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LOCAL SOLVABILITY IN L^p OF FIRST ORDER SEMILINEAR EQUATIONS

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Abstract

We prove existence of local solutions in L^p, $1 < p < \infty$, for first order complex semilinear equations in $\mathbb{R}^n$ whose linear principal part satisfies condition $(\mathcal{P})$ and has Lip^1 coefficients, assuming a Lipschitz condition on the nonlinear part. When the nonlinear part and the coefficients of the linear part are smooth, we find solutions in the Sobolev spaces L^p_s for any $s > n/p$. This extends to semilinear equations local solvability results that were known for linear equations.

Resumo

Neste trabalho demonstramos que existem soluções locais em L^p, $1 < p < \infty$, para equações semi-lineares complexas de primeira ordem em $\mathbb{R}^n$ cuja parte principal satisfaz a condição $(\mathcal{P})$, de Nirenberg-Treves, e cujos coeficientes e suas derivadas de primeira ordem são Lipschitz, assumindo uma condição de Lipschitz na parte não linear. Quando os coeficientes da parte linear e a parte não linear são suaves (i.e. C^∞), obtemos a existência de soluções nos espaços de Sobolev L^p_s para qualquer $s > n/p$. Isto estende para equações semi-lineares resultados já obtidos recentemente para equações lineares.

0. Introduction

Consider the semilinear equation

$$L(x, D)u = f(x, u), \tag{0.1}$$

where $L(x, D)$ is a linear partial differential operator of order 1 with complex coefficients defined on $\mathbb{R}^n$ and $f(x, \zeta)$ is a continuous complex function. We

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will assume that

$$L = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j} \tag{0.2}$$

is defined in $\mathbb{R}^n$, that a_j, $\partial a_j/\partial x_k$, $j = 1, \ldots, n$, $k = 1, \ldots, n$ are Lipschitz functions and that $|a_1(x)| > 0$. Write $A(x,\xi) = \sum_{j=1}^n \operatorname{Re} a_j(x)\,\xi_j$, $B(x,\xi) = \sum_{j=1}^n \operatorname{Im} a_j(x)\,\xi_j$. A null bicharacteristic of $A(x,\xi)$ is a curve satisfying the system of ordinary differential equations

$$\begin{cases} \dot{x} = \nabla_\xi A(x,\xi), & x(0) = x_0, \\ \dot{\xi} = -\nabla_x A(x,\xi), & \xi(0) = \xi_0, \end{cases}$$

with initial conditions verifying $A(x_0,\xi_0) = 0$ (notice that $\nabla_\xi A$ depends on x alone). The operator (0.2) satisfies the Nirenberg-Treves condition $(\mathcal{P})$ ([NT]) if $B(x,\xi)$ does not change sign along any null bicharacteristic of $A(x,\xi)$; this formulation is invariant under coordinate changes. Concerning the nonlinear part of the (0.1), we will assume that $f(x,\zeta)$ is a continuous, complex valued function defined in $\mathbb{R}^n \times \mathbb{C}$ satisfying a uniform Lipschitz condition

$$|f(x,\zeta_1) - f(x,\zeta_2)| \le K|\zeta_1 - \zeta_2|, \qquad x \in \mathbb{R}^n, \quad \zeta_1, \zeta_2 \in \mathbb{C}. \tag{0.3}$$

When (0.1) is a linear equation, i.e., when $f(x,u) = c(x)u$, it is known that $(\mathcal{P})$ implies its local solvability in L^p ([HP], [P], [HM]). However, for higher order equations, $(\mathcal{P})$ alone does not guarantee local solvability in L^p when $p \neq 2$ (*cf.* [HP] and the references therein). On the other hand, concerning higher order semilinear equations whose principal part satisfies $(\mathcal{P})$, a general local solvability theorem valid for $p = 2$, was recently proved ([S], [HS]).

In this paper we extend the results of [HP] and [P] to the semilinear case. We prove two theorems:

Theorem 0.1. *Assume that the linear partial differential operator $L(x,D)$ of order 1, given by (0.2) and defined on $\mathbb{R}^n$, satisfies $(\mathcal{P})$ and has complex coefficients whose first order derivatives are Lipschitz. Let $1 < p < \infty$. If $f(x,\zeta)$ is continuous and satisfies (0.3), there exists $\rho = \rho(p) > 0$ and $u \in L^p(\mathbb{R}^n)$ such that u satisfies (0.1) on $|x| < \rho$.*

When the coefficients of L and the function $f(x,\zeta)$ are smooth, it is possible to prove existence of solutions with arbitrary high regularity in the scale of Sobolev spaces $L^p_s(I\!R^n)$. In this case there is no need to assume the Lipschitz condition (0.3).

Theorem 0.2. *Assume that the linear partial differential operator $L(x,D)$ of order 1, given by (0.2) and defined on $I\!R^n$, satisfies ($\mathcal{P}$) and has smooth complex coefficients. Let $1 < p < \infty$. If $f(x,\zeta)$ is smooth and $s > n/p$, there exists $\rho = \rho(s,p) > 0$ and $u \in L^p_s(I\!R^n)$ such that u satisfies (0.1) on $|x| < \rho$.*

The paper is organized as follows: in Section 1 we prove the first theorem relying on solvability results known for complex linear equations (see Theorem 1.1 and Corollary 1.2); the rest of the paper is concerned with the proof of Theorem 0.2. In Section 2 we consider a slightly more general linear part (real linear rather than complex linear) which includes complex conjugation of the unknown function and indicate how to obtain an analogue of Corollary 1.2; in Section 3 we perform standard but essential reductions on the semilinear equation and in Section 4 we finally are able to construct solutions for the reduced equation, which is what is needed in order to conclude the proof of Theorem 0.2.

1. Proof of Theorem 0.1

Let ${}^tL$ denote the formal transpose of the operator L given by (0.2) that satisfies the hypothesis of Theorem 0.1. We shall need

Theorem 1.1. *Assume that L is given by (0.2) where a_j, $\partial a_j/\partial x_k$, $j = 1,\dots,n$, $k = 1,\dots,n$, are complex Lipschitz functions. If L satisfies condition ($\mathcal{P}$), then given $1 < q < \infty$, there exist constants C, $\rho_0 > 0$ such that*

$$\|u\|_q \le C\rho\|{}^tLu\|_q, \qquad u \in C^\infty_c(B_\rho), \tag{1.1}$$

for all $0 < \rho \le \rho_0$, where B_ρ denotes the ball of radius ρ centered at the origin.

The a priori estimate (1.1) was first proved in [P], [HP] for smooth coefficients and later extended to the case of Lip^1 coefficients in [HM]. A standard duality argument involving the Hahn-Banach theorem then gives, if $1/p+1/q=1$,

Corollary 1.2. *Under the hypotheses of the theorem, for any $g \in L^p(\mathbb{R}^n)$ and $0<\rho<\rho_0$, there exists $u \in L^p(\mathbb{R}^n)$ such that*

i) $\|u\|_p \le C\rho\|g\|_p$,

ii) $Lu = g$ *in the sense of distributions in* B_ρ

We will now state and prove a slightly stronger version of Theorem 0.1.

Theorem 1.3. *Assume that the linear partial differential operator $L(x,D)$ of order 1, given by (0.2) and defined on $\mathbb{R}^n$, satisfies $(\mathcal{P})$ and has complex coefficients whose first order derivatives are Lipschitz. Let $1<p<\infty$. If $f(x,\zeta)$ is continuous and satisfies (0.3), there exists $\rho=\rho(p)>0$ such that for every $g \in L^p(\mathbb{R}^n)$ there exists $u \in L^p(\mathbb{R}^n)$ satisfying*

$$L(x,D)u = f(x,u)+g \quad \text{on} \quad |x|<\rho. \tag{1.2}$$

It is clear that (1.2) reduces to (0.1) if we choose $g=0$ in (1.2), so Theorem 0.1 in the introduction is a particular case of the result just stated.

Proof. Of course, (1.2) should be understood in the weak sense so what it really means is that

$$\langle u, {}^tL\phi\rangle = \langle f(x,u)+g,\phi\rangle, \quad \forall\phi \in C_c^\infty(B_\rho). \tag{1.3}$$

We will construct a Cauchy sequence $u_k \in L^p(\mathbb{R}^n)$ such that

$$\langle u_{k+1}, {}^tL\phi\rangle = \langle f(x,u_k)+g,\phi\rangle, \quad \forall\phi \in C_c^\infty(B_\rho). \tag{1.4}$$

Since (1.3) and (1.4) only depend on the values of $|x|<\rho$ and ρ will be taken small, we may assume without loss of generality that $f(x,\zeta)\equiv 0$ for $|x|>1$, in particular, $f(x,0) \in C_c(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. Set $A = \|f(x,0)\|_p$ and $B = \|g\|_p$. If $u \in L^p(\mathbb{R}^n)$ we have, in view of (0.3),

$$\|f(x,u)\|_p \leq \|f(x,u) - f(x,0)\|_p + A \leq K\|u\|_p + A,$$

so $f(x,u) \in L^p(\mathbb{R}^n)$. Invoking Corollary (1.2) we find u_0 such that

$$\begin{aligned} Lu_0 &= f(x,0) + g \quad \text{on} \quad |x| < \rho, \\ \|u_0\|_p &\leq C\rho(A+B). \end{aligned}$$

Thus, choosing ρ small, we may assume that $\|u_0\|_p \leq 1$. Next, we find v_1 by solving the equation

$$\begin{aligned} Lv_1 &= f(x,u_0) - f(x,0) \quad \text{on} \quad |x| < \rho, \\ \|v_1\|_p &\leq C\rho\|f(x,u_0) - f(x,0)\|_p \leq CK\rho, \end{aligned}$$

and set $u_1 = u_0 + v_1$. Hence, $Lu_1 = f(x,u_0) + g$ and $\|u_1 - u_0\|_p \leq CK\rho$. Assume inductively that for some $k \geq 1$ and $j = 1, \ldots, k$, functions u_j, have been determined so as to satisfy

$$\begin{aligned} &Lu_j = f(x,u_{j-1}) + g \quad \text{on} \quad |x| < \rho, \\ &\|u_j - u_{j-1}\|_p \leq (CK\rho)^j. \end{aligned}$$

Then, solving

$$\begin{aligned} Lv_{k+1} &= f(x,u_k) - f(x,u_{k-1}) \quad \text{on} \quad |x| < \rho, \\ \|v_{k+1}\|_p &\leq C\rho\|f(x,u_k) - f(x,u_{k-1})\|_p \leq (CK\rho)^{k+1}, \end{aligned}$$

and setting $u_{k+1} = u_k + v_{k+1}$, it is clear that we have completed the required inductive step. Thus, there exists an infinite sequence u_k satisfying (1.4) such that $\|u_k - u_{k-1}\|_p \leq (CK\rho)^k$. If we choose $\rho < 1/CK$, u_k converges to a certain $u \in L^p(\mathbb{R}^n)$, furthermore,$\|f(x,u) - f(x,u_k)\|_p \leq K\|u - u_k\|_p \to 0$ as $k \to \infty$. Letting $k \to \infty$ in (1.4) we see that u satisfies (1.3). The method also gives an estimate $\|u\|_p \leq \|u_0\|_p + (CK\rho)(1 - CK\rho)^{-1} \leq C\rho(A + B + K(1 - CK\rho)^{-1}) = O(\rho)$ for the norm of the solution. The proof is complete.

As we have seen, the proof of Theorem 1.3 is based on the contraction principle and the key fact that allows it to work is the presence of the factor ρ at our disposal in Corollary 1.2 i). When the coefficients of L are smooth, it is still possible to solve locally $Lu = g$ in L^p_s when $g \in L^p_s$ for any s, but for large values of s it is no longer possible to make the ratio $\|u\|_{p,s}/\|g\|_{p,s}$ small by shrinking the neighborhood in which the equation holds. A substitute for the contraction method is then bounded iteration allied to a compactness argument. We start by studying the linearization of the equation in the next section. Observe that the linearization of $\zeta \to f(x,\zeta)$ at ζ_0 is of the form $\zeta \to c(x,\zeta_0)\zeta + d(x,\zeta_0)\bar{\zeta}$ because we are not assuming that f depends holomorphically on ζ.

2. Solutions for the linear part

In this section we consider the operator

$$Lu = \sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j} + c(x)u + d(x)\bar{u} = L_0 u + cu + d\bar{u}, \tag{2.1}$$

defined in $\mathbb{R}^n$, where the coefficients $a_j(x)$, $j = 1, \ldots, n$, $c(x)$ and $d(x)$ are smooth functions and that $|a_1(x)| > 0$. We will assume that the principal part L_0 satisfies condition $(\mathcal{P})$. Although L is not complex linear because of the presence of the term $\bar{u}$, it is real linear. Hence, it will be convenient in the sequel to consider all complex linear spaces equipped with the underlying structure of real spaces and think of functions and distributions as $\mathbb{R}^2$-valued rather than complex valued. Of course, the standard duality pairing between complex valued functions and distributions is based on the product of complex numbers whereas that between $\mathbb{R}^2$-valued functions and distributions relies on the inner product of $\mathbb{R}^2$. Hence, the formal transpose of a differential operator will change according to the paring that is being used. However, if we denote by ${}^t\tilde{L}$ the transpose induced by the inner product of $\mathbb{R}^2$, it is easy to verify that $|{}^tL\phi(x)| = |{}^t\tilde{L}\phi(x)|$, $\phi \in C_c^\infty(\mathbb{R}^n)$, $x \in \mathbb{R}^n$. For that reason, ${}^tL$ and ${}^t\tilde{L}$ can be interchanged in all the estimates below, which will be just stated for the complex transpose ${}^tL$. From Theorem 1.1 we have the estimate

$$\|\phi\|_q \leq C\rho\|{}^tL_0\phi\|_q, \qquad u \in C_c^\infty(B_\rho), \tag{2.2}$$

for all $0 < \rho \leq \rho_0$, where B_ρ denotes the ball of radius ρ centered at the origin. Now (2.2) implies, since the coefficients of L_0 are smooth, the existence of convenient estimates in $L^q_s = L^q_s(\mathbb{R}^n)$ for any $s \in \mathbb{R}$ and $1 < q < \infty$. Given $s \in \mathbb{R}$ there exists $C = C(q, s, \epsilon)$ and $\rho = \rho(q, s, \epsilon) > 0$ such that

$$\|\phi\|_{q,s} \leq \epsilon \|{}^t L_0 \phi\|_{q,s} + C\|\phi\|_{q,s-1}, \qquad \phi \in C^\infty_c(B_\rho). \tag{2.3}$$

The derivation of (2.3) from (2.2) will be left to the reader (the computations for the case $q = 2$ can be found in [HS]); it is standard and involves the use of commutators with the pseudo-differential operator Λ^s with symbol $(1+|\xi|^2)^{s/2}$, so that $\|u\|_{q,s} = \|\Lambda^s u\|_{q,0} = \|\Lambda^s u\|_q$ for u in the Schwartz space $\mathcal{S}$, and $L^q_s = \Lambda^{-s} L^q$. The relevant fact is that Λ^s is a global pseudo-differential operator of order s and type $(1,0)$ and maps isomorphically L^q_{t+s} onto L^q_t for any t, $s \in \mathbb{R}$ and $1 < q < \infty$. Since we can take advantage of an arbitrary $\epsilon > 0$ in (2.3), it is clear that treating $L = L_0 + cI + dJ$ $(Ju = \bar{u})$ as a perturbation of L_0 we obtain

$$\|\phi\|_{q,s} \leq \epsilon \|{}^t L \phi\|_{q,s} + C\|\phi\|_{q,s-1}, \qquad \phi \in C^\infty_c(B_\rho). \tag{2.4}$$

The fact that L is of principal type implies that (*cf.* [NT,II,p.469]) ${}^t L D^\alpha \delta \neq 0$ for any multi-index α and it is well known (*cf.* [NT,II,p.468]) that this, together with estimate (2.4) (with $-s$ in the place of s), implies that for small $\rho = \rho(q, s) > 0$ and certain $C = C(q, s) > 0$ we have

$$\|\phi\|_{q,-s} \leq C\|{}^t L \phi\|_{q,-s}, \qquad \phi \in C^\infty_c(B_\rho). \tag{2.5}$$

Then, (2.5) implies by a standard application of the Hahn-Banach theorem and identification of the dual of L^p_s with L^q_{-s}, $1 < p < \infty$, $1/p + 1/q = 1$,

Proposition 2.1 *Let L be as above. For every $s \in \mathbb{R}$, $1 < p < \infty$ there exists $\rho = \rho(p, s) > 0$ and $C = C(p, s) > 0$ such that for every $f \in L^p_s$ there exists a function (distribution) $u \in L^p_s$ satisfying*

$$\begin{aligned} Lu &= f \quad \text{in} \quad B_\rho, \\ \|u\|_{p,s} &\leq C\|f\|_{p,s}, \end{aligned} \tag{2.6}$$

3. Standard reductions

We now discuss some standard reductions for the equation

$$L(x, D)u = f(x, u) \tag{3.1}$$

that we want to solve locally in L^p_s for some $s > n/p$ that we fix once for ever; we are assuming that the hypotheses of Theorem 0.2 hold. Thus, $L = L_0$ is given by (0.2) and $f(x,\zeta)$ is smooth. We may assume without loss of generality that $f(x,\zeta) = 0$ if $|x| \geq 1$ and then write

$$f(x,\zeta) = \phi(x) + f_1(x,\zeta)$$

where

$$\phi \in C^\infty_c(B_1) \quad \text{and} \quad f_1(x,0) = 0.$$

We may solve formally (3.1) by the power series method because, being of principal type, L_0 possesses noncharacteristic directions. That means that we may determine a formal power series $u = \sum u_\alpha x^\alpha$ such that

$$L_0(x,D)u - f_1(x,u) = \sum \phi_\alpha x^\alpha$$

where $\sum \phi_\alpha x^\alpha$ is the Taylor series of ϕ. Let $u_0 \in C^\infty_c(B_1)$ have Taylor series equal to $\sum u_\alpha x^\alpha$ at the origin, which is possible by a well known lemma of Borel. It follows that

$$L_0(x,D)u_0 - f_1(x,u_0) \overset{\text{def}}{=} \phi_0 = \phi - \eta \tag{3.2}$$

where $\eta \in C^\infty_c(B_1)$ vanishes to infinite order at the origin. Let $\rho \in C^\infty_c(B_1)$, $\rho(x) = 1$ if $|x| \leq 1/2$ and consider for $0 < \delta < 1$ the function

$$\phi_\delta(x) = \phi_0(x) + \rho(x/\delta)\eta(x).$$

The following properties are easily verified:

i) $\phi_\delta \in C^\infty_c(B_1)$;

ii) $\phi_\delta(x) = \phi(x) + (\rho(x/\delta) - 1)\eta(x) = \phi(x)$ if $|x| \leq \delta/2$;

iii) $\|\phi_\delta - \phi_0\|_{p,s} = \|\rho(\cdot/\delta)\eta\|_{p,s} \to 0$ as $\delta \to 0$ because η vanishes of infinite order at the origin.

To solve (3.1) in a neighborhood of the origin, it is enough to solve the modified equation

$$L_0(x,D)u = \phi_\delta + f_1(x,u)$$

which writing $u = u_0 + v$ and taking account of (3.2) is equivalent to

$$\begin{aligned} L_0(x,D)v &= \phi_\delta + f_1(x,u_0+v) - L_0(x,D)u_0 \\ &= \phi_\delta + f_1(x,u_0+v) - f_1(x,u_0) + f_1(x,u_0) - L_0(x,D)u_0 \\ &= \phi_\delta - \phi_0 + f_1(x,u_0+v) - f_1(x,u_0) \\ &= \psi_\delta + f_2(x,v) \end{aligned} \tag{3.3}$$

where $\|\psi_\delta\|_{p,s} = \|\phi_\delta - \phi_0\|_{p,s} \to 0$ as $\delta \to 0$ and $f_2(x,0) = 0$. We may further write

$$f_2(x,\zeta) = c(x)\zeta + d(x)\bar{\zeta} + f_3(x,\zeta).$$

Notice that f_3 is a Taylor remainder of f_2 and can be expressed as $f_3(x,\zeta) = \sum \xi_i \xi_j c_{ij}(x,\zeta)$, where the ξ_i's are the components of the real and imaginary parts of ζ. Thus, (3.3) becomes

$$L'v = L_0(x,D)v - c(x)v - d(x)\bar{v} = \psi_\delta + f_3(x,v). \tag{3.4}$$

It is at this point that we make crucial use of our assumption that $s > n/p$. Thus, we may apply the standard estimates for the composition and product of functions in Sobolev spaces (see [M] or [AH]) to the quadratic form f_3 and obtain

$$\|f_3(x,w)\|_{p,s} \leq C\|w\|_{p,s}^2, \qquad w \in L_s^p, \quad \|w\|_{p,s} \leq 1. \tag{3.5}$$

4. Proof of Theorem 0.2

To simplify the notation, we drop the subindexes in equation (3.4), and write $\psi = \psi_\delta$, $f = f_3$, so that we must solve

$$L'v = \psi + f(x,v) \tag{4.1}$$

keeping in mind that we may assume that $\|\psi\|_{p,s}$ is conveniently small and that (3.5) gives

$$\|f(x,w)\|_{p,s} \leq C\|w\|_{p,s}^2, \qquad w \in L_s^p, \quad \|w\|_{p,s} \leq 1. \tag{4.2}$$

We will now define inductively a pair of sequences $u_j, v_j \in L_s^p$, satisfying

$$\begin{aligned} &u_0 = 0, \qquad v_0 = 0, \\ &u_k = \psi + f(x,v_{k-1}), \qquad L'v_k = u_k \quad \text{on} \quad |x| < \rho. \end{aligned} \tag{4.3}$$

We will use the fact that $L'u = g$ may be solved for $|x| < \rho$ with a solution u satisfying the estimate $\|u\|_{p,s} \leq C\|g\|_{s,p}$, for a fixed constant C (this follows from an application of Proposition 2.1). Thus, we may then assume that v_k is chosen so as to verify $\|v_k\|_{p,s} \leq C\|u_k\|_{p,s}$. We shall assume without loss of generality that $C > 1$ and that it is the same constant that appears in estimate (4.2). We choose a number $\delta > 0$ such that $2C^3\delta < 1$, in particular, we also have $C\delta < 1$. We may also assume that $\|\psi\|_{p,s} < \delta/2$.

Lemma 4.1. *For $k = 1, 2, \ldots$ the estimates*

$$\|u_k\|_{p,s} < \delta \quad \text{and} \quad \|v_k\|_{p,s} < 1 \tag{4.4}$$

hold.

Proof. It is obvious that (4.4) is valid for $k = 0$. Assume that it holds for k, in particular, it follows from (4.2) that $\|f(x, v_k)\|_{p,s} \leq C\|v_k\|^2_{p,s}$. Then, $\|u_{k+1}\|_{p,s} = \|\psi + f(x, v_k)\|_{p,s} < \delta/2 + C\|v_k\|^2_{p,s} \leq \delta/2 + C(C\|u_k\|_{p,s})^2 < \delta/2 + C^3\delta^2 < \delta$. Furthermore, $\|v_{k+1}\|_{p,s} \leq C\|u_{k+1}\|_{p,s} < C\delta < 1$. This completes the induction.

We now recall that $\psi(x)$ and $f(x, \zeta)$ vanish identically for $|x| > 1$, so (4.3) implies that $u_k \in C^0_c(B_1)$. If we multiply each each v_k by $\chi(x)$ where χ is a smooth cut-off function equal to one in the ball of radius 1 and supported in the ball of radius 2 and keep calling it v_k, the new sequence will still verify (4.3) and the estimate $\|v_k\|_{p,s} < 1$ in (4.4) will turn into $\|v_k\|_{p,s} < K$ because $\|\chi(x)v_k\|_{p,s} \leq K\|v_k\|_{p,s}$. Thus, we may assume that the v_k are supported in a fixed ball and are uniformly bounded in L^p_s. Passing to subsequences we may as well assume that $v_k \rightharpoonup v$ weakly in L^p_s and, by compactness, that $v_k \to v$ strongly in L^p_t for some $n/p < t < s$. By Sobolev's inequalities, $v_k \to v$ uniformly. If ϕ is a test function supported in $|x| < \rho$, we get from (4.3) that

$$\langle v_k, {}^tL'\phi\rangle = \langle \psi + f(x, v_{k-1}), \phi\rangle$$

and letting $k \to \infty$ we obtain $L'v = \psi + f(x, v)$ in the weak sense, as we wished to prove.

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