Matemática Contemporânea, Vol. 10, 99–121
http://doi.org/10.21711/231766361996/rmc105

ON A NEW CONCEPT OF ASYMPTOTIC STABILITY FOR RIEMANN SOLUTIONS

Hermano Frid *

Abstract

We introduce a new concept of asymptotic stability for Riemann solutions and apply it to scalar inviscid and viscous conservation laws with non-convex flux functions and to a class of degenerate systems of conservation laws.

Resumo

Neste artigo introduzimos uma nova noção de estabilidade assintótica para soluções de Riemann de sistemas de leis de conservação. Provamos então a estabilidade assintótica de soluções de Riemann para leis de conservação escalares em uma dimensão de espaço, nos casos não-viscoso e viscoso, e para soluções de Riemann de uma classe de sistemas degenerados.

1. Introduction

We are concerned with the study of the asymptotic behavior of solutions to initial value problems for conservation laws, or viscous conservation laws, whose initial data are disturbances of given Riemann data. More specifically, let $u(x,t)$ be a solution to the system of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \tag{1}$$

or to the viscous system of conservation laws,

$$\partial_t u + \partial_x f(u) = \partial_x^2 u, \tag{2}$$

*1991 *Mathematics Subject Classification.* Primary:35B40, 35B35; Secondary:35K55
Key words and phrases. Conservation laws, Riemann problems, asymptotic stability

satisfying the initial condition

$$u(x,t)|_{t=0} = u_0(x), \tag{3}$$

and let $u^*(x,t) = V(x/t)$ be the solution of the Riemann problem given by (1) and

$$u^*(x,t)|_{t=0} = u_0^*(x) := \begin{cases} u_L, & \text{if } x < 0, \\ u_R, & \text{if } x > 0. \end{cases} \tag{4}$$

Assume $u_0(x) = u_0^*(x) + h(x)$, with h representing a disturbance decaying to 0 as $|x| \to \infty$ sufficiently fast. Setting $\xi = x/t$, we are interested in answering the question whether

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |u(\xi t, t) - V(\xi)|\, dt = 0 \tag{5}$$

in L^1_{loc} as a sequence of measurable functions indexed by T, of the variable $\xi \in I\!R$.

Definition 1. We say that the Riemann solution is *weakly asymptotically stable* under the perturbation h for (1) or (2), according to the case, if (5) holds.

The motivation for considering the limit in (5) to study asymptotic stability of Riemann solutions comes from the following observation established before in [4,5].

Lemma 1. *Suppose $u(x,t)$ and $V(x/t)$ are uniformly bounded in $I\!R \times [0,\infty)$. Then, (5) implies that the scaling sequence $u^\varepsilon(x,t)$, given by*

$$u^\varepsilon(x,t) = u(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon}), \tag{6}$$

converges in $L^1_{loc}(I\!R \times [0,\infty))$ to $V(x/t)$.

Here we will be mostly concerned with the case of scalar conservation laws. In section 2 we treat the case of BV solutions to (1), in the scalar case. We show that under fairly general hypotheses on f and on the initial disturbance h we have (5).

In section 3, we obtain the same result for a class of degenerate systems of conservation laws. In this connection, we recall that the asymptotic behavior of solutions of the same system treated in section 3, under more restrictive assumptions, were studied in [13]. In section 4, we consider the viscous equation (2), and show that the techniques applied in section 2, for BV solutions of (1), may also be applied to show the validity of (5) for solutions of (2), in the scalar case. The result obtained here, for the viscous equation, improves on the results known so far in the sense that it establishes the asymtotic stability (in the above sense) of the Riemann solution as a whole (*cf.* [9, 15]).

The study of asymptotic stability of Riemann waves for solutions of viscous conservation laws began with the pioneering work of Il'in & Oleinik in 1961 [8], whose results were extended to systems of conservation laws in the 80's by Goodman [6] and Matsumura-Nishihara [14], giving rise to a number of papers on the subject (see, e.g., [21], [11], and the references therein). The usual concept of asymptotic stability means convergence in $L^p(\mathbb{R})$, for some $p \in [1,\infty]$, as $t \to \infty$, of $u(\,\cdot\,,t)$ to a traveling wave of the viscous system (viscous shock wave) or to a viscous rarefaction wave of the viscous system of conservation laws associated with it. It is easy to see that this convergence implies (5), with V equals to the corresponding shock or rarefaction wave of the inviscid system, whenever u and V are bounded in $\mathbb{R}\times[0,\infty)$. This justifies the name "weak" asymptotic stability. This weaker concept has, therefore, the good features of implying the strong convergence of the whole scaling sequence, as given by Lemma 1, and being an actual extension of the notion usually adopted so far. It has also the advantage of allowing results of asymptotic stability not known yet for the usual definition, as the result for the whole Riemann solution of a general scalar conservation laws in one space variable, which we present here.

We end this introduction with some final comments more. The ideas set forth in this note are currently being extended to a much more general context, in collaboration with Gui-Qiang Chen [3], who helped the author with many valuable suggestions during the preparation of this work. There is also a very active line of research in the analogous problem of asymptotic stability of traveling waves for equations of reaction-diffusion and dispersive type, where the spectral analisys has played a decisive role. In this connection, we refer to [17], [20]. We are not sure about the utility of the concept introduced here for this last class of equations, since they require a different qualitative approach, as one may realize from simple examples like the KPP waves [17]. However, we do think that it can be of effective use in the field of conservation laws, among other reasons, by its ability to provide an unified treatment for linear and nonlinear waves.

2. BV solutions of scalar conservation laws

We will treat the case of BV solutions of (1), (3), when (1) is a scalar conservation law. We will prove the following.

Theorem 1. *Let (1) be a scalar conservation law with $f \in C^2(I\!\!R)$ and $h \in BV(I\!\!R) \cap L^1(I\!\!R)$. Let $u(x,t)$ be a solution of (1), (3) satisfying the entropy condition: for every entropy-flux pair (η, q), with η convex, and all $\phi \in C_0^1(I\!\!R_+^2)$, with $\phi \geq 0$, we have*

$$\iint_{I\!\!R_1^2} \{\eta(u(x,t))\phi_t + q(u(x,t))\phi_x\}\, dxdt \geq 0. \tag{7}$$

Then (5) holds for a.e. $\xi \in I\!\!R$. In particular, the Riemann solution is weakly asymptotically stable for (1).

Proof. We consider domains of the type:

$$E_+^{\xi,T} = \{\, (x,t) \in I\!\!R_+^2 \mid 0 \leq t \leq T, \xi t \leq x \leq +\infty \,\}, \tag{8}$$

$$E_-^{\xi,T} = \{\, (x,t) \in I\!\!R_+^2 \mid 0 \leq t \leq T, -\infty \leq x \leq \xi t \,\}. \tag{9}$$

The fact that $u(x,t) \in BV(I\!\!R \times [0,T))$ together with (7) imply, upon making ϕ approximate the characteristic function of $E_\pm^{\xi,T}$, for η, q, a convex entropy-flux pair, that

$$
\begin{aligned}
-\int_0^\infty \eta(u_0(x))\,dx \quad &+\int_{\xi T}^\infty \eta(u(x,T))\,dx \\
&-\int_0^T (-\xi\eta+q)(u(\xi t,t))\,dt \le 0,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
-\int_{-\infty}^0 \eta(u_0(x))\,dx \quad &+\int_{-\infty}^{\xi T} \eta(u(x,T))\,dx \\
&+\int_0^T (-\xi\eta+q)(u(\xi t,t))\,dt \le 0,
\end{aligned} \tag{11}
$$

provided that $\eta(u_R)=0$, in the first case, and $\eta(u_L)=0$, in the second one. We will make use of three families of entropy-flux pairs $\pi_i(u,\bar u)=(\eta_i(u,\bar u),q_i(u,\bar u))$, $i=1,2,3$, where $\bar u$ is to be viewed as a parameter:

$$\eta_1(u,\bar u)=|u-\bar u|,\quad q_1(u,\bar u)=\operatorname{sign}(u-\bar u)(f(u)-f(\bar u)), \tag{12}$$

$$\eta_2(u,\bar u)=H(u-\bar u)(u-\bar u),\quad q_2(u,\bar u)=H(u-\bar u)(f(u)-f(\bar u)), \tag{13}$$

$$\eta_3(u,\bar u)=H(\bar u-u)(\bar u-u),\quad q_3(u,\bar u)=H(\bar u-u)(f(\bar u)-f(u)). \tag{14}$$

Here $H(s)$ denotes the well known Heaviside function, i.e., $H(s)=0$, $s<0$, $H(s)=1$, $s>1$.

It is well known that the solution of (1), (3) for $h\in BV$, satisfying (7), is unique, belongs to $BV(\mathbb{R}\times[0,T))$, for all $T>0$, and is uniformly bounded [22]. So, we have, in particular, $a\le u(x,t)\le b$, with $a\le\min\{u_L,u_R\}\le\max\{u_L,u_R\}\le b$. To show the type of techniques involved in proving (5) we start with the simple case where f is a convex function. In this case, for the solution of (1), (4) we have only two possibilities: if $u_L>u_R$, a shock wave with inclination s given by the Rankine-Hugoniot relation

$$s(u_L-u_R)=f(u_L)-f(u_R); \tag{15}$$

and if $u_L<u_R$, a rarefaction wave where for $f'(u_L)\le x/t\le f'(u_R)$ we have $V(x/t)=f'^{-1}(x/t)$.

Set,

$$
\begin{aligned}
\xi_m &= \min_{u\in[a,b]} f'(u),\\
\xi_M &= \max_{u\in[a,b]} f'(u).
\end{aligned}
$$

So, let us consider separately each one of the two cases:

(i) $u_L > u_R$, *shock wave.*

So, for the case of a shock wave, we start by taking ξ in the interval $(\xi_M, +\infty)$. We take the entropy-flux pair $\pi_1(u, u_R)$, given by (12), with $\bar{u} = u_R$. By (10), we have

$$\int_0^T |u - u_R|(\xi - \frac{f(u) - f(u_\xi)}{u - u_\xi}(\xi t, t))\,dt \le \int_0^\infty |u_0(x) - u_R|\,dx. \tag{16}$$

Dividing by T and making $T \to \infty$ we get

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |u - u_R|(\xi - \frac{f(u) - f(u_R)}{u - u_R})(\xi t, t)\,dt = 0, \tag{17}$$

since the integrand of the left-hand side of (16) is non-negative. We could then conclude the convergence we wish to prove, for $\xi \in (\xi_M, +\infty)$, but we will proceed with a reasoning which will be systematically used in what follows. We consider the measures μ_T^ξ given by

$$\langle \mu_T^\xi, h\rangle = \frac{1}{T}\int_0^T h(u(\xi t, t))\,dt, \tag{18}$$

for all $h \in C([a, b])$. They are probability measures on $[a, b]$ and, so, given any subsequence of μ_T^ξ, we can find a further subsequence converging to a certain $\mu^\xi \in I\!\!P([a, b])$, the set of probability measures on $[a, b]$. Now, by (17), we have

$$\langle \mu^\xi, |u - u_R|(\xi - \frac{f(u) - f(u_R)}{u - u_R})\rangle = 0. \tag{19}$$

This is possible only if $\mu^\xi = \delta_{u_R}$, the Dirac measure concentrated at u_R. Since this holds for any subsequence of μ_T^ξ, we then have $\mu_T^\xi \rightharpoonup \delta_{u_R}$. So we get (5) for $\xi_M < \xi < +\infty$.

For $\xi \in (f'(u_L), \xi_M)$, let u_ξ be the only point so that

$$f'(u_\xi) = \xi. \tag{20}$$

We take the pair $\pi_2(u, u_\xi)$, given by (13). We observe that we must have $u_\xi > u_L$. So, we can apply (11), to get

$$\begin{aligned}\int_0^T H(u - u_\xi)(u - u_\xi)(-\xi \ &+ \tfrac{f(u) - f(u_\xi)}{u - u_\xi})(\xi t, t)\,dt \\ &\le \int_{-\infty}^0 H(u_0(x) - u_\xi)(u_0(x) - u_\xi)\,dx.\end{aligned} \tag{21}$$

Again, given any subsequence of μ_T^ξ, we obtain a subsequence converging to a certain $\mu^\xi \in \mathbb{P}([a,b])$. For this μ^ξ, we have by (21)

$$\langle \mu^\xi, H(u-u_\xi)(u-u_\xi)(-\xi + \frac{f(u)-f(u_\xi)}{u-u_\xi})\rangle = 0, \tag{22}$$

since the integrand in the left-hand side of (21) is non-negative. This implies that $\operatorname{supp}\mu^\xi \subseteq [a, u_\xi]$. Now, by (16), we also have

$$\langle \mu^\xi, |u-u_R|(\xi - \frac{f(u)-f(u_R)}{u-u_R})\rangle \le 0. \tag{23}$$

Then, since $\operatorname{supp}\mu^\xi \subseteq [a, u_\xi]$, there must be $\mu^\xi = \delta_{u_R}$. For $\xi \in (s, f'(u_L))$, we first consider the entropy-flux pair $\pi_2(u, u_L)$. Then, again using (11), we get

$$\begin{aligned}\int_0^T H(u-u_L)(u-u_L)(-\xi \quad &+ \tfrac{f(u)-f(u_L)}{u-u_L})(\xi t, t)\, dt \\ &\le \int_{-\infty}^0 H(u_0(x)-u_L)(u_0(x)-u_L)\, dx.\end{aligned} \tag{24}$$

and, so, as above, for μ^ξ obtained from μ_T^ξ in the same way, we have

$$\langle \mu, H(u-u_L)(u-u_L)(-\xi + \frac{f(u)-f(u_L)}{u-u_L})\rangle = 0. \tag{25}$$

This implies $\operatorname{supp}\mu^\xi \subseteq [a, u_L]$ and, hence, (23) implies $\mu^\xi = \delta_{u_R}$.

For $\xi \in (f'(u_R), s)$, we first take the entropy pair $\pi_3(u, u_R)$, given by (14), with $\bar{u} = u_R$. So, applying (10) we get, for μ^ξ obtained as above, $\operatorname{supp}\mu^\xi \subseteq [u_R, b]$. Then, taking the pair $\pi_1(u, u_L)$ and applying (11), we get

$$\langle \mu^\xi, |u-u_L|(-\xi + \frac{f(u)-f(u_L)}{u-u_L})\rangle \le 0, \tag{26}$$

which implies $\mu^\xi = \delta_{u_L}$ and, therefore $\mu_T^\xi \rightharpoonup \delta_{u_L}$. The cases $\xi \in (\xi_m, f'(u_R))$ and $\xi \in (-\infty, \xi_m)$ are analogous to the cases $\xi \in (f'(u_L), \xi_M)$ and $\xi \in (\xi_M, +\infty)$, respectively. This concludes the proof of (5) in the case of the shock waves.

(ii) $u_L < u_R$, rarefaction waves.

For $-\infty < \xi < \xi_m$, we consider the entropy-flux pair $\pi_1(u, u_L)$ and apply (11) to get

$$\int_0^T |u-u_L|(-\xi + \frac{f(u)-f(u_L)}{u-u_L})(\xi t, t)\, dt \le \int_{-\infty}^0 |u_0(x)-u_L|\, dx. \tag{27}$$

So, we obtain as above $\mu_T^\xi \rightharpoonup \delta_{u_L}$. For $\xi_m \le \xi < f'(u_L)$, we consider first the the entropy-flux pair $\pi_3(u, u_\xi)$, with $u_\xi < u_L$, given by (20). Then, applying (10), for μ^ξ, we get

$$\langle \mu^\xi, H(u_\xi - u)(u_\xi - u)(\xi - \frac{f(u_\xi) - f(u)}{u_\xi - u})\rangle = 0. \tag{28}$$

This implies $\operatorname{supp} \mu^\xi \subseteq [u_\xi, b]$. We also have

$$\langle \mu^\xi, |u - u_L|(-\xi + \frac{f(u) - f(u_L)}{u - u_L})\rangle \le 0,$$

by (27), and, so, we must have again $\mu^\xi = \delta_{u_L}$, which implies $\mu_T^\xi \rightharpoonup \delta_{u_L}$.

For $f'(u_L) < \xi < f'(u_R)$, we use the pairs $\pi_2(u, u_\xi)$ and $\pi_3(u, u_\xi)$, again with u_ξ given by (20). Using (10), with (η, q) given by $\pi_3(u, u_\xi)$, and (11), with (η, q) given by $\pi_2(u, u_\xi)$, we obtain, for μ^ξ as above, (28) and, also,

$$\langle \mu^\xi, H(u - u_\xi)(u - u_\xi)(-\xi + \frac{f(u) - f(u_\xi)}{u - u_\xi}\rangle = 0, \tag{29}$$

and, hence, $\mu^\xi = \delta_{u_\xi}$, which implies $\mu_T^\xi \rightharpoonup \delta_{u_\xi}$.

The case $\xi > f'(u_L)$ is analogous to the case $\xi < f'(u_L)$. Therefore, we get (5) also in the case of rarefaction waves, and, then, we have proved the weak asymptotic stability of the Riemann solutions for scalar conservation laws with convex flux function.

We now consider the case when f is not a convex function. Let g denote the convex envelope of f on the interval $[u_L, u_R]$, if $u_L < u_R$, or the concave envelope of f on the interval $[u_R, u_L]$, if $u_R < u_L$. That is, g is the maximum among al

l convex functions on $[u_L, u_R]$, whose graphs lie below the graph of f, in the first case, and the minimum among all the concave functions on $[u_R, u_L]$, whose graphs lie above the graph of f, in the second case. It is well known that, in any case, the solution of the Riemann problem will be either a single shock wave or a composition of success

ive one- or two-sided contact discontinuities connecting the extremes of intervals where g is linear, separated by rarefaction waves connecting the extremes of the intervals where the graphs of f and g coincide (see fig. 1).

Let us assume, to fix ideas, $u_L < u_R$. We first show that, for a.e. $\xi \in I\!R$, we have $\operatorname{supp}\mu^\xi \subseteq [u_L, u_R]$. Indeed, by Sard's Theorem (see, *e.g.*, [18]), the set $\mathcal{N}$ consisting of those ξ such that $f''(u_\xi) = 0$ for some u_ξ satisfying $f'(u_\xi) = \xi$, *i.e.*, the set of ξ that are not regular values of f', has zero Lebesgue measure. Let us take $\xi \in \mathbb{R} - \mathcal{N}$. Consider the finitely many points $a \le u_\xi^1 < u_\xi^2 < \ldots < u_\xi^N \le u_L$, such that $f'(u_\xi^k) = \xi$, $k = 1, \ldots, N$, and also the finitely many points $u_R \le \tilde{u}_\xi^M < \tilde{u}_\xi^{M-1} < \ldots < \tilde{u}_\xi^1 \le b$, such that $f'(\tilde{u}_\xi^j) = \xi$, $j = 1, \ldots, M$. Beginning with u_ξ^1, we must have either $f'(u) \le \xi$, for $\xi \in [a, u_\xi^1]$, or $f'(u) \ge \xi$, for u in the same interval. We take the pair $\pi_3(u, u_\xi^1)$. If $f'(u) \le \xi$, for $\xi \in [a, u_\xi^1]$, we apply (10), if $f'(u) \ge \xi$, in that interval, we apply (11). In either case, we conclude $\operatorname{supp}\mu^\xi \subseteq [u_\xi^1, b]$. Next, we take the pair $\pi_3(u, u_\xi^2)$. Now, we must have either $f'(u) \le \xi$, for $u \in [u_\xi^1, u_\xi^2]$, or $f'(u) \ge \xi$, in this same interval. So, using the pair $\pi_3(u, u_\xi^2)$ and applying (10) or (11), we, then, obtain that $\operatorname{supp}\mu^\xi \subseteq [u_\xi^2, b]$. Continuing this process up to u_ξ^M, we get $\operatorname{supp}\mu^\xi \subseteq [u_\xi^M, b]$. Now, we take the pair $\pi_3(u, u_L)$ and argue the same way once more to get $\operatorname{supp}\mu^\xi \subseteq [u_L, b]$. Analogously, we take the pair $\pi_2(u, \tilde{u}_\xi^1)$ and apply (10) or (11), according to whether $f'(u) \le \xi$, for $u \in [\tilde{u}_\xi^1, b]$, or $f'(u) \ge \xi$, in this same interval. So, we get $\operatorname{supp}\mu^\xi \subseteq [u_L, \tilde{u}_\xi^1]$. Arguing the same way, taking su

ccessively the pairs $\pi_2(u, \tilde{u}_\xi^2)$, $\pi_2(u, \tilde{u}_\xi^3)$, …,$\pi_2(u, \tilde{u}_\xi^M)$, and, finally, $\pi_2(u, u_R)$, we arrive at $\operatorname{supp}\mu^\xi \subseteq [u_L, u_R]$.

Now, for $\xi \in (-\infty, g'(u_L))$, we take the pair $\pi_1(u, u_L)$, and observe that

$$\xi < \frac{f(u) - f(u_L)}{u - u_L}, \qquad \text{if } u \in [u_L, u_R].$$

So, using (11), we get $\mu^\xi = \delta_{u_L}$, and, so, $\mu_T^\xi \rightharpoonup \delta_{u_L}$. For $\xi \in (g'(u_R), +\infty)$, we take the pair $\pi_1(u, u_R)$ and observe that

$$\xi > \frac{f(u) - f(u_R)}{u - u_R}, \qquad \text{if } u \in [u_L, u_R].$$

So, we apply (10) to conclude that $\mu^\xi = \delta_{u_R}$, and, therefore, $\mu_T^\xi \rightharpoonup \delta_{u_R}$. Finally, we consider $\xi \in (g'(u_L), g'(u_R))$ such that if $g'(u_\xi) = \xi$, then $g''(u_\xi) \neq 0$. There are only finitely many ξ's in $(g'(u_L), g'(u_R))$ which do not satisfy this condition. So, for ξ satifying this condition, we take the pairs $\pi_2(u, u_\xi)$, $\pi_3(u, u_\xi)$, with u_ξ

given by $g'(u_\xi) = \xi$ (observe that we also have (20)), and notice that

$$\xi > \frac{f(u) - f(u_\xi)}{u - u_\xi}, \qquad \text{if } u \in [u_L, u_\xi),$$

and

$$\xi < \frac{f(u) - f(u_\xi)}{u - u_\xi}, \qquad \text{if } u \in (u_\xi, u_R].$$

So, applying (10) for $\pi_2(u, u_\xi)$, and (11) for $\pi_3(u, u_\xi)$, we obtain $\mu_\xi = \delta_{u_\xi}$, and, therefore, $\mu_T^\xi \rightharpoonup \delta_{u_\xi}$. This concludes the proof of the theorem.

□

3. A class of degenerate systems of conservation laws

In this section we study the weak asymptotic stability for BV solutions of initial value problems for systems of conservation laws of the type

$$\partial_t u + \partial_x(g(|u|)u) = 0, \tag{30}$$

where $u \in I\!R^n$ and $g : (0, \infty) \to I\!R$ is a C^1 function satisfying $g'(r) \geq 0$. We assume that we are given the initial data (3) with $u_0(x) = u_0^*(x) + h(x)$, where $u_0^*(x)$ is given by (4) and $h \in BV(I\!R) \cap L^1(I\!R)$. We also assume that the initial data satisfy:

$$|u_0(x)| \geq r_0 > 0, \tag{31}$$

and

$$\langle u_0(x), v_0 \rangle > 0, \tag{32}$$

for all $x \in I\!R$, for some $r_0 > 0$ and some constant vector $v_0 \in I\!R^n$. Without loss of generality, we may assume $v_0 = (0, \dots, 1)$, the n-th element of the cannonical basis. We denote $r = |u|$ and $\Theta = (\theta_1, \dots, \theta_{n-1})$, where θ_k, $k = 1, \dots, n-1$, are given by

$$\begin{aligned} u_k &= r \cos\theta_k, \qquad k = 1, \dots, n-1 \\ u_n &= r(1 - \cos^2\theta_1 - \ldots - \cos^2\theta_{n-1})^{1/2}. \end{aligned}$$

So, we can write $u \equiv (r, \Theta)$ for each vector u in the halph-space determined by (32).

The vector function u is said to be an *admissible weak solution* of (30), (3), if it satisfies:

- for all $\phi \in C_0^1(I\!R \times [0,\infty))$ we have

$$\iint_{I\!R\times[0,\infty)} \{u(x,t)\phi_t \; + u(x,t)g(|u(x,t)|)\phi_x\}\,dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)\,dx = 0; \tag{33}$$

- for any pair (η, q), with η a convex function on $[0,\infty)$, q given by

$$q(r) = \int^r \eta'(s)sg(s)\,ds,$$

and for all $\psi \in C_0^1(I\!R, [0,\infty))$, with $\psi \geq 0$, we have

$$\iint_{I\!R\times[0,\infty)} \{\eta(r(x,t))\psi_t \; + q(r(x,t))\psi_x\}\,dxdt + \int_{-\infty}^{\infty} \eta(r_0(x))\psi(x,0)\,dx \geq 0, \tag{34}$$

where $r(x,t) = r(u(x,t))$, $r_0(x) = r(u_0(x))$.

Condition (34) implies that the function $r(x,t)$ must be a weak solution of the problem

$$\partial_t r + \partial_x f(r) = 0, \tag{35}$$

$$r(x,t)|_{t=0} = r_0(x), \tag{36}$$

where we set $f(r) = rg(r)$. In particular, there exists a Lipschitz function $y(x,t)$ satisfying

$$\begin{aligned} \partial_x y &= r(x,t), \\ \partial_t y &= -f(r(x,t)), \\ y(x,0) &= \int_0^x r_0(x)\,dx. \end{aligned}$$

Now, if we choose ϕ in (33) in the form $\phi(x,t) = \varphi(y(x,t))\chi^\varepsilon(t)$, with φ any function in $C_0^1(I\!R)$ and χ^ε a suitable sequence in $C_0^1([0,\infty))$ approaching the characteristic function of $[0,T]$, when $\varepsilon \to 0$, for given $T > 0$, we will get from (34)

$$\int_{-\infty}^{\infty} \cos(\theta_k(x(y,T),T))\varphi(y)\,dy = \int_{-\infty}^{\infty} \cos(\theta_k(x(y,0),0))\varphi(y)\,dy, \qquad k = 1,\dots,n-1, \tag{37}$$

where we use the notation $\theta_k(x,t) = \theta_k(u(x,t))$. These equations imply that $\theta_k(x,t)$ must be determined by the formulas

$$\theta_k(x(y,t),t) = \theta_k(x(y,0),0), \qquad k = 1,\dots,n-1. \tag{38}$$

As a consequence, for any continuous function $H \in C(S^{n-1})$, it holds (*cf.* [19, 1, 2])

$$\begin{aligned} \iint_{\mathbb{R}\times[0,\infty)} & H(\Theta(x,t))\{r(x,t)\phi_t + f(r)\phi_x\}\,dxdt \\ & + \int_{-\infty}^{\infty} H(\Theta(u_0(x)))r_0(x)\phi(x,0)\,dx = 0. \end{aligned} \tag{39}$$

The system (30) is non-strictly hyperbolic (if $n > 2$) and its first $n-1$ eigenvalues are linearly degenerated. Indeed, $\lambda_1(u) = \dots = \lambda_{n-1}(u) = g(r)$, and, so $\partial_{\theta_k}\lambda \equiv 0$, $k = 1,\dots,n-1$. On the other hand, it is easy to see that θ_k, $k = 1,\dots,n-1$, as well as r, are Riemann invariants for (30). The n-th eigenvalue of (30) is $\lambda_n(u) = (rg(r))' = g(r) + rg'(r)$ and is associated with the decoupled equation (27). The solution of the Riemann problem (30), (4) is, then, simply formed by a contact discontinuity with inclination $g(r_L)$, connecting (r_L,Θ_L) to (r_L,Θ_R) followed by the scalar wave given by $\Theta \equiv \Theta_R$ and the solution of the Riemann problem for (35) with initial data

$$r(x,t)|_{t=0} = \begin{cases} r_L, & x < 0 \\ r_R, & x > 0, \end{cases} \tag{40}$$

where $r_L = r(u_L)$, $r_R = r(u_R)$ (see [10]).

It is well known that, given an initial data $u_0(x) \in BV(\mathbb{R})$ satisfying conditions (31), (32), one can prove the existence of a weak admissible solution for (30), (3), by using Glimm's method (see [19, 1, 2]). This solution will be in $BV(\mathbb{R}\times[0,\infty))$ and satisfies $\min_{x\in\mathbb{R}} r_0(x) \le r(x,t) \le \max_{x\in\mathbb{R}} r_0(x)$. We have the following result.

Theorem 2. *Let $u(x,t) \in BV(\mathbb{R}\times[0,\infty))$ be a weak admissible solution of (30), (3), with $u_0(x) = u_0^*(x) + h(x)$ for some $h \in BV(\mathbb{R})\cap L^1(\mathbb{R})$. Then (5) holds for a.e. $\xi \in \mathbb{R}$.*

Proof. From (39) and the fact that $u \in BV$, in the same way we have done in the scalar case, we get, for all $T > 0$

$$-\int_0^\infty H(\Theta_0(x))r_0(x)\,dx + \int_{\xi T}^\infty H(\Theta(x,T))r(x,T)\,dx \\ + \int_0^T H(\Theta(\xi t,t))r(\xi t,t)(\xi - g(r(\xi t,t)))\,dt = 0, \tag{41}$$

for $H \in C(S^{n-1})$ satisfying $H(\Theta_R) = 0$, and

$$-\int_{-\infty}^0 H(\Theta_0(x))r_0(x)\,dx + \int_{-\infty}^{\xi T} H(\Theta(x,T))r(x,T)\,dx \\ + \int_0^T H(\Theta(\xi t,t))r(\xi t,t)(-\xi + g(r(\xi t,t)))\,dt = 0, \tag{42}$$

for $H \in C(S^{n-1})$ satisfying $H(\Theta_L) = 0$, where $\Theta_L = \Theta(u_L)$, and $\Theta_R = \Theta(u_R)$. Let $r^*(x,t) = R(x/t)$ be the solution of the Riemann problem (35), (40), and let $e(r)$ be the convex envelope of $f(r)$ on $[r_L, r_R]$, if $r_L < r_R$, or the concave envelope of $f(r)$ on $[r_R, r_L]$, if $r_R < r_L$. Given $\xi \in \mathbb{R}$, we define the probability measures $\mu_T^\xi \in \mathbb{P}([0,\infty) \times S^{n-1})$ by

$$\mu_T^\xi = \frac{1}{T}\int_0^T G(r(\xi t,t), \Theta(\xi t,t))\,dt,$$

for any $G \in C([0,\infty) \times S^{n-1})$. Since the supports of all μ_T^ξ, $T > 0$, are contained in a fixed compact of $[0,\infty) \times S^{n-1}$, say K, the set $\{\mu_T^\xi\}_{T>0}$ is relatively compact in $\mathbb{P}(K)$, with the weak star topology of $\mathcal{M}(\mathcal{K})$. So, given any subsequence of μ_T^ξ, with $T \to \infty$, one can find a further subsequence, which we still label the same way, such that $\mu_T^\xi \rightharpoonup \mu^\xi$, for some $\mu^\xi \in \mathbb{P}(K)$. Now, by the result of the last section we must have

$$\mu^\xi = \nu^\xi(\Theta) \otimes \delta_{R(\xi)},$$

for some $\nu^\xi \in \mathbb{P}(S^{n-1})$.

Now, for $\xi < g_L = g(r_L)$, we observe first that $g_L \le e'(r_L)$, and, so, $R(\xi) = r_L$. Then, choosing $H(\Theta) = |\Theta - \Theta_L|$, from (42) we get for ν^ξ

$$\langle \nu^\xi, |\Theta - \Theta_L| \rangle (-\xi + g(R(\xi)))R(\xi) \le 0, \tag{43}$$

and, then, we must have $\nu^\xi = \delta_{\Theta_L}$, which gives $\mu_T^\xi \rightharpoonup \delta_{(r_L,T_L)}$. For $g_L < \xi < e'(r_L)$, we again have $R(\xi) = r_L$. So, choosing this time $H(\Theta) = |\Theta - \Theta_R|$, from

(42) we get for ν^ξ

$$\langle \nu^\xi, |\Theta - \Theta_R| \rangle (\xi - g(R(\xi)))R(\xi) \le 0, \tag{44}$$

and, then, we must have $\nu^\xi = \delta_{\Theta_R}$, which implies $\mu_T^\xi \rightharpoonup \delta_{(r_L,\Theta_R)}$. For $e'(r_L) < \xi < e'(r_R)$, such that ξ is not in the image by e' of the closed intervals where e is affine or g' vanishes, we have $\xi = e'(R(\xi)) > g(R(\xi))$. Then, again by (44), we get $\nu^\xi = \delta_{\Theta_R}$, and, therefore, $\mu_T^\xi \rightharpoonup \delta_{(R(\xi),\Theta_R)}$. Finally, for $\xi > e'(r_R)$, we have $R(\xi) = r_R$ and, since $e'(r_R) \ge g(r_R)$, (44) gives again $\nu^\xi = \delta_{\Theta_R}$ and, hence, $\mu_T^\xi \rightharpoonup \delta_{(r_R,\Theta_R)}$, concluding the proof of the theorem.

□

4. The Viscous Equation

In this section we prove the weak asymptotic stability condition (5) for solutions of (2), (3), in the scalar case.

Theorem 3. *There exists a smooth classical solution to (2), (3), $u(x,t)$, in the half space $I\!R \times (0,\infty)$, with $f \in C^2(I\!R)$ and $u_0(x) = u_0^*(x) + h(x)$, where u_0^* is given by (4) and $h \in BV(I\!R) \cap L^1(I\!R)$. Further, (5) holds for a.e. $\xi \in I\!R$.*

Proof. The existence of a smooth classical solution of (2), (3), under the assumptions on the equation and the initial data, follows from the techniques and results in [16] (see also [12]). The basic tools are the maximum principle, the fact that one is able to prove that any local solution has the total variation in x bounded by the total variation of the initial data, and a regularity result for the local solution (see [16]). It remains then to prove the validity of (5), for a.e. $\xi \in I\!R$. We begin by proving a fact about the solution of (2), (3) which will be useful in the proof of (5).

Lemma 2. *Let $u(x,t)$ be the classical solution of (2), (3). Then, there exists $T > 0$ and $C = C(T) > 0$ such that, for $0 < t \le T$, one has*

$$\|u_x(t)\|_\infty \le \frac{C}{\sqrt{t}} \|u_0\|_\infty. \tag{45}$$

Moreover, given any $t_1 > 0$*, one has*

$$u_x \in L^\infty(I\!\!R \times [t_1, +\infty)). \tag{46}$$

Proof. Following [7] one can easily prove that for sufficiently small $T > 0$, the operator

$$S(v) = K(t) * u_0 - \int_0^t K_x(t-s) * f(v(s))\, ds$$

is a contraction in

$$B_{r;0,T} = \{\, v \in L^\infty(I\!\!R \times [0,T]) \,:\, \|v - \bar u\|_\infty < r\,\},$$

where K is the well known heat kernel, $r > \|u_0 - \bar u\|$ and $\bar u$ is a constant state. Therefore, there exists one and only one fixed point of S in B_r. On the other hand, by the maximum principle, $u(x,t) \in B_{r;0,T}$ and it is clearly a fixed point of S. Hence, $u(x,t)$ is the only fixed point in $B_{r;0,T}$. Again following [7], one can also easily prove that if T is small enough, there exists $C = C(T) > 0$ such that if v satisfies

$$\|v_x(t)\|_\infty \le \frac{C}{\sqrt{t}}\|u_0\|_\infty, \qquad 0 < t \le T, \tag{47}$$

then $S(v)$ also satisfies (47), $0 < t \le T$. Now, the fixed point of S, $u(x,t)$, can be obtained as limit in L^∞ of the sequence $u^n = S^n(\bar u)$. By what was said above, u^n satisfies (47) for all $n \in I\!\!N$. Hence, given any $0 < t_0 < T$, we have

$$\|u_x^n\|_{L^\infty(I\!\!R\times[t_0,T])} \le \frac{C}{\sqrt{t_0}}\|u_0\|_\infty. \tag{48}$$

We claim that, given such $0 < t_0 < T$, we also have

$$\|u_x\|_{L^\infty(I\!\!R\times[t_0,T])} \le \frac{C}{\sqrt{t_0}}\|u_0\|_\infty, \tag{49}$$

where C is the same as in (48). This can be achieved as follows. By the compactness of the ball of radius $(C/\sqrt{t_0})\|u_0\|_\infty$ in the weak $*$ topology of $L^\infty(I\!\!R \times [t_0,T])$, we obtain, passing to a subsequence if necessary, that $u_x^{n_k} \rightharpoonup \tilde v$ for some $\tilde v \in L^\infty(I\!\!R \times [t_0,T])$, with $\|\tilde v\|_\infty \le (C/\sqrt{t_0})\|u_0\|_\infty$. It is then easy to prove that $\tilde{v} = \partial_x u$ as distributions over $I\!R \times (t_0, T)$, and therefore we conclude (49). Now, to obtain (45), we argue as follows. We let t_0 run over all the rational numbers contained in $(0, T]$. For each such t_0, we obtain as above (49). In particular, for each such t_0, one has

$$\|u_x(t_0)\|_\infty \le \frac{C}{\sqrt{t_0}} \|u_0\|_\infty. \tag{50}$$

Therefore, by density and continuity, one finally concludes (45).

We now prove (46). We first observe that the numbers T and C for which (45) holds depend only on bounds on the initial data and on the derivative of f. In particular, (45) holds for any solution of the Cauchy problem (2), (3), as long as the initial data admit a common uniform bound. So, given any $\tau \in [t_1, \infty)$, we take $0 < \delta < \min\{t_1, T\}$, where T is as above. We then apply the argument above for the solution $u(x,t)$ in the time-interval $[\tau - \delta, \tau - \delta + T]$. Namely, we consider the operator

$$S(v)(t) = K(t) * u(\tau - \delta) - \int_{\tau-\delta}^{t} K_x(s) * f(v(t-s))\, ds,$$

for $t > \tau - \delta$. Since $u(x,t)$ is uniformly bounded, we again have that S is a contraction, now in $B_{r;t_*,t^*}$, with $t_* = \tau - \delta$, $t^* = \tau - \delta + T$. Arguing exactly as in the proof of (45), one gets

$$\|u_x(t)\|_\infty \le \frac{C}{\sqrt{t - t_*}} \|u_0(t_*)\|_\infty,$$

for $t \in (t_*, t^*]$, where C is the same as above. In particular, one obtains

$$\|u_x(\tau)\|_\infty \le \frac{C}{\sqrt{\delta}} r,$$

which gives (46) and the lemma is proved.

□

Now, given any entropy-flux pair (η, q) with η convex, non-negative, from (2) we get

$$\partial_t \eta(u) + \partial_x q(u) = \partial_x(\nabla \eta(u) \partial_x u) - \nabla^2 \eta(u)(\partial_x u, \partial_x u), \tag{51}$$

and, so, integrating over $E_+^{\xi,T}$, and $E_-^{\xi,T}$, we get

$$\begin{aligned} -\int_0^\infty \eta(u_0(x))\,dx + \int_{\xi T}^\infty \eta(u(x,T))\,dx - \int_0^T (-\xi\eta + q)(u(\xi t,t))\,dt \\ \le \int_0^T \nabla\eta(u(\xi t,t))\partial_x u(\xi t,t)\,dt \end{aligned} \tag{52}$$

$$\begin{aligned} -\int_{-\infty}^0 \eta(u_0(x))\,dx + \int_{-\infty}^{\xi T} \eta(u(x,T))\,dx + \int_0^T (-\xi\eta + q)(u(\xi t,t))\,dt \\ \le \int_0^T \nabla\eta(u(\xi t,t))\partial_x u(\xi t,t)\,dt, \end{aligned} \tag{53}$$

provided that $\eta(u_R) = 0$, in the first case, and $\eta(u_L) = 0$, in the second one. We will prove the following:

Lemma 3. *Let $u(x,t)$ be as in the statement of Theorem 3. Then,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |u_x(\xi t,t)|\,dt = 0, \qquad \text{for a.e. } \xi \in I\!R. \tag{54}$$

Once we have proved Lemma 3, since, by the maximum principle, the solution of (2), (3) is uniformly bounded, we get that the right-hand members of (52), (53) divided by T, converge to 0, and the remaining of the proof of Theorem 3 follows exactly as the proof of the Theorem 1. Now, Lemma 3 will follow from:

Proposition 1. *Given ξ_1, $\xi_2 \in I\!R$ and $0 < \theta < 1$, we have*

$$\int_0^\infty \int_{\xi_1 t}^{\xi_2 t} \frac{u_x^2(x,t)}{(1+t)^{1+\theta}}\,dxdt < +\infty. \tag{55}$$

Before proving Proposition 1 we show how it implies Lemma 3.

Proof of Lemma 3. Indeed, by (55), we have

$$\int_{\xi_1}^{\xi_2} \int_0^\infty \frac{u_x^2(\xi t,t)}{(1+t)^{1+\theta}} t\,dtd\xi < +\infty.$$

Then, for a.e. $\xi \in I\!R$ we must have

$$\int_0^\infty \frac{u_x^2(\xi t, t)}{(1+t)^\theta}\, dt < +\infty.$$

So, we have for T sufficiently large

$$\frac{1}{T}\int_0^T u_x^2(\xi t, t)\, dt \le \frac{2^\theta}{T^{1-\theta}}\int_0^T \frac{u_x^2(\xi t, t)}{(1+t)^\theta}\, dt \le \frac{2^\theta}{T^{1-\theta}}\int_0^\infty \frac{u_x^2(\xi t, t)}{(1+t)^\theta}\, dt,$$

and then we get

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |u_x(\xi t, t)|^2\, dt = 0. \tag{56}$$

Now, by Jensen's inequality, we have

$$\left(\frac{1}{T}\int_0^T |u_x(\xi t, t)|\, dt\right)^2 \le \frac{1}{T}\int_0^T |u_x(\xi t, t)|^2\, dt,$$

and this with (54) implies (55).

□

Proof of Proposition 2. Given a non-negative strictly convex entropy η, with flux q, we have

$$\eta_t + q_x = (\nabla \eta u_x)_x - \nabla^2 \eta(u_x, u_x). \tag{57}$$

Dividing (57) by $(1+t)^{1+\theta}$, we get

$$\left(\frac{\eta}{(1+t)^{1+\theta}}\right)_t + \frac{(1+\theta)\eta}{(1+t)^{2+\theta}} + \left(\frac{q}{(1+t)^{1+\theta}}\right)_x = \left(\frac{\nabla \eta u_x}{(1+t)^{1+\theta}}\right)_x - \frac{\nabla^2 \eta u_x^2}{(1+t)^{1+\theta}}.$$

Integrating over $0 < t < T$, $\xi_1 t < x < \xi_2$, we get

$$\int_{\xi_1 T}^{\xi_2 T} \frac{\eta(u(x,T))}{T^{1+\theta}}\, dx + \int_0^T \int_{\xi_1 t}^{\xi_2 t} \frac{\eta}{(1+t)^{2+\theta}}\, dxdt + \int_0^T \frac{[(-\xi\eta + q)(u(\xi t, t))]_{\xi_1}^{\xi_2}}{(1+t)^{1+\theta}}\, dt$$

$$= \int_0^T \frac{[\nabla \eta u_x(\xi t, t)]_{\xi_1}^{\xi_2}}{(1+t)^{1+\theta}}\, dt - \int_0^T \int_{\xi_1 t}^{\xi_2 t} \frac{\nabla^2 \eta(u_x, u_x)}{(1+t)^{1+\theta}}\, dxdt.$$

We notice that, since u is bounded, all the terms in the left hand side of the above equation are bounded. As to the first term of the right hand side, we use the estimates of Lemma 2 and the fact that u is uniformly bounded, to conclude that it is also bounded. Now, using also the fact that η is strictly convex, we get

$$\int_0^T \int_{\xi_1 t}^{\xi_2 t} \frac{u_x^2(x,t)}{(1+t)^{1+\theta}}\, dxdt < A,$$

for some $A > 0$, independent of T. Then, (55) follows and this concludes the proof of the proposition.

□

Now the proof of Theorem 3 follows exactly as the proof of Theorem 1.

□

Acknowledgements: This work was done while the author was visiting the Department of Mathematics of the Northwestern University, and he would like to thank its direction, falculty and staff members for the warm hospitality. In particular to Gui-Qiang Chen, whose interest in these ideas served as a strong encouragement. The author gratefully acknowledges the financial support from CNPq-Brazil, proc. 201110/94-6. Finally, I would like to mention that the notion of asymptotic stability introduced in this note was in part motivated by the invariance of finite difference schemes for conservation laws under an equal scaling in the time and space mesh lenghts. This fact was first brought to my attention by I-Shih Liu and I would like to thank him for that.

References

[**1**] Bonnefille, M., *Propagation des oscillations dans deux classes de systèmes hyperboliques (*2×2 *et* 3×3*)*, Comm. Partial Diff. Eqs. 13 (1988), 905–925.

[**2**] Chen, G.-Q., *The method of quasidecoupling for discontinuous solutions to conservation laws*, Arch. Rational Mech. Anal. 121 (1992), 131–185.

[**3**] Chen, G.-Q., Frid, H., *Asymptotic stability and decay of solutions to conservation laws*, in preparation.

[**4**] Frid, H., Liu, I-S., *Oscillation waves in Riemann problems inside elliptic regions for conservation laws of mixed type*, Estudos e Comunicações do IM-UFRJ 49, 1994. To appear in ZAMP – J. Appl. Math. Phys, (1995).

[**5**] Frid, H., Liu, I-S., *Oscillation waves in Riemann problems for phase transitions*, Estudos e Comunicações do IM-UFRJ 50, 1994. To appear in Quaterly of Appl. Math..

[**6**] Goodman, J., *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Rat. Mech. Anal. 95 (1986), 325–344.

[**7**] Hoff, D., Smoller, J., *Solutions in the large for certain nonlinear parabolic systems*, Ann. Inst. Henri Poincaré 2 (1985), 213–235.

[**8**] Il'in, A. M., Oleinik, O. A., *Asymptotic behavior of the solutions of the Cauchy problem for certain quasilinear equations for large time*, Amer. Math. Soc. Translations, Ser. 2 42 (1964), 19–23.

[**9**] Jones, C., Gardner, R., Kapitula, T., *Stability of travelling waves for non-convex viscous conservation laws*, Comm. Pure Appl. Math. 46 (1993), 505–526.

[**10**] Keyfitz, B., Kranzer, H., *A system of nonstrictly hyperbolic conservation laws arising in elasticity theory*, Arch. Rational Mech. Anal. 72 (1980), 219–241.

[**11**] Kawashima, S., Matsumura, A., *Stability of shock profiles in viscoelasticity with non-convex constitutive relations*, Comm. Pure Appl. Math. 67 (1994), 1547–1569.

[12] Kruzkov, S. N., *First-order quasilinear equations in several independent variables*, Math. USSR Sb. 10 (1970), 217–243.

[13] Liu, T.-P., Wang, C.-H., *On a nonstrictly hyperbolic system of conservation laws*, J. Diff. Eqs. 57 (1985), 1–14.

[14] Matsumura, A., Nishihara, K., *On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math. 2 (1985), 17–25.

[15] Matsumura, A., Nishihara, K., *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys. 165 (1994), 83–96.

[16] Oleinik, O., *Discontinuous solutions of nonlinear differential equations*, Amer. Math. Soc. Transl. 26 (1957), 95–172.

[17] Sattinger, D. H., *On the stability of waves of nonlinear parabolic systems*, Advances in Mathematics 22 (1976), 312–355.

[18] Schwartz, J., Nonlinear Functional Analysis. Courant Inst. Math. Sci. Notes (1963/64).

[19] Serre, D., *Solutions à variations bornées pour certains systèmes hyperboliques de lois de conservation*, J. Diff. Eqs. 68 (1987), 137–168.

[20] Smoller, J., Shock Waves and Reaction-Diffusion Equations, Second Edition, Spriger-Verlag, New York, 1994.

[21] Szepessy, A., Xin, Z., *Nonlinear stability of viscous shock waves*, Arch. Rational Mech. Anal. 122 (1993), 53–103.

[22] Vol'pert, A. I., *The spaces BV and quasilinear equations*, Math. USSR Sb. 2 (1967), 225–267.

Instituto de Matemática
Universidade Federal do Rio de Janeiro
Caixa Postal 68530, CEP 21945-970, RJ, Brazil

Received January 23, 1996
Revised Abril 19, 1996