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ON COMPOSITIONS OF ISOMETRIC IMMERSIONS

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The starting goal of our work was to understand local and global isometric immersions with low codimension of the n-dimensional round sphere $\mathbf{S}^n$ into Euclidean space $\mathbf{R}^{n+p}$. For codimension $p = 2$ and dimension $n \geq 4$, this problem was considered by O'Neill ([O'N]), Erbacher ([Er]), Henke ([He]), Whitt ([Wh]) and Moore ([Mo]). Erbacher proved that in a neighborhood of a non-umbilical point the immersion is necessarily a composition of the standard inclusion of $\mathbf{S}^n$ into $\mathbf{R}^{n+1}$ with a local isometric immersion of $\mathbf{R}^{n+1}$ into $\mathbf{R}^{n+2}$. A simple example provided by Henke shows that for an umbilical point this may no longer be the case.

In fact, earlier work due to O'Neill suggested that a similar result as Erbacher's should hold up to codimension $p = n-2$ under suitable regularity assumptions which extend the umbilical-free hypothesis. Recently, it was shown that for codimension p=n-1 there exist many local isometric immersions which are nowhere compositions (see $[DT]_2$).

In the process of getting a solution to the above problem we realized the possibility of setting a more general rigidity question. Namely, to provide conditions on isometric immersions $f : \mathbf{M}^n \hookrightarrow \mathbf{Q}_c^{n+1}$ and $g : \mathbf{M}^n \hookrightarrow \mathbf{Q}_c^{n+p}, p \geq 2$, which imply that g is necessarily (locally or globally) a composition. Here $\mathbf{M}^n$ is a connected n-dimensional Riemannian manifold, $\mathbf{Q}_c^N$ denotes a complete simply connected manifold of constant sectional curvature c, and by g being a *composition* we mean that there exists an isometric immersion $h : U \hookrightarrow \mathbf{Q}_c^{n+p}$ of an open subset $U \subset \mathbf{Q}_c^{n+1}$ containing $f(\mathbf{M}^n)$ so that $g = h \circ f$.

Let us denote by $p_f(x)$ the number of nonzero principal curvatures of the hypersurface f at $x \in \mathbf{M}^n$. Without making any regularity assumption we proved the following result:

Theorem 1: *Let $f : \mathbf{M}^n \hookrightarrow \mathbf{Q}_c^{n+1}$ and $g : \mathbf{M}^n \hookrightarrow \mathbf{Q}_c^{n+p}, p \geq 2$, be isometric immersions with $p_f(x) \geq p+2$ everywhere. If $p \geq 6$, assume further that $\mathbf{M}^n$ does not contain an open $(n-p+2)$-ruled subset for both f and g. Then there exists an open and dense subset $V \subset \mathbf{M}^n$ so that g/V is locally a composition.*

In fact, by imposing further conditions than in Theorem 1, we were able to prove a global and more general result (see $[\mathrm{DT}]_2$). We were also able to apply Theorem 1 to some problems where the existence of a "hidden composition" turned out to be the main tool.

Recall that the *first normal space* $N_1^g(x)$ of an isometric immersion $g : \mathbf{M}^n \hookrightarrow \mathbf{Q}_c^{n+p}$ at $x \in \mathbf{M}^n$ is the normal subspace defined by

$$N_1^g(x) = \operatorname{span}\{\alpha_g(X,Y) : X, Y \in T_x\mathbf{M}^n\},$$

where α_g stands for the vector valued second fundamental form of g. From now on set $s_g(x) = \dim N_1^g(x)$.

Do Carmo and Dajczer considered in ([dC-D]) the question whether a Riemannian manifold $\mathbf{M}^n$ can be isometrically immersed into two space forms $\mathbf{Q}_c^{n+1}$ and $\mathbf{Q}_{\tilde{c}}^{n+p}$ with $c < \tilde{c}$. Given $f : \mathbf{M}^n \hookrightarrow \mathbf{Q}_c^{n+1}$ and $g : \mathbf{M}^n \hookrightarrow \mathbf{Q}_{\tilde{c}}^{n+p}$, they showed that if $s_g(x) \leq n-3$ at $x \in \mathbf{M}$, then there exists an umbilical subspace $\mathbf{U}(x) \subset \mathbf{T}_x\mathbf{M}$ for both immersions with $\dim \mathbf{U}(x) \geq n - s_g(x)$. It is a well known fact that the umbilical subspaces form an integrable smooth distribution with umbilical leaves on any open subset of $\mathbf{M}$ where they have constant dimension. In particular, a complete understanding of the geometric origin of the common umbilical foliation follows from the following result:

Theorem 2: *Suppose that $\mathbf{M}^n, n \geq 4$, admits isometric immersions $f : \mathbf{M}^n \hookrightarrow \mathbf{Q}_c^{n+1}$ and $g : \mathbf{M}^n \hookrightarrow \mathbf{Q}_{\tilde{c}}^{n+p}, c < \tilde{c}$, with $s_g(x) \leq n-3$ everywhere. Set $G = i \circ g$, where i is the inclusion of $\mathbf{Q}_{\tilde{c}}^{n+p}$ into $\mathbf{Q}_c^{n+p+1}$. Then there exists an open and dense subset $V \subset \mathbf{M}$ so that G/V is locally a composition.*

By the above, the immersions f and g can always be locally produced as the transversal intersection of the image of a local embedding of $\mathbf{Q}_c^{n+1}$ into $\mathbf{Q}_c^{n+p+1}$ with $\mathbf{Q}_{\tilde{c}}^{n+p}$ contained in $\mathbf{Q}_c^{n+p+1}$ as an umbilical hypersurface. The umbilical foliation is the intersection with $\mathbf{Q}_{\tilde{c}}^{n+p}$ of the leaves of the relative nullity foliation of h and has at least dimension $n - s_g$. The latter follows from the fact that the leaves of the relative nullity foliation of h have at least dimension $n - s_h + 1$ and $s_h = s_g$ from our assumption.

Using the classical Cartan-Schouten theory for conformally flat hypersurfaces for dimension $n \geq 4$, we obtain from the do Carmo-Dajczer result that only conformally flat manifolds can be realized as hypersurfaces of two space forms of different sectional curvature. After extending Theorem 2 to the case where $s_g = n - 2$, we were able to show that this is no longer the case for dimension $n = 3$.

Classifying Euclidean pseudoumbilical submanifolds of codimension 2 has been attempted by Yano-Ishihara ([Y-I]) and Otsuki ([Ot]). Recall that an isometric immersion $g : \mathbf{M}^n \hookrightarrow \mathbf{R}^{n+p}$ is said to be pseudoumbilical if the (normalized) mean curvature vector $\mathbf{H}$ is an umbilical vector field. If in addition $\mathbf{H}$ is parallel in the normal bundle of g, then it is easy to show that $g(\mathbf{M}^n)$ is contained in an umbilical hypersurface of $\mathbf{R}^{n+p}$ as a minimal submanifold. Among several related results, we obtained the following complete description:

Theorem 3: *Any umbilical-free pseudo-umbilical isometric immersion* $g : \mathbf{M}^n \hookrightarrow \mathbf{R}^{n+2}, n \geq 3$, *with non-parallel mean curvature vector is a composition* $g = h \circ f$. *Here* $f : \mathbf{M}^n \hookrightarrow \mathbf{R}^{n+1}$ *is a rotation hypersurface with axis* e_{n+1} *and principal curvatures* $\gamma, \beta > 0$ *of multiplicities* 1 *and* $n-1$, *respectively, satisfying* $-(n-1)\beta < \gamma < \beta$, *and* $h = Id \times d : \mathbf{R}^n \times \mathbf{R} \hookrightarrow \mathbf{R}^{n+2}$ *is the right cylinder over the unit speed plane curve* $d = d(s)$ *so that*

$$(\gamma - \beta)(\gamma + (n-1)\beta) + (k \cos\theta)^2 = 0,$$

where $k(s)$ *is the curvature of* $d(s)$ *and* $\theta(s)$ *is the angle between* e_{n+1} *and the principal direction correspondent to* γ.

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