

## SUBMANIFOLDS OF CONSTANT NON POSITIVE CURVATURE

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The  $n$ -dimensional hyperbolic submanifolds  $M^n$  of the euclidean space  $\mathbf{R}^{2n-1}$  are in correspondence with solutions of a system of differential equations called the *intrinsic generalized sine-Gordon equation*. Similarly, flat submanifolds  $M^n$  of the unit sphere  $S^{2n-1}$ , correspond to solutions of the *intrinsic generalized wave equation*. [A], [BT].

In this note we consider particular solutions of these equations, which depend only on one variable, and we obtain the associated submanifolds as being generated by curves, in such a way that each point of the curve describes an  $(n-1)$ -dimensional torus. These are called *toroidal submanifolds*.

In the case of constant negative curvature the associated submanifolds are generated by curves which are given explicitly in terms of a family of elliptic functions when  $n \leq 3$ . For  $n \geq 4$ , the submanifold is generated by a tractrix in  $\mathbf{R}^n$ . These provide the classification of the hyperbolic toroidal submanifolds. (see Theorem 2.1 Proposition 2.1) As an immediate consequence, one concludes that there are no complete toroidal submanifolds  $M^n$  in  $\mathbf{R}^{2n-1}$ , with constant sectional curvature  $-1$ . This last result is also contained in Aminov's theorem [A2].

Similarly, the flat  $n$ -dimensional submanifolds of  $S^{2n-1}$  which correspond to the particular solutions of the intrinsic generalized wave equation, are the Clifford torus and the toroidal submanifolds generated by a family of curves contained in a two-dimensional sphere (see Theorem 2.1). Moreover, these give the classification of the flat toroidal submanifolds of  $S^{2n-1}$ . In particular, we

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conclude that the Clifford torus is the only complete flat toroidal submanifold of the sphere  $S^{2n-1}$ .

The classical Hilbert's theorem states that the hyperbolic plane  $H^2$  cannot be realized isometrically in  $\mathbf{R}^3$ . It has been conjectured for a long time that the hyperbolic  $n$ -space cannot be isometrically immersed in  $\mathbf{R}^{2n-1}$ ,  $n \geq 2$ . Partial results in this direction were obtained by Xavier [X] and Aminov [A2]. The latter considers solutions of the intrinsic generalized sine-Gordon equation which do not depend on one independent variable and shows that the hyperbolic submanifolds associated to such solutions are not complete. The difficulty in proving the conjecture lies in the study of global solutions of the system of differential equations (highly nonlinear) which depend on all independent variables.

In order to state our results we recall some preliminary results. We denote by  $M^n(K)$  an  $n$ -dimensional manifold of constant sectional curvature  $K$ . The following theorem for submanifolds  $M^n(K) \subset \bar{M}^{2n-1}(\bar{K})$ ,  $K < \bar{K}$ , is well known. (See [C], [M]).

**Theorem A** *Let  $M^n(K)$  be a riemannian manifold isometrically immersed in  $\bar{M}^{2n-1}(\bar{K})$ , such that  $K < \bar{K}$ . Then there exist local coordinates  $(x_1, \dots, x_n)$  such that the first and second fundamental forms are given by*

$$I = \sum_{i=1}^n a_{1i}^2 dx_i^2 \quad II = \sum_{i=1, j=2}^n g_{ji} a_{1i} dx_i^2 e_{n+j-1} \quad (1)$$

where  $\sum_{i=1}^n a_{1i}^2 = 1$  and  $e_{n+j-1}$  is an orthonormal frame normal to  $M$ .

From now on we normalize the curvatures by considering  $\bar{K} - K = 1$ . Under the conditions of Theorem A, one can show ([A], [T], [TT]) that the  $n \times n$  matrix function  $a = (a_{ij})$  satisfies the following system of equations

$$aa^t = I, \quad (2)$$

$$\frac{\partial a_{1i}}{\partial x_j} = a_{1j} h_{ji} \quad i \neq j, \quad (3)$$

$$\frac{\partial h_{ij}}{\partial x_i} + \frac{\partial h_{ji}}{\partial x_j} + \sum_{s \neq i, s \neq j} h_{si} h_{sj} = -K a_{1i} a_{1j}, \quad i \neq j, \tag{4}$$

$$\frac{\partial h_{ij}}{\partial x_s} = h_{is} h_{sj} \quad i, j, s \text{ distinct}, \tag{5}$$

$$\frac{\partial a_{ji}}{\partial x_i} = a_{ji} h_{ii} \quad i \neq l, \quad j \geq 2, \tag{6}$$

where the off diagonal matrix function  $h = (h_{ij})$  is defined by (3).

When  $K = 0$ , this is the *generalized wave equation* (GWE) and when  $K = -1$ , this is the *generalized sine-Gordon equation* (GSGE) (see [T] [TC] [TT]).

The above equations are equivalent to the Gauss and Codazzi equations. As a consequence of the fundamental theorem for submanifolds, given a matrix function  $a$ , which satisfies the equations (2)-(6), defined on a simply connected open subset  $\Omega \subset \mathbb{R}^n$ , there exists an isometric immersion  $X : \Omega \subset \mathbb{R}^n \rightarrow \bar{M}^{2n-1}(\bar{K})$  whose first and second fundamental forms are given by (1).

In the two-dimensional case, the Codazzi equation (6) is a consequence of the Gauss equation (4) and (5). Motivated by this result, intrinsic generalizations for the sine-Gordon and the wave equations were introduced in [BT] (see also [A]) as stated in the following result.

**Theorem B** *Let  $\Omega$  be a simply connected open subset of  $\mathbb{R}^n$ , with coordinates  $x_1, \dots, x_n$  endowed with a riemannian metric  $g = g_{ij}$  of constant sectional curvature  $K$ . Suppose  $g$  is diagonal and  $\text{tr } g \equiv 1$ . Then the smooth function  $v : \Omega \rightarrow \mathbb{R}^n$ ,  $v = (v_1, \dots, v_n)$  defined by  $v_i^2 = g_{ii}$  satisfies*

$$v v^t = 1 \tag{7}$$

$$\frac{\partial v_i}{\partial x_j} = v_j h_{ji} \quad i \neq j, \tag{8}$$

$$\frac{\partial h_{ij}}{\partial x_i} + \frac{\partial h_{ji}}{\partial x_j} + \sum_{s \neq i, s \neq j} h_{si} h_{sj} = -K v_i v_j, \quad i \neq j, \tag{9}$$

$$\frac{\partial h_{ij}}{\partial x_s} = h_{is} h_{sj}, \quad i, s, j \text{ distinct}, \tag{10}$$

where  $1 \leq i, j, s \leq n$  and  $h_{ij}$  is an off diagonal matrix function given by (7). Conversely, given a solution of (7)-(10) such that  $v_i(x) \neq 0$ , for all  $x \in \Omega$ ,

then  $g_{ij} = \delta_{ij}v_i^2$  defines a metric on  $\Omega$  which satisfies the above conditions.

As it was shown in [BT], whenever  $v_i(x) \neq 0$ , for all  $x \in \Omega$  and  $1 \leq i \leq n$ , then  $v$  and  $h$  satisfy

$$\frac{\partial v_i}{\partial x_i} = - \sum_{j \neq i} v_j h_{ij} \quad (11)$$

$$\frac{\partial h_{ji}}{\partial x_i} + \frac{\partial h_{ij}}{\partial x_j} + \sum_{s \neq i, s \neq j} h_{js} h_{is} = 0. \quad (12)$$

The system (7)-(12) is called the *Intrinsic Generalized Wave Equation* (IGWE) when the constant  $K = 0$  and the *Intrinsic Generalized Sine-Gordon Equation* (IGSGE) when  $K = -1$ .

In what follows, we consider solutions  $v_i$  of these equations, which define metrics as in Theorem B. Therefore, the intrinsic equations reduce to (7)-(10). Under these conditions, the relation between the generalized equations is stated in the following theorem (see [A] and Theorem 2 [BT]).

### Theorem C

- (i) If  $a$  is a solution of the GWE, then each row of  $a$ , whose elements do not vanish, is a solution of the IGWE.
- (ii) Suppose  $a$  is a solution of the GSGE. Then the first row of  $a$  is a solution of the IGSGE, whenever its elements do not vanish.
- (iii) Conversely, if  $v$  is a solution of the IGWE (resp. IGSGE) whose coordinate functions do not vanish on a simply connected domain  $\Omega \subset \mathbb{R}^n$ , then there exists a solution  $a$  on  $\Omega$  for the GWE (resp. GSGE) whose first row is  $v$ .

It follows from the above results that the solutions  $v$  of the IGWE (resp. IGSGE), which do not vanish on a simply connected domain  $\Omega \subset \mathbb{R}^n$  are in correspondence with the isometric immersions  $X : \Omega \rightarrow S^{2n-1}$  (resp.  $X : \Omega \rightarrow \mathbb{R}^{2n-1}$ ) of constant curvature  $K = 0$  (resp.  $K = -1$ ). Such an immersion is determined up to rigid motions and is called *the immersion associate to  $v$* .

As an example we consider the solution of the IGSGE

$$v_1 = \operatorname{tgh} x_1 \quad v_j = c_j \operatorname{sech} x_1 \quad j \geq 2$$

where  $c_j \neq 0$  and  $\sum_{j=2}^n c_j^2 = 1, x_1 > 0$ . The associated immersion is

$$(\operatorname{tgh} x_1 - x_1, c_2 \operatorname{sech} x_1 \cos x_2, c_2 \operatorname{sech} x_1 \sin x_2, \dots, c_n \operatorname{sech} x_1 \cos x_n, c_n \operatorname{sech} x_1 \sin x_n).$$

This submanifold is generated by a tractrix and each point of this curve describes a torus of dimension  $n - 1$  in  $\mathbf{R}^{2n-2}$ .

Similarly, the Clifford torus

$$X(x_1 \dots x_n) = (c_1 \cos x_1, c_1 \sin x_1, \dots, c_n \cos x_n, c_n \sin x_n)$$

is the immersion associated to the constant solution  $v_i = c_i \neq 0, \sum_{i=1}^n c_i^2 = 1$ , of the IGWE.

We introduce the definition of toroidal submanifold inspired by the examples above.

**Definition**

a) Let

$$\alpha(x_1) = (f_1(x_1), \dots, f_n(x_1)) \quad x_1 \in I \subset \mathbf{R}$$

be a parametrization of a regular curve in  $\mathbf{R}^n, n \geq 3$  such that  $f_i(x_1) \neq 0, i \geq 2, \forall x_1 \in I$ . The submanifold, which up to a rigid motion of  $\mathbf{R}^{2n-1}$ , is given by

$$(f_1(x_1), f_2(x_1) \cos x_2, f_2(x_1) \sin x_2, \dots, f_n(x_1) \cos x_n, f_n(x_1) \sin x_n),$$

is called a *toroidal submanifold*  $M^n$  of  $\mathbf{R}^{2n-1}$  generated by the curve  $\alpha$ .

b) Let

$$\beta(x_1) = (f_0(x_1), f_1(x_1), \dots, f_n(x_1)) \quad x_1 \in I \subset \mathbf{R}$$

be a parametrization of a regular curve in  $\mathbf{R}^{n+1}, n \geq 3$ , such that  $f_i(x_1) \neq 0, i \geq 2, \forall x_1 \in I$ . The submanifold, which up to a rigid motion of  $\mathbf{R}^{2n}$  is given by

$$(f_0(x_1), f_1(x_1), f_2(x_1) \cos x_2, f_2(x_1) \sin x_2, \dots, f_n(x_1) \cos x_n, f_n(x_1) \sin x_n)$$

is called a *toroidal submanifold*  $M^n$  of  $\mathbf{R}^{2n}$  generated by the curve  $\beta$ .

A toroidal submanifold is generated by a curve in such a way that each point of the curve describes a flat  $(n-1)$ -dimensional torus  $T^{n-1}$  contained in  $\mathbf{R}^{2n-2}$ .

## 1. Hiperbolic submanifolds of Euclidean space

The solutions of the IGSGE which depend only on one independent variable are given in the following result in terms of elliptic functions. These solutions are invariant by the local symmetry group of the equation (see [TW]).

**Proposition 1.1** *Let  $v = (v_1, \dots, v_n), n \geq 2$ , be a solution of the IGSGE which depends only on  $x_1$ , such that  $v_i(x_1) \neq 0, 1 \leq i \leq n$  for  $x_1$  in an open interval  $I \subset \mathbf{R}$ .*

(i) *If  $n=2$ , then*

$$\begin{aligned}(v_1')^2 &= (1 - v_1^2)(1 + c - v_1^2), \\ v_2^2 &= 1 - v_1^2,\end{aligned}\tag{13}$$

where  $c \in \mathbf{R}$  and  $1 + c > 0$ .

(ii) *If  $n = 3$ , then*

$$\begin{aligned}(v_1')^2 &= \left(\frac{b^2}{c^2} - v_1^2\right)(1 + c^2 - b^2 - v_1^2), \\ v_2^2 &= \frac{c^2}{1 + c^2} \left(\frac{b^2}{c^2} - v_1^2\right), \\ v_3^2 &= \frac{1}{1 + c^2} (1 + c^2 - b^2 - v_1^2),\end{aligned}\tag{14}$$

where  $b \neq 0, c \neq 0$  and  $1 + c^2 - b^2 > 0$ .

(iii) *If  $n \geq 4$  then*

$$\begin{aligned}v_1 &= \pm \operatorname{tgh}(x_1 - a), \\ v_j &= c_j \operatorname{sech}(x_1 - a), \quad j \geq 2\end{aligned}\tag{15}$$

where  $a, c_j \neq 0$  are real numbers and  $\sum_{j=2}^n c_j^2 = 1$ .

We observe that when  $b^2 = c^2$ , the equations (14) give  $v$  in terms of elementary functions exactly as in (15). Moreover, for each solution given in Proposition 1.1 there exists a value  $x_0 \in \mathbf{R}$  such that  $v_1(x_0) = 0$ .

One can show that the isometric immersions  $M^n \hookrightarrow \mathbf{R}^{2n-1}$  of constant curvature  $K \equiv -1$  associated to these solutions are, up to rigid motions, toroidal submanifolds.

**Theorem 1.1** *The submanifolds  $M^n \subset \mathbf{R}^{2n-1}$  with constant sectional curvature  $K \equiv -1$ , associated to the solutions of the IGSGE given in Proposition 1.1 are, up to a rigid motion.*

(i) *The surface of rotation generated by the curve*

$$\left( -\int v_1^2 dx_1, \frac{v_2}{\sqrt{1+c}} \right), \quad \text{if } n = 2,$$

where  $v_1, v_2$  are defined by (13).

(ii) *The toroidal submanifolds generated by the curves*

$$\left( -\int v_1^2 dx_1, \frac{v_2}{\sqrt{1+c^2-b^2}}, \frac{cv_3}{b} \right), \quad \text{if } n = 3,$$

where  $v_i, 1 \leq i \leq 3$ , are given by (14) and

(iii) *The toroidal submanifold generated by the curve*

$$\left( \operatorname{tgh} x_1 - x_1, c_2 \operatorname{sech} x_1, \dots, c_n \operatorname{sech} x_1 \right), \quad \text{if } n \geq 4,$$

where  $\sum_{j=2}^n c_j^2 = 1$ .

**A sketch of the proof:** Let  $v = (v_1, \dots, v_n)$  be a solution of the IGSGE as in Proposition 1.1. By Theorem C there exists a solution  $(a_{ij})$  of the GSGE such that  $a_{1j} = v_j$ , for  $1 \leq j \leq n$ . Moreover,  $a_{ij}, 2 \leq i \leq n$ , also depend only on  $x_1$ .

The submanifold associated to  $v$  is determined by the first and second fundamental forms given by

$$I = \sum_{i=1}^n a_{1i}^2 dx_i^2 \quad \text{and} \quad II = \sum_{j=2, i=1}^n (a_{ji} a_{1i} dx_i^2) e_{n+j-1}$$

where  $e_{n+j-1}, 2 \leq j \leq n$ , is an orthonormal basis for the normal bundle. The proof of the theorem follows from the fundamental theorem for submanifolds of the euclidean space, which says that the immersion  $X : \Omega \subset \mathbf{R}^n \rightarrow \mathbf{R}^{2n-1}$ , where  $\Omega$  is simply connected, is determined, up to a rigid motion, by solving the following system of differential equations for the vector fields  $X_{x_i}$  and  $e_{n+s-1}$ , where  $1 \leq i \leq n, 2 \leq s \leq n$ :

$$\begin{aligned} X_{x_i x_j} &= \sum_{k=1}^n \Gamma_{ij}^k X_k + \sum_{r=2}^n a_{1i} a_{ri} \delta_{ij} e_{n+r-1}, \quad 1 \leq i, j \leq n \\ e_{n+s-1, x_i} &= -\frac{a_{si}}{a_{1i}} X_{x_i}, \quad 2 \leq s \leq n. \end{aligned}$$

We can show that these submanifolds characterize the toroidal submanifolds of  $\mathbf{R}^{2n-1}$  with constant curvature  $-1$ . Consequently, we conclude that these are no complete toroidal submanifolds  $M^n \subset \mathbf{R}^{2n-1}$  with  $K \equiv -1$ .

### Theorem 1.2

a) The submanifolds  $M^n \subset \mathbf{R}^{2n-1}, n \geq 3$ , given by Theorem 1.1 are, up to a rigid motion, the only toroidal submanifolds of  $\mathbf{R}^{2n-1}$  with constant sectional curvature  $K \equiv -1$ .

b) There are no complete toroidal submanifolds  $M^n \subset \mathbf{R}^{2n-1}$ , with  $K \equiv -1$ .

## 2. Flat submanifolds of the unit sphere

We obtain the results analogous to those in section 1, by considering the correspondence between flat submanifolds of the sphere and solutions of the IGWE. The solutions of the IGWE which depend only on one independent variable are given by the following result.

**Proposition 2.1** Let  $v = (v_1, \dots, v_n), n \geq 2$ , be a solution of the IGWE which depends only on  $x_1$ , such that  $v_i(x_1) \neq 0, \forall 1 \leq i, \leq n$ , for  $x_1$  in an open interval  $I \subset \mathbf{R}$ . Then there exists  $j_0, 2 \leq j_0 \leq n$  such that

$$v_1 = \sqrt{1 - c^2} \sin(\lambda x_1 - a)$$



$$\begin{aligned} v_{j_0} &= \pm\sqrt{1-c^2} \cos(\lambda x_1 - a) \\ v_j &= c_j, \quad j \geq 2, j \neq j_0 \end{aligned} \tag{16}$$

where  $\sum_{j=3}^n c_j^2 = c^2$ ,  $c_j \neq 0$ ,  $\lambda, a, c_j \in \mathbf{R}$  and  $|c| < 1$ . When  $\lambda = 0$ , then  $I = \mathbf{R}$  and  $a \neq l\pi/2$  for any integer  $l$ ; when  $\lambda \neq 0$ , then  $x_1 \in I$  such that  $l\pi/2 < \lambda x_1 - a < (l+1)\pi/2$ .

We observe that when  $\lambda = 0$ , the solution given by (16) is constant. The flat isometric immersions in the unit sphere,  $M^n \subset S^{2n-1} \subset \mathbf{R}^{2n}$  associated to the solutions above are toroidal surfaces generated by a family of curves of  $S^2$  and the Clifford torus. More precisely:

**Theorem 2.1** *The flat submanifolds  $M^n \subset S^{2n-1}$ , associated to the solutions of the IGWE given in Proposition 2.1 are up to a rigid motion,*

(i) *The Clifford torus*

$$(c_1 \cos x_1, c_1 \sin x_1, \dots, c_n \cos x_n, c_n \sin x_n)$$

where  $v_i = c_i \neq 0$ ,  $1 \leq i \leq n$ , whenever  $\lambda = 0$ .

(ii) *The toroidal submanifolds generated by the curves*

$$(\delta f_0, \delta f_1, \delta f_2, c_3, \dots, c_n), \quad \delta = \sqrt{1-c^2}$$

where  $r^2 = \lambda^2 + 1$ ,  $c^2 = \sum_{j=3}^n c_j^2$  and

$$\begin{aligned} f_0 &= \frac{\lambda}{r} \sin r x_1 \cos(\lambda x_1 - a) - \cos r x_1 \sin(\lambda x_1 - a) \\ f_1 &= \frac{\lambda}{r} \cos r x_1 \cos(\lambda x_1 - a) + \sin r x_1 \sin(\lambda x_1 - a) \\ f_2 &= -\frac{1}{r} \cos(\lambda x_1 - a) \end{aligned}$$

whenever  $\lambda \neq 0$ .

The proof of this theorem follows from the fundamental theorem for submanifolds of the sphere. One can show that, up to a rigid motion, the immersion

is given by

$$X = (\delta f_0, \delta f_1, \delta f_2 \cos r x_2, \delta f_2 \sin r x_2, c_3 \cos x_3, c_3 \sin x_3, \dots, c_n \cos x_n, c_n \sin x_n),$$

**Remark 2.1** The family of curves which generate the toroidal submanifolds of Theorem 2.1 (ii) is contained on a two-dimensional sphere. Moreover, we can show the following:

**Theorem 2.2**

- a) The submanifolds  $M^n \subset S^{2n-1}$  given by Theorem 2.1 are, up to a rigid motion, the only toroidal flat submanifolds of  $\mathbb{R}^{2n}$  contained in  $S^{2n-1}$ .  
b) The only complete toroidal flat submanifolds  $M^n \subset S^{2n-1}$  is the Clifford torus.

Complete details of the proofs will appear elsewhere.

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