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IMMERSIONS OF KAHLER MANIFOLDS

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Let $f : M^{2m} \to N$ be an isometric immerson of a real $2m$ Kahler manifold into a Riemannian manifold. We say that f is (p, q)-geodesic if the standard type decomposition of the second fundamental form α gives $\alpha^{(p,q)} \equiv 0$.

For example, for a surface M $(m = 1)$ f is $(2,0)$-geodesic if the expression of α in isotermic parameters has the dz^2 coefficient identically zero. It means that f is umbilical.

The concept of $(1,1)$-geodesic maps is very important and it was introduced in the literature under various names as pluriharmonic maps and circular maps in [R], [U], [D-G], [D-T] and [D-R]. When N is a Kahler manifold, the concept of $(1,1)$-geodesic is intermediate between minimality and holomorphicity.

Our purpose is to present some results which show that some conditions on the curvature tensor of the ambient space N can give obstructions to the existence of minimal and $(1,1)$-geodesic immersions.

The first result in this direction is due to Dajczer and Rodrigues [D-R].

Theorem 1. *Let $f : M^{2m} \to E^n(c)$ be an isometric immersion of M into a space of constant curvature c.*

i) *If $c = 0$ then f is minimal if and only if f is $(1,1)$ -geodesic.*

ii) *If $c < 0$ then f minimal implies $m = 1$ (M is a surface).*

iii) *If $c > 0$ then f $(1,1)$-geodesic implies $m = 1$.*

This result can be generalized in several ways. For example, for conformally flat manifolds, for symmetric manifolds and for pinched Riemannian manifolds [F-R-T].

For conformally flat manifolds we have the following result: Let

$$r = \inf\{Ricc_x^N(v,v), x \in M, \| v \|_x = 1\},$$

$$R = \sup\{Ricc_x^N(v,v), x \in M, \| v \|_x = 1\},$$

$$s = \inf_{x\in M} S(N)_x \quad \text{and} \quad S = \sup_{x\in M} S(N)_x,$$

where $Ricc^N$ is the *Ricci* tensor of the manifold N and $S(N)$ is the normalized scalar curvature of N.

Theorem 2. *Let $f : M^{2m} \to N^n$ be an isometric immersion of M into a conformally flat Riemannian manifold.*

i) *If $S(N) > 0$, $r/S > \frac{n}{2(n-1)}$ and f is (1,1)-geodesic, then $m = 1$.*

ii) *If $S(N) < 0$, $R/s > \frac{n}{2(n-1)}$ and f is minimal, then $m = 1$.*

Theorem 1 i) can be extended to immersions in symmetric manifolds of non compact type. For immersions in pinched manifolds, we have the following result:

Theorem 3. *Let $f : M^{2m} \to N$ be an isometric immersion.*

i) *If the sectional curvatures K_N of N satisfy $\frac{1}{4} < K_N \leq 1$ and f is (1,1)-geodesic, then $m = 1$.*

ii) *If the sectional curvatures K_N of N satisfy $-\frac{1}{4} > K_N \geq -1$ and f is minimal, then $m = 1$.*

The following theorem of Dajczer and Thorbergson [D-T], shows that the above result can not be extended to the case where $\frac{1}{4} \leq K_N \leq 1$.

Theorem 4. *Let $f : M^{2m} \to P(c)$ an isometric immersion of M into the space $P(c)$ of holomorphic curvature c.*

i) *If $c > 0$ and f is (1,1)-geodesic then either $m = 1$ or f is holomorphic or antiholomorphic ($\pm$ holomorphic).*

ii) *If $c < 0$ and f is minimal then either $m = 1$ or f is $\pm$ holomorphic.*

This result can be generalized to immersions in complex Grassmannians [F-R-T].

Theorem 5. *Let $f : M^{2m} \to G_p(\mathbb{C}^{p+q})$ be an isometric immersion of M into the complex Grassmannian of p subspaces of $\mathbb{C}^{p+q}$. If f is (1,1)-geodesic, and $m \geq (p-1)(q-1)+1$ then f is $\pm$ holomorphic.*

In fact, each complex symmetric space N admits a number $n(N)$ such that if $f : M^{2m} \to N$ is (1,1)-geodesic and $m > n(N)$ then f is $\pm$ holomorphic. This number $n(N)$ is equal to the dimension of the biggest subspace L of $TN \otimes \mathbb{C}$, $L \not\subset T^{(1,0)}N$, $L \not\subset T^{(1,0)}N$ such that $[X, Y] = 0 \; \forall \; X, Y \in L$ where $[\ ,\ ]$ comes from the structure of homogeneous manifold of N and $T^{(1,0)}N(T^{(1,0)}N)$ is the subspace of $(1,0)$-vectors ($(0,1)$Vectors) of $TN \otimes \mathbb{C}$. Udagawa [U] gave another description of $n(N)$.

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