Matemática Contemporânea, Vol. 1, 3–16
http://doi.org/10.21711/231766361991/rmc12

CONSTANT MEAN CURVATURE SURFACES BOUNDED BY A PLANE CURVE

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1. Introduction

Let M be a surface with smooth boundary ∂M and $x : M \to R^3$ be an immersion with constant mean curvature. Let Γ be a jordan curve on the plane $x_3 = 0$. Assume that x restricted to ∂M is a diffeomorphism onto Γ. "To determine all such immersions" is a problem that has received the recent attention of several geometers, such as M. Koiso, Ricardo Earp, Fabiano Brito, Harold Rosenberg, William H. Meeks and the author.

I must point out that, even when Γ is a circle, this problem is still unsolved although several partial results have been obtained in [K], [BEMR], [BE 1], [BE 2], [B].

M. Koiso [K] transformed this question into the following: how does a surface of constant mean curvature inherit a certain symmetry from its boundary? She proved that when x is an embedding and $x(M)$ does not intersect the plane $x_3 = 0$ outside of the region bounded by Γ, then, whenever Γ has a line of symmetry, $x(M)$ is symmetric with respect to the plane containing that line and perpendicular to the plane $x_3 = 0$.

The author [B] studied the case when Γ is a circle and $x(M)$ is contained in a sphere of radius $R = 1/ \mid H \mid$ and showed, without any further hypothesis, that $x(M)$ must be a spherical cap.

In [BEMR], [BE 1] and [BE 2] several partial results have been obtained for the embedded case under the hypothesis that, locally around Γ, $x(M)$ lies in one of the closed regions determined by the plane of Γ.

In this talk I will show how to prove an extension of my result for the case in which $x(M)$ is contained in a cylinder of radius $1/\mid H\mid$. In fact there is no advantage in considering the case of surfaces in R^3 instead of considering the case of hypersurfaces in R^{n+1}, for the purpose of this problem. Thus we will treat the more general case.

2. Preliminary Results

Let M be a n-dimensional manifold with smooth boundary ∂M. Let $x : M \to R^{n+1}$ be an immersion. Consider M endowed with the induced metric so that x becomes an isometry. $<.,.>$ will be used to represent the standard inner product of R^{n+1} and also the metric on M. We will assume the immersion x has constant mean curvature $H \neq 0$. It is worthwhile to remember that, in this case, M is orientable and so, we may choose a globally defined unit vector field $N : M \to R^{n+1}$. If Δ represents the Laplacian on M then

$$\Delta x = nHN. \tag{1}$$

For a proof of this fact see, for example, [L]. We observe that this is a well known formula that holds without the assumption of H being constant. As a consequence, the following equation is obtained.

$$(1/2)\Delta < x, x >= nH < x, N > +n. \tag{2}$$

If $\{e_1, \ldots, e_n\}$ is an orthonormal tangent frame field in an open set of M and $\theta_1, \ldots, \theta_n$ are its corresponding dual forms, then

$$dx = \sum_{i=1}^{n} \theta_i e_i \tag{3}$$

and

$$dN = \sum_{i,j=1}^{n} h_{ij} \theta_i e_j, \tag{4}$$

where $h_{ij} = h_{ji}$ are the coefficients of the second fundamental form of the immersion x with respect to this frame. It follows from (3) and (4) that

$$grad \mid x \mid^2 = 2\sum_{i-j}^{n} < e_i, x > e_i, \tag{5}$$

$$grad < x, N >= \sum_{i,j=1}^{n} h_{ij} < x, e_j > e_i. \tag{6}$$

Furthermore, if U represents any fixed vector of R^{n+1}, we also obtain

$$grad < x, u >= \sum_{i=1}^{n} < e_i, U > e_i, \tag{7}$$

$$grad < N, U >= \sum_{i,j=1}^{n} h_{ij} < e_j, U > e_i. \tag{8}$$

We will consider on M the standard volume element dM associated to the chosen orientation, given by

$$dM(X_1, \ldots, X_n) =< N, X_1 \wedge \ldots \wedge X_n >,$$

for any vector fields $X_1, \ldots, X_n$ tangent to M. We also consider on ∂M the volume element dS associated to the induced orientation.

Consider in R^{n+1} a fixed hyperplane P, and, on this hyperplane, a smooth hypersurface Γ which is the boundary of a compact subset D of P. It follows that Γ is itself compact and it is also orientable.

Theorem 2.1 *Let M be and orientable n-dimensional Riemannian manifold and $x : M \to R^{n+1}$ be an immersion that, restricted to ∂M, is a diffeomorphism between ∂M and Γ. Let U represent a constant unit vector of R^{n+1} and $\tilde{U}$ a unit vetor normal to the hyperplane P that contains Γ. Let N be a unit normal vector field along M. Then:*

$$\mid \int_M < U, N > dM \mid = \mid < U, \tilde{U} > \mid A(D)$$

where $A(D)$ represents the area of the compact set D.

Proof: We assume that the origin of R^{n+1} is a point of D, so that we have the following expression for the area of D:

$$A(D) = \frac{1}{n}\int_{\partial D} < x, \upsilon > dS \tag{9}$$

where υ is a unit vector field perpendicular to Γ in the hyperplane P. Denote by η the outward unit normal vector field along ∂M. It is clear that, for any $p \epsilon \partial M, N(p)$ and $\eta(p)$ are perpendicular to Γ, are orthogonal to each other and belong to the plane determined by $\tilde{U}$ and υ. Without loss of generality we assume that $\upsilon(p), \tilde{U}$ and $N(p), \eta(p)$ define the same orientation in the plane they span. Since $x(p)$ has no component in the direction of $\tilde{U}$ then we obtain the following elementary identity

$$< N, U >< \eta, x > - < \eta, U >< N, x > = < U, \tilde{U} >< x, \upsilon > .$$

Consequentely

$$\int_{\partial D}(< N, U >< \eta, x > - < \eta, U >< N, x >)dS = < U, \tilde{U} > nA(D). \tag{10}$$

Now we are going to express the left hand side of this equality in a different form.

$$\begin{aligned}
&\int_{\partial D}(< N, x >< \eta, U > - < N, U >< \eta, x >)dS = \\
= \ &\int_{\partial M}(< N, x > \eta[< x, U >] - (1/2) < N, U > \eta[| x |^2])dS = \\
= \ &\int_M(< N, x > \Delta < x, U > -(1/2) < N, u > \Delta | x |^2)dM + \\
+ \ &\int_M < grad < N, x >, grad < x, U >> dM - \\
- \ &\int_M < grad < N, U >, (1/2) grad | x |^2 > dM.
\end{aligned}$$

Now, using equations (1), (2), (5), (6), (7) and (8) we obtain

$$\int (< N, U >< \eta, x > - < \eta, U >< N, x >)dS = n\int_M < N, U > dM. \tag{11}$$

Therefore the theorem is proved.

Corollary 2.2 *Under the same hypothesis as in the theorem and assuming that x has constant mean curvature we have:*

a) $\mid \int_{\partial M} < \eta, U > dM \mid = \mid H \mid \mid < U, \tilde{U} > \mid nA(D);$

b) $\mid H \mid \leq L(\Gamma)/nA(D)$

where $L(\Gamma)$ means the volume of Γ.

Proof. Using that H is constant, equation (1), Stokes theorem and equation (3) we obtain:

$$\begin{aligned} nH \int_M < N, U > dM &= \int_M < nHN, U > dM = \\ &= \int_M \Delta < x, U > dM = \int_{\partial M} < \eta, U > dS. \end{aligned} \quad (12)$$

Now, (a) follows from theorem (2.1). To prove (b) first observe that:

$$\mid < \eta, U > \mid \leq 1.$$

It follows from (a) that

$$L(\Gamma) \geq \int_{\partial M} \mid < \eta, U > \mid dS \geq \mid \int_{\partial M} < \eta, U > dS \mid = \mid H \mid \mid < U, \tilde{U} > \mid nA(D).$$

Choosing $U = \tilde{U}$ the result is obtained.

Corollary 2.3 *Under the same hypothesis as in the theorem and assuming that x has constant mean curvature and that $\Gamma = S^{n-1}(1)$ then,*

$$\mid H \mid \leq 1.$$

Furthermore, if $U = \tilde{U}$, we obtain

$$\mid \int_{\partial M} < \eta, U > dM \mid = \mid H \mid vol(S^{-1}(1)).$$

Proof: This is just a consequence of the fact that, if $\Gamma = S^{n-1}(1)$ then D is a ball or radius one and $A(D) = L(D)/n$.

3. The Main Result

In this section we prove the following theorem:

Theorem 3.1 *Let M^n be a n-dimensional manifold with smooth boundary ∂M. Let $x : M \to R^{n+1}$ be an immersion with constant mean curvature $H \neq 0$ such that, restricted to ∂M, x is a diffeomorphism onto the Euclidean sphere $S^{n-1}(1)$ of the hyperplane $x_{n+1} = 0$. If $x(M)$ is contained in a closed solid cylinder of radius $1/\mid H \mid$ then x describes a spherical cap.*

Proof. From Corollary (2.3) we know that $\mid H \mid \leq 1$. Choose the normal vector field N of M so that $H > 0$. Let C be the solid cylinder of radius $1/H$ that contains $x(M)$ and let $\alpha(t) = p_0 + tv$ be a parametric description of its rotation axis. Consider, on this axis, the direction of v as the upward direction, and the direction of $-v$ as the downward direction. Let S_u be a closed hemisphere of radius $1/H$ whose equator lies in ∂C and whose center lies in the region of C below S_u. First of all, move S_u upward until it does not intersect $x(M)$. This is possible since both sets are compact. Now we move S_u downward until it touches $x(M)$ for the first time. In this position $x(M)$ lies completely in the closed convex region of C below S_u. We want to apply maximum principle to compare S_u and $x(M)$. For a good reference on the maximum principle for the equation $H = const.$ see [S].

Lemma 3.2 *Under the hypothesis of the theorem, if there is a point q interior to M such that $x(q)$ belongs to S_u and if $x(M)$ lies below S_u then $x(M)$ is a spherical cap.*

Proof (of the lemma) Let V be a small neighborhood of q such that x restricted to V is an embedding. Set $U = x(V)$ and $p = x(q)$. Since U lies below S_u and p belongs to $S_u \cap U$, then S_u and U are tangent at p. This is true even when p is a boundary point of S_u. In this last case, the point p will also belong to the boundary of the cylinder. Hence U will be tangent to the boundary of the cylinder at p and, hence, will be tangent to S_u at p. Now we are in position to apply maximum principle, provided that the unit normal vector fields of U and of S_u agree at p. If they do, then $S_u \cap U$ must contain an open set. By analyticity of the solutions of the equation $H = const$ we conclude that $x(M)$ must be a subset of the sphere of radius $1/H$ and, therefore, it is a spherical cap. If the normal vector fields of U and of S_u do not agree at p, then they must differ by a minus sign (since U and S_u are tangent at p). In this case we reflect S_u with respect to its tangent hyperplane at p. Now we apply the maximum principle to U and to this reflected surface to conclude that they must have a common open set. By analyticity we obtain that $x(M)$ is contained in a sphere of radius $1/H$, where this sphere lies fully above the hyperplane tangent to S_u at p. Since $x(M)$ lies, by hypothesis, entirely below it, then we have reached a contradiction. Therefore this case cannot occur. Thus the lemma is proved.

Lemma 3.3 *Under the hypothesis of the theorem, if there is a point q of ∂M such that $x(q)$ belongs to the interior of S_u being $T_{x(q)}M = T_{x(q)}S_u$ and assuming that $x(M)$ lies below S_u then $x(M)$ is a spherical cap.*

Proof (of the lemma) This lemma can be proved in the same way as the previous one. The extra hypothesis guarantees that $x(M)$ and S_u are comparable as in lemma (3.2).

From these two lemmas it follows that either $x(M)$ is a spherical cap or S_u touches $x(M)$ only at points of $S_{n-1}(1)$. These points are either points of ∂S_u or points of S_u where $x(M)$ and S_u are not tangent.

Let S_d be a closed hemisphere of radius $1/H$ whose equator lies in ∂C and whose center lies in the region of C above S_d. Observe that S_d can be translated upward or downward along C until it touches S_u along its equator to form a sphere of radius $1/H$ contained in C.

We can move S_d downward until it has no point in common with $x(M)$. Then move it up until it touches $x(M)$ the first time. The same argument used for S_u works for S_d to prove that $x(M)$ is either a spherical cap or S_d touches $x(M)$ only at points of $S^{n-1}(1)$ which are points either of ∂S_d or points of S_d where $x(M)$ and S_d are not tangent.

Lemma 3.4 *S_d and S_u do intersect.*

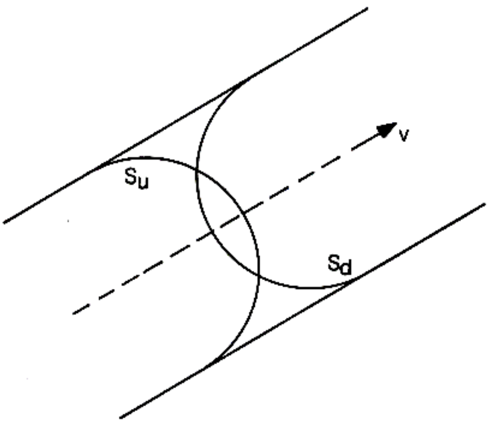

Fig. 1

Proof (of the lemma). Observe that if the hyperplane P that contains $S^{n-1}(1)$ intersects S_u along a full sphere $S^{n-1}(r)$, then $S^{n-1}(1)$ will be contained in the ball bounded by this sphere and so, in the convex hull of S_u. Since S_d must also intercept $S^{n-1}(1)$ then it intersects the convex hull of S_u and then S_u itself. This is an extreme case. Another extreme case occurs when the hyperplane P contains a line parallel to the axis of the cylinder. In this case P intersects S_u and S_d along two complementary hemispheres S_1 and S_2 of radius $r \geq 1$. If S_u and S_d do not intercept, then neither do S_1 and S_2, and there is no hope that $S^{n-1}(1)$ can be tangent to both S_1 and S_2 at some points p_1 and p_2. Hence, we reach a contradiction and conclude that S_u and S_d must intercept.

In the remaining cases, the plane P always cuts S_u and S_d in two spherical caps S_1 and S_2 of radius r_1 and r_2 respectively, with $r_1 \geq 1$ and $r_2 \geq 1$. If S_u and S_d do not intersect, then neither do S_1 and S_2. In fact one can find a hyperplane perpendicular to the axis of the cylinder such that the center of S_1 is above this hyperplane and the center of S_2 is below it. Indeed, if X represents a general point of R^{n+1}, then S_u, S_d and P have the equations:

$$S_u: \quad | X - P_u | = R \text{ and } < X - P_u, v > \geq 0,$$

$$S_d: \quad | X - P_d | = R \text{ and } < X - P_d, v > \leq 0,$$

$$P: \quad < X - P_0, \tilde{U} > = 0.$$

Represent by C_u the center of S_1 and by C_d the center of S_2. Then

$$C_u = P_u + \lambda_u \tilde{U},$$

$$C_d = P_d + \lambda_d \tilde{U},$$

where

$$\lambda_u = < P_0, \tilde{U} > - < P_u, \tilde{U} >,$$

$$\lambda_d = < P_0, \tilde{U} > - < P_d, \tilde{U} > .$$

If S_u and S_d do not intersect then we must have

$$P_u = P_d + av \quad a > 0.$$

Hence we obtain

$$\lambda_u = \lambda_d - a < v, \tilde{U} >$$

and

$$C_u = C_d + a(v - < v, \tilde{U} > \tilde{U}).$$

The term inside the parenthesis is simply the projection of v on the plane P. Since $< v - < v, \tilde{U} > \tilde{U}, v > = 1 - < v, \tilde{U} >^2 \geq 0$, we see that C_u lies above C_d. This proves the claim. We observe that both S_1 and S_2 are spherical caps whose boundary lies in the boundary of S_u and S_d respectively. Hence the boundaries lie in parallel hyperplanes of R^{n+1} and of P. It is also clear that the centers of S_1 and S_2 determine a line perpendicular to their boundaries. If p_1 belongs to $S^{n-1}(1) \cap S_1$ and P_2 belongs to $S^{n-1}(1) \cap S_2$ then the line through p_1 and the center of $S^{n-1}(1)$ will intercept the commom (revolution) axis of S_1 and S_2 at the center C_u of S_1, and the line through p_2 and the center of $S^{n-1}(1)$ will intersect the commom (revolution) axis of S_1 and S_2 at the center C_d of S_2. Since r_1 and r_2 are larger than or equal to one, then we must conclude that C_u lies below C_d which is a contradiction.

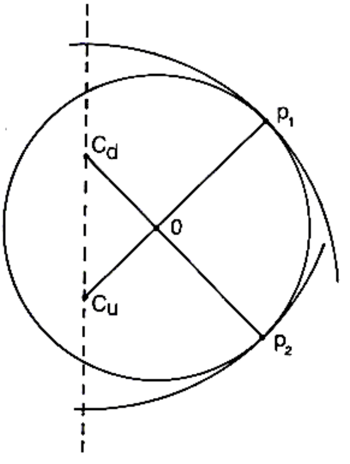

Fig. 2

This concludes the proof of the lemma 3.4.

From this lemma we conclude the following: there is a ball of radius $1/H$ that contains $x(M)$, and hence, there is a solid cylinder of radius $1/H$, whose axis is perpendicular to the hyperplane P, that contains $x(M)$.

We now repeat the entire procedure described above to this new cylinder to conclude, at the end, that: if $x(M)$ is not a spherical cap, then the image of the interior of M by x must lie in the interior of the intersection B of the regions above S_d and below S_u. Furthermore $S^{n-1}(1) = S_u \cap S_d$

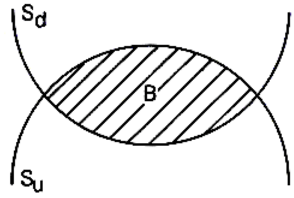

Fig. 3

It is clear that ∂B can be writen as $T_u \cap T_d$, where T_u is contained in S_u and T_d is contained in S_d. T_u and T_d are also spherical caps of radius $1/H$ and $S^{n-1}(1)$ is their common boundary.

If η_1 and η_2 are, respectively, the outward unit normal vector field of T_u and T_d along their common boundary, then it is elementary to see that

$$|<\eta_1,\tilde{U}>|=|<\eta_2,\tilde{U}>|=|H|. \tag{13}$$

As before, let η represent the outward unit normal vector field to M along ∂M. The condition that the image of interior of M, through x, is contained in B implies that, for $i=1,2$,

$$|<\eta,\tilde{U}>|\leq|<\eta_i,\tilde{U}>|. \tag{14}$$

Hence

$$|<\eta,\tilde{U}>|\leq|H|. \tag{15}$$

Now, using Corollary (2.3), one obtains

$$\begin{aligned} |H|\, vol(S^{n-1}(1)) &= \left|\int_{\partial M} <\eta,\tilde{U}> dS\right| \\ &\leq \int |<\eta,\tilde{U}>| dS \leq |H|\, vol(S^{n-1}(1)) \end{aligned}$$

Since the first and the last term in this chain of inequalities are the same we have equality for all terms.

In particular,

$$|<\eta, \tilde{U}>| = | H | . \tag{16}$$

But this implies that η must coincide with either η_1 or η_2. Hence $x(M)$ is tangent to either T_u or T_d along their common boundary. Now we may apply maximum principle to conclude that $x(M)$ must coincide with either T_u or T_v. But this is impossible since the image of the interior of M lies in the interior of B. Hence we have reached a contradiction. Therefore $x(M)$ is a spherical cap.

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