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MINIMAL SURFACES OF FINITE TOTAL CURVATURE IN $\mathbb{H} \times \mathbb{R}$

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We dedicate this work to Renato Tribuzy, on his 60'th birthday.

1 Introduction

We consider complete minimal surfaces Σ in $\mathbb{H} \times \mathbb{R}$, $\mathbb{H}$ the hyperbolic plane. Let $C(\Sigma)$ denote the total curvature of Σ, $C(\Sigma) = \int_\Sigma K dA$, K the intrinsic curvature of Σ. We shall prove that $C(\Sigma)$ is an integer multiple of 2π, when it is finite. We give examples of such Σ with total curvature $-2\pi m$, m any non-negative integer.

In $\mathbb{R}^3$, complete minimal surfaces of finite total curvature have total curvature an integer multiple of -4π. This results from the Gauss map of the surface, that extends meromorphically to the conformal compactification. In $\mathbb{H} \times \mathbb{R}$, we have no conformal Gauss map. We have the holomorphic quadratic differential of the harmonic height function: projection on the $\mathbb{R}$ factor of $\mathbb{H} \times \mathbb{R}$.

Now we describe a simply connected example. Let Γ be an ideal polygon in $\mathbb{H}$ with $m+2$ vertices at infinity, $2m+2$ sides, $A_1, B_1, A_2, B_2, ..., A_{m+1}, B_{m+1}$. Let D be the convex hull of Γ.

In [3], the authors find necessary and sufficient conditions on the "lengths" of the A_i and B_j which ensure the existence of a minimal graph $u : D \longrightarrow \mathbb{R}$, taking the values $+\infty$ on each A_i and $-\infty$ on each B_j.

They prove the graph of such a u is complete and of total curvature $-2\pi m$.

The Γ obtained from the $m+2$ roots of unity satisfies the "length" conditions. Thus this gives examples of total curvature $-2\pi m$ for each integer $m \geqslant 1$.

For $m=0$, take $\Sigma = \gamma \times \mathbb{R}$, γ a complete geodesic of $\mathbb{H}$. It would be interesting to construct non-simply connected examples of finite total curvature. For example an annulus of total curvature -4π.

2 Preliminaries

We consider $X : \Sigma \to \mathbb{H} \times \mathbb{R}$ a minimal surface conformally embedded in $\mathbb{H} \times \mathbb{R}$, $\mathbb{H}$ the hyperbolic plane. We denote by $X = (F, h)$ the immersion where $F : \Sigma \longrightarrow \mathbb{H}$ is the vertical projection to $\Sigma = \Sigma \times (0)$, and $h : \Sigma \longrightarrow \mathbb{R}$ the horizontal projection. We consider local conformal parameters $z = x + iy$ on Σ. The metric induced by the immersion is of the form $\mathrm{d}s^2 = \lambda^2(z)|\,\mathrm{d}z|^2$.

If $\mathbb{H}$ is isometrically embedded in $\mathbb{L}^3$ the Minkowski space, the mean curvature vector is (see B.Lawson [8], page 8)

$$2\overrightarrow{H} = (\Delta X)^{T_X(\mathbb{H}\times\mathbb{R})} = ((\Delta F)^{T_F\mathbb{H}}, \Delta h) = 0$$

Then F is a harmonic map and h is a real harmonic function. In the following, we will use the unit disk model for $\mathbb{H}$. We will note $(D, \sigma^2(u)|\,\mathrm{d}u|^2)$ the disk with the hyperbolic metric $\sigma^2(u)|\,\mathrm{d}u|^2$. We will denote $|v|^2_\sigma = \sigma^2|v|^2$, $\langle v_1, v_2\rangle_\sigma = \sigma^2\langle v_1, v_2\rangle$ where $|v|$ and $\langle v_1, v_2\rangle$ stands for the standard norm and inner product in $\mathbb{R}^2$. The harmonic map equation in the complex coordinate $u = u_1 + iu_2$ of D (see [12], page 8) is

$$F_{z\bar{z}} + 2(\log \sigma \circ F)_u F_z F_{\bar{z}} = 0 \qquad (1)$$

where $2(\log \sigma \circ F)_u = 2\bar{F}(1 - |F|^2)^{-1}$. In the theory of *harmonic maps* there are two global objects to consider. One is the holomorphic *quadratic Hopf* *differential* associated to F:

$$Q(F) = (\sigma \circ F)^2 F_z \bar{F}_z (\mathrm{d}z)^2 := \phi(z)(\mathrm{d}z)^2 \tag{2}$$

The function ϕ depends on z, whereas $Q(F)$ does not. An other object is the *complex coefficient of dilatation* (see Alhfors [1]) of a quasi-conformal map, which does not depend on z, a conformal parameter on Σ:

$$a = \frac{\overline{F_{\bar{z}}}}{F_z}$$

Since we consider conformal immersions, we have

$$|F_x|_\sigma^2 + (h_x)^2 = |F_y|_\sigma^2 + (h_y)^2$$

$$\langle F_x, F_y \rangle_\sigma + h_x.h_y = 0$$

hence $(h_z)^2(\mathrm{d}z)^2 = -Q(F)$ (see [10]).

Then the zeroes of Q are double and we can define η as the holomorphic one form $\eta = \pm 2i\sqrt{Q}$. The sign is chosen so that:

$$h = \mathrm{Re} \int \eta \tag{3}$$

When X is a conformal immersion then the unit normal vector n in $\mathbb{H}\times\mathbb{R}$ has third coordinate:

$$\langle n, \frac{\partial}{\partial t} \rangle = n_3 = \frac{|g|^2 - 1}{|g|^2 + 1}$$

where

$$g^2 := -\frac{F_z}{\overline{F_{\bar{z}}}} = -\frac{1}{a} \tag{4}$$

Then we define the function ω on Σ (which has poles where Σ is horizontal) by $n_3 = \tanh\omega$. By identification we have

$$\omega = \frac{1}{2} \ln \frac{|F_z|}{|F_{\bar{z}}|} \tag{5}$$

Using the equations above (2),(4) we can express the differential $\mathrm{d}F$ independently of z by:

$$\mathrm{d}F = F_{\bar{z}}\,\mathrm{d}\bar{z} + F_z\,\mathrm{d}z = \frac{1}{2\sigma\circ F}\overline{g^{-1}\eta} - \frac{1}{2\sigma\circ F}g\eta \tag{6}$$

The metric $\mathrm{d}s^2 = \lambda|\,\mathrm{d}w|^2$ is given [10] in a local coordinate z by:

$$\mathrm{d}s^2 = (|F_z|_\sigma + |F_{\bar{z}}|_\sigma)^2|\,\mathrm{d}z|^2 \tag{7}$$

Thus combining equations (6) and (7), we derive the metric in terms of g and η by

$$\mathrm{d}s^2 = \frac{1}{4}(|g|^{-1} + |g|)^2|\eta|^2 = 4\cosh^2\omega|Q| \tag{8}$$

We remark that the zeroes of Q correspond to the poles of ω so that the immersion is well defined. Moreover the zeroes of Q are points of Σ, where the tangent plane is horizontal.

It is a well known fact (see [12] page 9) that harmonic mappings satisfy the Böchner formula:

$$\Delta_0 \ln\frac{|F_z|}{|F_{\bar{z}}|} = -2K_{\mathbb{H}}J(F) \tag{9}$$

where $J(F) = \sigma^2\left(|F_z|^2 - |F_{\bar{z}}|^2\right)$ is the Jacobian of F with $|F_z|^2 = F_z\overline{F_z}$. Hence taking into account (2), (4), (5) and (9):

$$\Delta_0\omega = 2\sinh(2\omega)|Q| \tag{10}$$

where Δ_0 denote the laplacian in the euclidean metric $|\,dz|^2$. From this we deduce

$$\Delta_\Sigma \omega = n_3$$

where Δ_Σ is the Laplacian in the metric ds^2.

The Gauss curvature is given by:

$$K_\Sigma = K(X_x, X_y) + K_{ext} = -\tanh^2\omega - \frac{|\nabla\omega|^2}{4\cosh^4\omega|Q|}$$

(the sectional curvature of the tangent plane to Σ at a point z is $-n_3^2$.) The total curvature is defined by

$$C(\Sigma) = \int_\Sigma K_\Sigma \, dA$$

3 Minimal surfaces of finite total curvature

Theorem 3.1. *Let X be a complete minimal immersion of Σ in $\mathbb{H}\times\mathbb{R}$ with finite total curvature. Then*

a) Σ is conformally $\overline{M} - \{p_1...., p_n\}$, a Riemann surface punctured in a finite number of points.

b) Q is holomorphic on M and extends meromorphically to each puncture. If we parametrize each puncture p_i by the exterior of a disk of radius R_0, and if $Q(z) = z^{2m_i}(dz)^2$ at p_i then $m_i \geqslant -1$.

c) The third coordinate of the unit normal vector $n_3 \longrightarrow 0$ uniformly at each puncture.

d) The total curvature is a multiple of 2π:

$$\int(-KdA) = 2\pi\left(2 - 2g - 2k - \sum_{i=1}^{n} m_i\right).$$

Proof. The proof of this theorem uses arguments of harmonic diffeomorphisms theory as can be found in the work of Han, Tam, Treibergs and Wan [4], [5], [13], and Minsky [11].

The conformal type is an application of Huber's theorem ([7]). Σ is conformally a compact Riemann surface minus a finite number of points (the ends).

b) We consider $M(r_0) = M - \cup_i D(p_i, r_0)$; the surface minus a finite number of disks removed around the punctures p_i. Around each puncture we consider a conformal parametrization of the punctured disk $D^*(p_i, r_0)$. We parametrize these ends by the exterior of the disk of radius R_0 in $\mathbb{C}$. In this parameter we express the metric as $\mathrm{d}s^2 = \lambda^2|\,\mathrm{d}z|^2$ with $\lambda^2 = 4\cosh^2\omega|\phi|$ in a conformal parameter z. Then $-K\lambda^2 = \Delta_0 \ln\lambda$ where $\Delta_0 = 4\partial^2_{z\bar{z}}$.

Let us define $u = \ln\cosh^2\omega$, a subharmonic function by Böchner's formula:

$$\Delta_0 u = 8\sinh^2\omega|\phi| + \frac{2|\nabla\omega|^2}{\cosh^2\omega} \geqslant 0.$$

The function u is globally defined, since ω is globally defined on Σ.

Step 1: We prove that the holomorphic quadratic differential Q has a finite number of zeroes on M.

Since the zeroes are isolated, Q has a finite number of zeroes on the compact part $M(r)$. Then we assume that there is a disk $D^*(p_i, r)$ which contains an infinite number of zeroes of Q, $\{z_i\}$. We parametrize conformally this disk on the exterior of the disk of radius R_0. In this parameter $Q(z) = \phi(z)(\mathrm{d}z)^2$ and if Δ_0 is the laplacian in the flat metric $|\,\mathrm{d}z|^2$ at the puncture:

$$\Delta_0 \ln|\phi| = \sum 2\pi\delta_{z_i}$$

Then with $-K\lambda^2 - \frac{1}{2}\Delta_0 \ln|\phi| = \frac{1}{2}\Delta_0 u$ we have on the annulus $C(R) = \{R_0 \leqslant |z| \leqslant R\}$:

$$\int_{C(R)}(-K\,\mathrm{d}A) - m\pi \quad = \tfrac{1}{2}\int_{C(R)} \Delta_0 u \geqslant 0$$

Then m has to be finite and $\int_{C(r)} \Delta_0 u \leqslant C_0$

Step 2: An upper bound.

$$\begin{aligned}\int_{C(R)} \Delta_0 u = \int_{\partial C(R)} \frac{\partial u}{\partial n} \quad &= \int_0^{2\pi} \frac{\partial u}{\partial R} R\,\mathrm{d}\theta - \int_0^{2\pi} \frac{\partial u}{\partial R} R_0\,\mathrm{d}\theta \\ &= R\frac{\mathrm{d}}{\mathrm{d}R}\int_0^{2\pi} u(R,\theta)\,\mathrm{d}\theta - R_0\int_0^{2\pi} \frac{\partial u}{\partial r}\,\mathrm{d}\theta \leqslant 2C_0.\end{aligned}$$

Now let $I(r) := \int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta$. Then

$$\frac{\mathrm{d}}{\mathrm{d}R} I(R) \leqslant \frac{C_1}{R}$$

$$I(R) - I(R_0) \leqslant C_1 \ln\frac{R}{R_0}.$$

Then for $R >> R_0$ large we have, with $a > 0, b > 0$:

$$I(r) \leqslant a\ln r + b.$$

Step 3: Since ϕ has a finite number of zeroes, we prove at each puncture

$$\cosh^2\omega|\phi| \leqslant \beta|z|^{\alpha}|\phi|.$$

To prove ϕ extends meromorphically to the punctures, we will use a theorem of Osserman [9] (recall that the metric is complete). For $R > R_0$, ϕ is without zeroes and for $|z| = R$ large enough, u is subharmonic, hence

$$\begin{aligned} u(z) &\leqslant \frac{4}{\pi R^2}\int_{B(z,R/2)} u \\ &\leqslant \frac{4}{\pi|z|^2}\int_{B(0,3|z|/2)-B(0,|z|/2)} u \\ &\leqslant \frac{4}{\pi|z|^2}\int_{|z|/2}^{3|z|/2} I(r) r dr \leqslant \frac{4}{\pi|z|^2}\int_{|z|/2}^{3|z|/2} (a\ln r + b) r dr \\ &\leqslant \alpha \ln|z| + \beta \end{aligned}$$

Then

$$2\ln\lambda = u + \ln|\phi| \leqslant \alpha\ln|z| + \beta + \ln|\phi|$$

and

$$\lambda^2 = \cosh^2\omega|\phi| \leqslant e^{\beta}|z|^{\alpha}|\phi|$$

Thus the function ϕ extends meromorphically to the puncture by Osserman [9].

Step 4: We now prove that the function $\phi(z) = z^{2m}$ with, $m \geqslant -1$ at each puncture.

If $m \leqslant -2$, then we can conformally parametrize the end on the punctured disk by $w = 1/z$. Then $Q(w) = \psi(w)(\mathrm{d}w)^2$ with $\phi(1/w) = w^4\psi(w)$, where $\psi(w)$ has a pole of order $2m+4$. If $2m+4 \leqslant 0$, the following integral is finite:

$$\int_{D(p_i,r)} |\phi| \, \mathrm{d}z < \infty$$

Now, by the finite total curvature hypothesis we will show the area of the end is finite:

$$\int_D -K_\Sigma \,\mathrm{d}A = \int_D \Delta_0 \ln\lambda = \int_D 8\sinh^2\omega|\phi| + \int_D \frac{2|\nabla\omega|^2}{\cosh^2\omega}$$
$$= \int_D 8\cosh^2\omega|\phi| - \int_D 8|\phi| + \int_D \frac{2|\nabla\omega|^2}{\cosh^2\omega}$$

Hence

$$\mathrm{Area}(D) = \int_D \cosh^2\omega|\phi| < \infty.$$

But a complete end of Σ has infinite area by the monotonicity formula (see [2]).

c) Now we prove that $n_3 \longrightarrow 0$ uniformly at each puncture. We adapt estimates on positive solutions of *sinh*-Gordon equations by Minsky [11], Wan [13] and Han [5] to our context.

At each puncture we can choose R_0 such that $\phi(z)$ is without zeroes on $|z| \geqslant R_0/2$. Since $\phi(z)$ is without zeroes, the minimal surface is transverse to horizontal sections $\mathbb{H}\times\{c\}$ and we parametrize locally simply connected subdomains of the end by $w = \frac{i}{2}(x_3 + ix_3^*) = \int\sqrt{\phi}\,\mathrm{d}z$ so that $|\,\mathrm{d}w|^2 = |\phi(z)||\,\mathrm{d}z|^2$ is a flat metric. If we consider $z \in C_{R_0}$, then on the disk $D(z,|z|/2)$, we have the conformal coordinate $w = \int\sqrt{\phi(z)}\,\mathrm{d}z$, with the flat metric $|\,\mathrm{d}w|^2 = |\phi||\,\mathrm{d}z|^2$. In this metric, under the hypothesis that $m \geqslant -1$, the disk $D(z,|z|/2)$ contains a ball of radius at least $c\ln|z|$, where c is independent of z.

The function ω satisfies the *sinh*-Gordon equation

$$\Delta_{|\phi|}\omega = 2\sinh 2\omega$$

where $\Delta_{|\phi|}$ is the laplacian in the flat metric $|\,\mathrm{d}w|^2$. For $|z| \geqslant R_0$, we can find a disk of radius at least $r = c\ln|z|$ around z in the $|\,\mathrm{d}w|^2$ metric. When z is large, the radius r diverges to $+\infty$.

Then for R_0 large enough we can find a disk with radius 1 in the $|\,\mathrm{d}w|^2$ metric around any point z with $|z| \geqslant R_0$. On this disk $D_{|\phi|}(z,1)$, we consider the hyperbolic metric given by (w is $w-z$ in the following step):

$$\mathrm{d}\sigma^2 = \mu^2|\,\mathrm{d}w|^2 := \frac{4}{(1-|w|^2)^2}|\,\mathrm{d}w|^2$$

Then μ takes infinite values on $\partial D_{|\phi|}(z,1)$ and since the curvature of this metric is $K=-1$, the function $\omega_2 = \ln\mu$ satisfies the equation

$$\Delta_{|\phi|}\omega_2 = e^{2\omega_2} \geqslant e^{2\omega_2} - e^{-2\omega_2} = 2\sinh 2\omega_2$$

Now we apply a maximun principle to bound ω above as in Wan [13]. The same holds with ($\tilde{\omega} = -\omega$):

Let $\eta = \omega - \omega_2$. Then

$$\Delta\eta = e^{2\omega} - e^{-2\omega} - e^{2\omega_2} = e^{2\omega_2}(e^{2\eta} - e^{-4\omega_2}e^{-2\eta} - 1)$$

which can be written in the metric $\mathrm{d}\tilde{\sigma}^2 = e^{2\omega_2}|\,\mathrm{d}w|^2$, as

$$\Delta_{\tilde{\sigma}}\eta = e^{2\eta} - e^{-4\omega_2}e^{-2\eta} - 1.$$

Since ω_2 goes to $+\infty$ on the boundary of the disk, the function η is bounded above and attains its max at an interior point p_0, $\eta(p_0) = \bar{\eta}$ and $\Delta\bar{\eta} \leqslant 0$. At this point we have

$$e^{2\bar{\eta}} - e^{-4\omega_2}e^{-2\bar{\eta}} \leqslant 1.$$

Hence

$$e^{2\eta} \leqslant \frac{1+\sqrt{1+a^2}}{2}$$

where $a = e^{-2\omega_2(p_0)} \leqslant \sup \frac{1}{\mu^2} \leqslant \frac{1}{4}$. Then at any point of the disk

$$\omega \leqslant \omega_2 + \frac{1}{2}\ln\frac{1+\sqrt{1+1/4}}{2}.$$

The same estimate holds with $\widetilde{\omega} = -\omega$. Then at z (i.e. $w=0$) we have

$$|\omega(z)| \leqslant \ln 4 + \frac{1}{2}\ln\frac{1+\sqrt{1+1/4}}{2} := K_0$$

uniformly on $R \geqslant R_0$. Using this estimate we can apply a maximun principle as in Minsky [11]. For $|z|$ large, we can find a disk $D_{|\phi|}(z,r)$ with r large too.

We consider the function

$$F(x,y) = \frac{K_0}{\cosh r}\cosh\sqrt{2}x\cosh\sqrt{2}y.$$

Then $F \geqslant K_0 \geqslant \omega$ on $\partial D_{|\phi|}(z,r)$. Since $\Delta F = 4F$, we apply the maximun principle to have $\omega \leqslant F$. If p_0 is a point where $\omega(p_0) \geqslant F(p_0)$ is a minimun of $F-\omega$, then $0 \leqslant \omega(p_0) \leqslant \sinh\omega(p_0)$ and

$$\Delta(F-\omega) = 4F - 2\sinh 2\omega \leqslant 4(F(p_0)-\omega(p_0)) \leqslant 0.$$

Hence $\omega \leqslant F$ on the disk. We have $|\omega| \leqslant F$ by considering the same argument with $F+\omega$. Hence

$$|\omega(z)| \leqslant \frac{K_0}{\cosh r}$$

and $|\omega| \to 0$ uniformly at the puncture i.e. the tangent plane become vertical.

d) Now we compute the total curvature. We apply Gauss-Bonnet on the compact piece $M(r) = M - \sum_{1 \leqslant i \leqslant k} D(p_i, r)$ and we obtain

$$\int_{M(r)} K_\Sigma \, \mathrm{d}A + \int_{\partial M(r)} k_g = 2\pi(2 - 2g - k)$$

Here k_g is the geodesic curvature of $\partial M(r) = \Gamma_1 \cup ...\Gamma_k$ on the surface $M(r)$. Now consider a puncture p_i parameterized on $R \geqslant R_0$. We consider $w = x + iy$ a parametrization of the punctured disk (with $w = \int \sqrt{\phi}\,\mathrm{d}z$). In the w-plane, if $\phi(z) = z^{2m}$ there are $2m+2$ horizontal asymptotic directions i.e. directions with $\mathrm{Im}(w) = 0$ (diverging curves at zero level) which define some angular sector in $R \geqslant R_0$. Now for $C_1 >> 0$ large, we consider the "polygon" $\Gamma(C_1)$ which is the union of segments of curves $\mathrm{Re}(w) = \pm C_1$ and $\mathrm{Im}(w) = \pm C_1$, alternatively. At each change of direction the exterior angle is $\pi/2$. These curves, with $\Gamma_i = \{R = R_0\}$ bound an annulus $\Omega(r, C_1, p_i)$ and

$$\int_\Omega K_\Sigma \, \mathrm{d}A + \int_{\Gamma(C_1)} k_g - \int_{\Gamma_i} k_g = -(2m+2)\pi$$

Now we let $C_1 \to \infty$. If we prove $\int_{\Gamma(C_1)} k_g \to 0$, we will establish that

$$\int_M K_\Sigma \, \mathrm{d}A = 2\pi(2 - 2g - 2k - \sum_i m_i)$$

where $\phi(z) = z^{2m_i}$ at each p_i.

Now we prove $\int_{\Gamma(C_1)} k_g \to 0$. This fact comes from the exponential decreasing property of the function ω. First we prove

$$\int_{\mathrm{Im}(w)=C_1} k_g \, \mathrm{d}s \to 0$$

The curve $\gamma_1 = \{\mathrm{Im}(w) = C_1\}$ is a horizontal curve at level C_1, parameterized by $\mathrm{Re}(w) = x$. In Hauswirth [6], we find an expression of the curvature k_{γ_1} of the curve in $\mathbb{H}\times\mathbb{R}$ as function of ω. In the w variable (recall that the Hopf differential is $Q = \frac{1}{4}(\mathrm{d}w)^2$):

$$k_{\gamma_1}(x) = \frac{-\omega_y}{\cosh\omega}$$

Now we need a gradient estimate of ω. Schauder's estimate gives (with the exponential decreasing property of ω proved above):

$$|\omega|_{2,\alpha} \leqslant C(|\sinh\omega|_{0,\alpha} + |\omega|_0) \leqslant Ce^{-R}.$$

On the curve $x + iC_1$, we have $|\nabla\omega| \leqslant Ce^{-|C_1|}e^{-\sqrt{x^2C_1^{-2}+1}}$ and

$$\int_{\mathrm{Im}(w)=C_1} |k_g|\,\mathrm{d}s \leqslant \int_{\mathrm{Im}(w)=C_1} |k_{\gamma_1}|\,\mathrm{d}s = \int_{-\infty}^{+\infty} |\omega_y|dx \leqslant C|C_1|e^{-|C_1|}$$

which is converging to zero as $|C_1| \to +\infty$.

Now we prove

$$\int_{\mathrm{Re}(w)=C_1} k_g ds \to 0.$$

We compute the curvature k_{γ_2} in $\mathbb{H}\times\mathbb{R}$ of the curve $\gamma_2(y) = (F(C_1, y), y)$ with $|F'(C_1, y)|^2 := \sinh^2\omega(C_1, y)$ and $\gamma_2' = X_y$. We denote by ∇ the connexion in $\mathbb{H}\times\mathbb{R}$. For ν an unit vector field along γ_2 we have

$$k_{\gamma_2}\nu = \frac{1}{\cosh^2\omega}\nabla_{X_y}X_y - \frac{\omega_y\sinh\omega}{\cosh^3\omega}X_y \tag{11}$$

First recall that $Q(w) = \frac{1}{4}(\mathrm{d}w)^2$ and the second fundamental form is given by (see [6])

$$II = \omega_y \, \mathrm{d}x \otimes \mathrm{d}x - \omega_y \, \mathrm{d}y \otimes \mathrm{d}y - \omega_x \, (\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x).$$

Since the unit normal vector of Σ is given by $n = (\beta F_y, \tanh\omega)$ with $\beta = \frac{-1}{\cosh\omega \sinh\omega}$, we obtain

$$\nabla_{X_y} n = \frac{\omega_y}{\cosh^2\omega} \frac{\partial}{\partial t} + \nabla_{X_y}(\beta F_y) = \frac{\omega_y}{\cosh^2\omega} X_y + \frac{\omega_x}{\cosh^2\omega} X_x.$$

Hence, with $\nabla_{X_y} \frac{\partial}{\partial t} = 0$

$$\nabla_{X_y} X_y = \nabla_{X_y} F_y = \omega_y \tanh^{-1} \omega F_y - \omega_x \tanh\omega F_x$$

which give with (11):

$$k_{\gamma_2}\nu = \frac{-\omega_y \sinh\omega}{\cosh^3\omega} \frac{\partial}{\partial t} + \frac{\omega_y}{\sinh\omega \cosh^3\omega} F_y - \frac{\omega_x \sinh\omega}{\cosh^3\omega} F_x$$

and

$$|k_g|^2 \leqslant |k_{\gamma_2}|^2 = \frac{\omega_y^2 + \omega_x^2 \sinh^2\omega}{\cosh^4\omega} \to 0$$

Now one can argue as above to prove the result.

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