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INDEX OF SURFACES OF CONSTANT MEAN CURVATURE IN HYPERBOLIC SPACE

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We want to describe a basic problem which arises in the study of stability of surfaces with constant mean curvature.

Let us consider the simplest case of surfaces $x: M^2 \to R^3$ with constant mean curvature H. It is known that such surfaces are critical points of the problem of minimizing area keeping the volume fixed. More explicitly, let $D \subset M$ be a domain with compact closure $\overline{D}$, and let $f: \overline{D} \to R$ be a C^∞ function such that $f = 0$ on the boundary ∂D of D and $\int_D f dM = 0$. Consider the variation

$$x_t = x + tfN, \quad t \in (-\varepsilon, \varepsilon), \tag{1}$$

where N is a unit normal field along x. Denote by $A(t)$ the area of x_t in D. Then:

$$A'(0) = 0, \tag{2}$$

$$2A''(0) = -\int_D (f\triangle f + (4H^2 - 2K)f^2)dM = I(f). \tag{3}$$

Here $\triangle$ is the Laplacian and K is the Gaussian curvature of the metric induced by x.

From (2) it follows that surfaces with constant mean curvature are critical points of the area $A(t)$ for all variations of D that satisfy $\int_D f dM = 0$ (this is the infinitesimal version of the condition of preserving volume).

Eq. (3) has important consequences. We say that D is stable if $I(f) > 0$ for all f. If there is an f such that $I(f) < 0$, we say that D is unstable.

To measure the non-stability of D, we introduce the differential operator $L = \triangle + (4H^2 - 2K)$. Let $\mathcal{F}$ the set of C^∞ functions on $\overline{D}$ such that $f = 0$ an ∂D and $\int_D f dM = 0$. Let $g \in \mathcal{F}$ be such that $Lg + \lambda g = 0$, where $\lambda \in R$;

we say that g is an eigenfunction and λ is an eigenvalue of L. Notice that if g is an eigenfunction of L corresponding to a negative eigenvalue λ of L then

$$\begin{aligned} I(g) &= -\int_D g(\Delta g + (4H^2 - 2K)g)dM = -\int_D gLgdM \\ &= -\int_D g(-\lambda g) = \lambda \int_D g^2 < 0, \end{aligned}$$

that is, g is "direction of instability" of D. It well known from Analysis that the set of eigenvalues of L is discrete and each eigenspace is finite dimensional.

The number of negative eigenvalues, counted with multiplicities, is called the index of L in D and denoted by $\mathrm{Ind}_D L$. This is, in a certain sense, a measure of the instability of D. When M is complete we will define the index of L in M or simply, the index of M by

$$\mathrm{Ind}_M L = \sup_{D \subset M} \mathrm{Ind}_D L.$$

Notice that if M is complete and noncompact, $\mathrm{Ind}_M L$ may be infinite.

In the case that $x: M^2 \to R^3$ is minimal (i.e, $H = 0$) and the variations are unrestricted (i.e., they do not necessarely preserve volume) the following theorem has been proved.

Theorem 1. ([2]). *Let $x: M^2 \to R^3$ be a complete minimal surface. Then the index of M is finite if and only if the total curvature of M is finite.*

If we try to extend this theorem to surfaces with nonzero constant mean curvature H, the result is somewhat disappointing.

Theorem 2. ([4]). *Let $x: M^2 \to R^3$ be a complete surface with constant mean curvature $H \neq 0$. Then the index of M is finite if and only if M is compact.*

As often happens, the situation can be better understood if we go to the hyperbolic space. Let restrict ourselves to the hyperbolic space $H^3(-1)$ with sectional curvature -1. The definitions we have given before extend to surfaces $x: M^2 \to H^3(-1)$ with the only change that $I(f)$ is now given by

$$I(f) = -\int_D (f\Delta f + (-4 + 4H^2 - 2K)f^2)dM.$$

So that the corresponding operator L is

$$L = \Delta + (-4 + 4H^2 - 2K).$$

There is a general principle that surfaces in the hyperbolic space $H^3(-1)$ with $H \geq 1$ behave as surfaces in R^3 with $H \geq 0$. In this case, this is indeed a fact:

Theorem 3. ([1]). *Let $x: M^2 \to H^3(-1)$ be a complete surface with constant mean curvature $H \equiv 1$. Then*

$$\mathrm{Ind}_M < \infty \Leftrightarrow \int_M |K| \, dM < \infty.$$

Theorem 4. ([4]). *Let $x: M^2 \to H^3(-1)$ be a complete surface with constant mean curvature $H > 1$. Then M has finite index if and only if M is compact.*

Thus, if a new phenomenon is to occur, it will appear in the range $H < 1$. We conjecture that the following holds:

Conjecture: *Let $x: M^2 \to H^3(-1)$ be a complete surface with constant mean curvature H. Then*

$$\int_M (H^2 - 1 - K)dM < \infty \Rightarrow \; Ind\, M < \infty$$

and the converse is false.

We will present a number of pertinent remarks.

Remark: For $H = 0$, the conjecture has been shown to be true in the recent thesis of G. de Oliveira Filho [3].

Remark: The fact that the converse is false follows easily from some examples of rotation surfaces with $H =$ const. in hyperbolic space.

Remark: A result related to the conjecture was recently obtained by Sakaki, namely:

Let $x: M^2 \to R^3$ be a complete surface with constant $H < 1$. Assume that M is simply-connected and has no umbilics. Then

$$\text{Ind } M < c \int_M (H^2 - 1 - K)^{1/2} dM \int_M (H^2 - 1 - K)^{3/2} dM,$$

where c is a constant.

Remark: The integrand of (4) can be written as

$$H^2 - 1 - K = \frac{1}{4}(k_1 - k_2)^2$$

where k_1 and k_2 are the principal curvature of $x: M^2 \to H^3(-1)$. Notice that for $H = 1$ it agrees with the integrand that appears in Theorem 3. The number $\phi^2 = (k_1 - k_2)^2$ appears in various other questions of surfaces with constant mean curvature.

Remark: The above conjecture is related to proving that if $\int_M (k_1 - k_2)^2 dM < \infty$ then $(k_1 - k_2)^2$ is bounded on M. The general question of what the condition $\int_M (k_1 - k_2)^2 dM < \infty$ means for a complete non-compact surface with constant mean curvature (even in R^3) seems quite interesting.

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