\right) = \{(u, v) \in \mathbf{R}^2\,;\ v = 0 \text{ and } u \ge \frac{a}{2-a}u_0\};$$

Case $v_0 = 0$ and $u_0 < 0$:

$$\mathcal{R}^1(U_0) = \{(u,\, v) \in \mathbf{R}^2\,;\ v = 0 \text{ and } u_0 \le u \le 0\},$$
$$\mathcal{R}_-^2(U_0) = \emptyset,$$
$$\begin{aligned} \mathcal{S}^1(U_0) = \ & \mathcal{T}^1\left((\tfrac{a}{2-a})\right) \\ & \cup\{\ U = (u,\, v) \in \mathbf{R}_+^2\,;\ v = 0 \text{ and } \tfrac{a}{2-a}u_0 < u < u_0\}, \end{aligned}$$
$$\mathcal{S}_-^2(U_0) = \mathcal{T}_-^2(U_0),$$
$$\mathcal{C}\left((u_0,\, 0)\right) = \{(u, v) \in \mathbf{R}^2\,;\ v = 0 \text{ and } u \le \frac{a}{2-a}u_0\},$$
$$\mathcal{C}_-\left((u_0,\, 0)\right) = \{(u, v) \in \mathbf{R}^2\,;\ v = 0 \text{ and } u \ge \frac{2-a}{a}u_0\};$$

Case $v_0 = 0$ and $u_0 = 0$:

$$\mathcal{R}^1(U_0) = \mathcal{R}_-^2(U_0) = \emptyset,$$
$$\mathcal{S}^1(U_0) = \{(u,\, v) \in \mathbf{R}_+^2\,;\ v = -\sqrt{2-a}u\},$$

$$\mathcal{S}^2_-(U_0) = \{(u,\, v) \in \mathbf{R}^2_+\, ;\, v = \sqrt{2-au}\},$$

$$\mathcal{C}\,((0,\,0)) = \{(u,v) \in \mathbf{R}^2\, ;\, v = 0 \text{ and } u \le 0\},$$

$$\mathcal{C}_-\,((0,\,0)) = \{(u,v) \in \mathbf{R}^2\, ;\, v = 0 \text{ and } u \ge 0\}.$$

Proposition 4.1 (Solution of the Riemann problem) [10]. *Let*

$$\mathcal{W}^1(U_0) \stackrel{\text{def}}{=} \mathcal{R}^1(U_0) \cup \mathcal{S}^1(U_0) \cup \mathcal{C}(U_0)$$

(the 1–wave curve *of* U_0*) and*

$$\mathcal{W}^2_-(U_0) \stackrel{\text{def}}{=} \mathcal{R}^2_-(U_0) \cup \mathcal{S}^2_-(U_0) \cup \mathcal{C}_-(U_0)$$

(the reverse 2–wave curve *of* U_0*). Then for any* U_L, $U_R \in \mathbf{R}^2_+$, *there is a unique solution* $U(x,t) = U(x/t)$ *of (1.1) with initial data*

$$U(x,0) = \begin{cases} U_L, & x < 0 \\ U_R, & x > 0 \end{cases}$$

such that $U(x/t) \in \mathcal{W}^1(U_L)$ *for* $x/t < \gamma$ *and* $U(x/t) \in \mathcal{W}^2_-(U_R)$ *for* $x/t > \gamma$, *where* $-\infty \le \gamma \le \infty$ *is uniquely determined as a function of* U_L *and* U_R. *We denote this solution* U *by* $\Re(U_L, U_R)$.

Proposition 4.2 (Lateral Riemann problem). *For any* $U_R = (u_R, v_R) \in \mathbf{R}^2_+$ *there is a unique* $U_B = (u_B, v_B) \in \mathcal{R}^2_-(U_R) \cup \mathcal{S}^2_-(U_R)$ *such that* $\sqrt{a}u_B - v_B = 0$ *and the Riemann solution* $U = \Re(U_B, U_R)$ *satisfies* $U(0,t) = U_B$ *for all* $t > 0$, *that is,* U *satisfies the boundary condition (1.3) for all* $t > 0$.

Proof: The line $\sqrt{a}u - v = 0$ crosses any curve $\mathcal{R}^2_-(U_R) \cup \mathcal{S}^2_-(U_R)$ at exactly one point U_B. If $U_B \in \mathcal{R}^2_-(U_R)$ then the Riemann solution $U = \Re(U_B, U_R)$ is a 2–rarefaction wave. Since $\lambda_2(U_B) > 0$ we have $U(0,t) = U_B$ for all $t > 0$. On the other hand, if $U_B \in \mathcal{S}^2_-(U_R)$ then the Riemann solution $U = \Re(U_B, U_R)$ is a 2–shock wave with shock speed $s > \lambda_1(U_B) = 0$. Thus we have $U(0,t) = U_B$ for all $t > 0$.

□

To define the approximate solution and obtain a L^∞ *a priori* estimate, we prove that any compact convex region in $\mathbf{R}^2_+$ bounded by rarefaction curves is an invariant region of the Riemann problem for (1.1).

Lemma 4.3. *For arbitrary* $u_1 < 0,\ u_2 > 0$, *let* $S = S(u_1; u_2)$ *be the closed region defined by*

$$S \stackrel{\text{def}}{=} \{U \in \mathbf{R}^2_+\,;\ w_1(U) \geq w_1((u_1, 0)) \text{ and } w_2(U) \leq w_2((u_2, 0))\}.$$

Then S *is an invariant region of the Riemann problem for (1.1), that is, if* $U_L, U_R \in S$ *then* $\Re(U_L, U_R)(x, t) \in S$ *for all* $x, t \geq 0$.

To prove the above Lemma we will use the following Proposition.

Proposition 4.4. *Consider a* $n \times n$ *system of conservation laws*

$$U_t + F(U)_x = 0,\ \ x \in \mathbf{R},\ t > 0,\ \ U = (u, v) \in \mathbf{R}^2, \tag{4.1}$$

that is strictly hyperbolic and genuinely nonlinear in a neighborhood of a point $U_0 \in \mathbf{R}^2$. *Assume that* $dF(U_0)$ *is a symmetric matrix. Then we have the following formula for the third derivative at* U_0 *of the shock curve based at* U_0*:*

$$(\lambda_k - \lambda_j)\left\langle \dddot{U} - \ddot{r}_k,\, r_j \right\rangle \;=\; \dot{s}_k \left\langle (dr_k) r_k,\, r_j \right\rangle, \tag{4.2}$$

$j \neq k,\ j, k = 1, 2$. *Here* $\langle\, ,\, \rangle$ *denotes the usual inner product in* $\mathbf{R}^n$, $\lambda_1 < \lambda_2$ *are the eigenvalues of* dF, r_1, r_2 *are the respective eigenvectors normalized by* $\nabla\lambda_j \cdot r_j = 1,\ j = 1, 2$, *a dot over a letter means a derivative along the* k*–shock curve* $U = U(\epsilon)$ *based at* U_0 *with respect to a parameter* ϵ *such that* $U(0) = U_0,\ \dot{U}(0) = r_k$ *and* $\ddot{U}(0) = \dot{r}_k$ (cf. p.330 of [15]).

Proof: As in the proof of Corollary 17.13 in [15], we obtain the following equations evaluated at U_0:

$$3\ddot{s}_k\dot{U} + 3\dot{s}_k\ddot{U} + s_k\,\dddot{U} = \ddot{dF} + 2\dot{dF}\ddot{U} + dF\,\dddot{U}, \tag{4.3}$$

$$s_k = \lambda_k, \qquad \dot{U} = r_k, \qquad \ddot{U} = \dot{r}_k, \qquad 2\dot{s}_k = \dot{\lambda}_k, \tag{4.4}$$

$$\ddot{dF}r_k + 2\dot{dF}\dot{r}_k + dF\ddot{r}_k = \ddot{\lambda}_k r_k + 2\dot{\lambda}_k \dot{r}_k + \lambda_k \ddot{r}_k. \tag{4.5}$$

From (4.3)-(4.5) we get

$$(3\ddot{s}_k - \ddot{\lambda}_k)r_k - \dot{s}_k\dot{r}_k = (dF - \lambda_k)(\ddot{U} - \ddot{r}_k)$$

which gives (4.2) after taking the inner product with r_j.

□

Remark 4.5. If dF is symmetric in a neighborhood of U_0 then

$$\langle (dr_k)r_k,\, r_j\rangle = -\langle (dr_j)r_k,\, r_k\rangle\,,$$

as can be seen by differentiating $\langle r_j,\, r_k\rangle \equiv 0$ along r_k. So formula (4.3) becomes

$$(\lambda_j - \lambda_k)\left\langle \ddot{U} - \ddot{r}_k,\, r_j\right\rangle \;=\; \dot{s}_k\,\langle (dr_j)r_k,\, r_k\rangle\,, \tag{4.6}$$

$j \neq k,\;\; j,\, k \;=\; 1,\, 2$. Assume that $\lambda_j - \lambda_k$ and $\dot{s}_k$ have the same sign at U_0, and the j–rarefaction curve of U_0 is convex curve near U_0. Then formula (4.6) implies that, in a neighborhood of U_0, the k–shock curve of U_0 enters the and remains in the convex side of the j–rarefaction curve of U_0.

Proof of Lemma 4.3: Since all wave curves used in the construction of the Riemann solution $\Re(U_L, U_R)$ of (1.1) for any $U_L,\; U_R \;\in\; \mathbf{R}^2_+$ are contained in $\mathbf{R}^2_+$, we have that $\mathbf{R}^2_+$ is an invariant region, so it is enough to prove that

$$B_i \stackrel{\text{def}}{=} \{U \in \mathbf{R}^2_+\,;\, (-1)^i w_i(U) \leq c_i\},$$

is also an invariant region, where $c_i = (-1)^i w_i((u_i, 0)),\;\, (-1)^i u_i > 0,\;\, i = 1, 2$. To prove this it is enough to check the following (cf. [6]):

i) if $U_0 \in \partial B_2$ then $\mathcal{S}^1(U_0) \cup \mathcal{C}(U_0) \subset B_2$;
ii) if If $U_0 \in \partial B_1$ then $\mathcal{S}^2_-(U_0) \cup \mathcal{C}_-(U_0) \subset B_1$.

We recall that $\mathcal{C}(U_0) = \mathcal{C}_-(U_0) = \emptyset$ if $U_0 = (u_0, v_0)$ and $v_0 > 0$. From the fact that $r_2 \cdot (1, 0) > 0$ on the positive axis $\{v = 0,\, u > 0\}$, we have that w_2 restricted to it is an increasing function. Since $\mathcal{C}\,((u_0, 0))$ is a half line on the left of u_0 in the axis $\{v=0\}$, we have $\mathcal{C}_-((u_0,0)) \subset B_2$ if $(u_0,0) \in \partial B_2$. Similarly we prove that $\mathcal{C}_-((u_0,0)) \subset B_1$ for $(u_0,0) \in \partial B_1$.

Now let us prove that $\mathcal{S}^1(U_0) \subset B_2$ if $U_0 \in \partial B_2$ and $U_0 = (u_0, v_0)$ with $v_0 > 0$. We omit the proof that $\mathcal{S}^2_-(U_0) \subset B_1$ if $U_0 \in \partial B_1$ for it is similar. Take $k=1$ and $r_2 = \nabla w_2$ in (4.2). Then we have

$$(\lambda_1 - \lambda_2)\left\langle \dddot{U} - \ddot{r}_1,\, \nabla w_2 \right\rangle \;=\; \dot{s}_1 \left\langle (dr_1)r_1,\, \nabla w_2 \right\rangle. \tag{4.7}$$

Differentiating

$$\langle r_1,\, \nabla w_2 \rangle \equiv 0$$

along r_1 we have

$$\langle (dr_1)r_1,\, \nabla w_2 \rangle \;=\; -d^2 w_2(r_1, r_1). \tag{4.8}$$

Substituting (4.8) in (4.7) we obtain

$$(\lambda_2 - \lambda_1)\left\langle \dddot{U} - \ddot{r}_1,\, \nabla w_2 \right\rangle \;=\; d^2 w_2(r_1, r_1). \tag{4.9}$$

Now we note that for a parametrization $U = U(\epsilon),\ \epsilon \le 0$, of $\mathcal{S}^1(U_0)$ such that $U(0)=0,\ \dot{U}(0) = r_1$, and $\ddot{U}(0) = \dot{r_1}$, we have

$$w_2(U) = c_2 + \frac{1}{3}\dddot{w}_2(0)\epsilon^3 + O(\epsilon^4) \tag{4.10}$$

where we used that

$$U_0 \in \partial B_2 \;\text{ so } w_2(0) \equiv w_2(U(0)) = c_2 \equiv w_2((u_2, 0)),$$

$$\dot{w}_2(0) = \left\langle \nabla w_2,\, \dot{U}(0) \right\rangle = \langle \nabla w_2,\, r_1 \rangle = \frac{dw_2}{dr_1}(0) = 0,$$

and

$$\begin{aligned} \ddot{w}_2(0) &= d^2 w_2(\dot{U}(0),\, \dot{U}(0)) + \left\langle \nabla w_2,\, \ddot{U}(0) \right\rangle \\ &= d^2 w_2 \cdot (r_1, r_1) + \langle \nabla w_2,\, \dot{r}_1 \rangle = \frac{d^2 w_2}{dr_1^2}(0) = 0. \end{aligned}$$

Regarding the third derivative $\dddot{w}_2\,(0)$, from $\dfrac{d^3 w_2}{dr_1^3}(0) = 0$ we get

$$\dddot{w}_2\,(0) = \left\langle \dddot{U} - \ddot{r}_1,\, \nabla w_2 \right\rangle. \tag{4.11}$$

From (4.10) and (4.11) we have

$$w_2(U) = c_2 + \frac{1}{3}\left\langle \dddot{U} - \ddot{r}_1,\, \nabla w_2 \right\rangle \epsilon^3 + O(\epsilon^4). \tag{4.12}$$

By (4.9) we have that $\left\langle \dddot{U} - \ddot{r}_1,\, \nabla w_2 \right\rangle > 0$. Since $\epsilon < 0$, from (4.12) we have that $w_2(U) < 0$ for all $U \in \mathcal{S}^1(U_0)$ near U_0, that is, $\mathcal{S}^1(U_0) \subset B_2$ in a neighborhood of U_0. Now we assume by contradiction that there is a point $U_1 \in \mathcal{S}^1(U_0)$ such that $w_2(U_1) = c_2$. Then we have another point $U_2 \in \mathcal{S}^1(U_0)$ such that $\dot{w}_2(U_2) = \left\langle \nabla w_2,\, \dot{U}_2 \right\rangle = 0$, so $\mathcal{S}^1(U_0)$ is tangent to an integral curve of r_1 at U_2. Then by the *tangency rule* (cf. proof of in [9], and cf. Bethe–Wendroff's theorem [6, 17]) we have either $U_2 - U_0$ parallel to r_1 at U_2 or $\dot{s}(U_2) = 0$. But by Lemma 3.6 we have that $U - U_0$ is not parallel to $r_1(U)$ for any $U \in \mathcal{S}^1(U_0)$ (note that $v_0 \neq -\sqrt{2-a}u_0$ for $u_2 > 0$ and $U_0 \in \partial B_2$) and by Lemma 3.7, $\dot{s} \neq 0$ for all point in $\mathcal{S}^1(U_0)$. Thus we have a contradiction and finish the proof.

□

We can now define an appproximate solution of Godunov type. We fix an invariant region S such that $U_0(x) \in S$ for all $x > 0$ and the intersection of the wave curve $\mathcal{R}^2_-(U_R) \cup \mathcal{S}^2_-(U_R)$ with the line $\sqrt{a}u - v = 0$ is contained in S for all $U_R \in S$. Let

$$\{(j\Delta x, n\Delta t)\,;\ (j,n) \in \mathbf{N}^2\},\quad \mathbf{N} = \{1,2,\ldots\},$$

be a net in $\mathbf{R}_+ \times \mathbf{R}_+ = \{(x,t) \in \mathbf{R}^2\,;\ x \geq 0,\ t \geq 0\}$ such that $\delta \overset{\text{def}}{=} \Delta t/\Delta x$ is constant and satisfies the CFL condition

$$\delta \sup\{|\lambda_k(U)|\,;\ k = 1,2,\ U \in S\} < 1.$$

An approximate solution U^ε, $\varepsilon \overset{\text{def}}{=} \Delta t = \delta\Delta x$, is defined as follows: First we approximate the initial data U_0 by

$$U_0^\varepsilon(x) \overset{\text{def}}{=} \sum_{j=0}^{\infty} U_{0j}\chi_{(2j\Delta x,(2j+2)\Delta x]}(x),$$

where

$$U_{0j} \overset{\text{def}}{=} \frac{1}{2j\Delta x}\int_{2\Delta x}^{(2j+2)\Delta x} U_0(x)dx.$$

Since $U_0(x) \in S$ for all $x > 0$ and S is convex, we obtain $U_{0j} \in S$ for all $j \in \mathbf{N}$ and so $U_0^\varepsilon(x) \in S$ for all $x > 0$.

Now we suppose that U^ε is defined in some strip $\mathbf{R}_+ \times [0, n\Delta t)$ and $U^\varepsilon(x, n\Delta t) \in S$ for all $x \geq 0$, and we show how to define U^ε in $\mathbf{R}_+ \times [n\Delta t, (n+1)\Delta t)$. First we define U^ε on $\mathbf{R}_+ \times \{n\Delta t\}$ by

$$U_n^\varepsilon(x) \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} U_{nj}\chi_{(2j\Delta x,(2j+2)\Delta x]}(x),$$

where

$$U_{nj} \stackrel{\text{def}}{=} \frac{1}{2\Delta x}\int_{2j\Delta x}^{(2j+2)\Delta x} U^\varepsilon(x, n\Delta t - 0)dx.$$

Next we define U^ε in the mesh $((2j-1)\Delta x, (2j+1)\Delta x) \times (n\Delta t, (n+1)\Delta t)$, $j \geq 1$, by

$$U^\varepsilon(x,t) = \Re(U_{n\,j-1}, U_{nj})(x - j\Delta x, t - n\Delta t)$$

where $\Re(U_L, U_R)$ is defined in Proposition 4.1. Finally we define U^ε in $[0, \Delta x) \times (n\Delta t, (n+1)\Delta t)$ by

$$U^\varepsilon(x,t) = \Re(U_{Bn}, U_{0n})(x, t - n\Delta t)$$

where $U_{Bn} \in \mathcal{W}_-^2(U_{n0})$ satisfies $B(U_{Bn}) = 0$ and is given by Proposition 4.2.

By Lemma 4.3, we obtain that the sequence (U^ε) is bounded in L^∞. From [1, 2] we have that it converges a.e. to a solution of (1.1)–(1.2) as ε goes to zero. We devote the rest of this section to prove that U satisfies the boundary condition (1.3).

Let $B(u,v) \stackrel{\text{def}}{=} \sqrt{a}u - v$. If we denote the flux coordinates of (1.1) by q_1 and q_2, that is,

$$q_1(u,v) \stackrel{\text{def}}{=} \frac{1}{2}(au^2 + v^2),$$
$$q_2(u,v) \stackrel{\text{def}}{=} uv,$$

then B^2 is a linear combination of q_1 and q_2, namely, $B^2 = 2(q_1 - \sqrt{a}q_2)$. By Green's theorem we have [3]

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \Big(\eta(U^\varepsilon)\phi_t + B^2(U^\varepsilon)\phi_x\Big)\,dxdt \\
&= \int_0^\infty (\eta(U^\varepsilon)\phi)(x,0)dx + \int_0^\infty (B^2(U^\varepsilon)\phi)(0,t)dt \\
&+ \int_0^\infty \sum_{\text{shocks}} (s[\eta(U^\varepsilon)] - [B^2(U^\varepsilon)])\phi dt + L(\phi),
\end{aligned} \tag{4.13}$$

for all $\phi \in C_0^1(\mathbf{R}^2)$, where

$$L(\phi) \stackrel{\text{def}}{=} \int_0^\infty \sum_{n=1}^{\infty} \phi(x, n\Delta t)[\eta(U^\varepsilon)]_{t=n\Delta t-0}^{t=n\Delta t+0} dx, \tag{4.14}$$

and $\eta(U) = 2(u - \sqrt{a}v)$. By the construction of U^ε we have $B^2(U^\varepsilon(0,t)) = 0$, and by the Rankine–Hugoniot relation, we have

$$s([\eta(U^\varepsilon) - [B^2(U^\varepsilon)]) = 0.$$

Therefore if we prove that $L(\phi)$ goes to zero as $\varepsilon \to 0+$ for each $\phi \in C_0^1(\mathbf{R}^2)$, then we obtain by taking $\lim_{\varepsilon\to 0+}$ in (4.13) that $U = \lim_{\varepsilon\to 0+} U^\varepsilon$ is a solution to the IBVP (1.1)–(1.3), satisfying the boundary condition (1.3) in the the following sense:

$$\begin{aligned} \int_0^\infty \int_0^\infty \quad & 2(u - \sqrt{a}v)\phi_t + (\sqrt{a}u - v)^2 \phi_x dx dt \\ & + \int_0^\infty \left(2(u - \sqrt{a}v)\phi\right)(x, 0) dx = 0, \end{aligned} \tag{4.15}$$

for all $\phi \in C_0^1(\mathbf{R}^2)$. We remark that the equation (4.15) implies that

$$* - \lim_{\delta\to 0+} \frac{1}{\delta} \int_0^\delta \left(\sqrt{a}u(x,t) - v(x,t)\right)^2 dx = 0.$$

Indeed, if we take $\psi(x,t) = \xi(x)\zeta(t)$ in (4.15), where $\xi(x)$ is a smooth approximation of $(\delta - x)\chi_{[0,\delta]}$, and $\zeta \in C_0^1((0,\infty))$, then

$$\int_0^\infty \int_0^\delta (2u - \sqrt{a}v)(\delta - x)\zeta'(t) - (\sqrt{a}u - v)^2 \zeta(t) dx dt = 0,$$

so

$$\int_0^\infty \int_0^\delta (\sqrt{a}u - v)^2 \zeta(t) dx dt = O(\delta^2).$$

Next we prove that $\lim_{\varepsilon\to 0+} L(\phi) = 0$. We follow [3, 16], and use that

$$\int_{2j\Delta x}^{2(j+1)\Delta x} [\eta(U^\varepsilon)]_{t=n\Delta t+0}^{t=n\Delta t-0} dx = 0 \tag{4.16}$$

for all n, j. This a direct consequence of the definition of U_{nj}^ε and the linearity of η. From the definition of L in (4.14), and (4.16) we have

$$\begin{aligned} L(\phi) = \sum_{n=1,j=0}^{N} \int_{2j\Delta x}^{2(j+1)\Delta x} \quad & (\phi(x,\, n\Delta t) \\ & -\phi((2j+1)\Delta x,\, n\Delta t)) \left([\eta(U^\varepsilon)]_{t=n\Delta t+0}^{t=n\Delta t-0}\right) dx \end{aligned}$$

for some $N = N(\phi, \varepsilon) \in \mathbf{N}$. Then

$$\begin{aligned}
|L(\phi)| &\leq \| \phi_x \|_\infty \sum_{n=1,j=0}^{N} \int_{2j\Delta x}^{2(j+1)\Delta x)} \left| [\eta(U^\varepsilon)]_{t=n\Delta t+0}^{t=n\Delta t-0} \right| dx \\
&\leq \| \phi_x \|_\infty \sum_{n=1,j=0}^{N} \int_{2j\Delta x}^{2(j+1)\Delta x)} \left(\frac{(\Delta x)^2}{\sqrt{\Delta x}} + \sqrt{\Delta x} |[\eta(U^\varepsilon)]_{t=n\Delta t+0}^{t=n\Delta t-0}|^2 \right) dx \\
&= \sqrt{\Delta x} \| \phi_x \|_\infty \left\{ \sum_{n=1,j=0}^{N} (\Delta x)^2 + \sum_{n=1,j=0}^{N} \int_{2j\Delta x}^{2(j+1)\Delta x} \left| [\eta(U^\varepsilon)]_{t=n\Delta t+0}^{t=n\Delta t-0} \right|^2 dx \right\}.
\end{aligned} \tag{4.17}$$

The last term in braces is majored by a constant because $N = O\left(1/(\Delta x)^2\right)$ and because we have [3]

$$\sum_{n=1,j=0}^{N} \int_{2j\Delta x}^{2(j+1)\Delta x} \left| [\eta(U^\varepsilon)]_{t=n\Delta t+0}^{t=n\Delta t-0} \right|^2 dx \leq \text{const.},$$

due the existence of an strictly convex entropy for the system (), namely, $\eta(U) = |U|^2$. From (4.17) we have $\lim_{\varepsilon \to 0+} L(\phi) = 0$.

References

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Pui Tak Kan
Department of Mathematical Sciences
Indiana University–Purdue University
402 N. Blackford Street
Indianapolis, IN 46202
USA
e-mail: pkan@math.iupui.edu

Marcelo M. Santos
Departamento de Matemática
Universidade Federal da Paraíba
João Pessoa, PB 58 059-900
Brazil
e-mail: ccendm20@brufpb.bitnet

Zhouping Xin
Courant Institute of Mathematical Sciences
New York University
251 Mercer Street
New York, NY 10012
USA
e-mail: xinz@cims.nyu.edu

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