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PSEUDO-PARALLEL IMMERSIONS IN SPACE FORMS

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Dedicated to Professor Manfredo P. do Carmo on his 70th birthday

Abstract

In this note we introduce a new class of isometric immersions, the pseudo-parallel immersions, defined as the extrinsic analogue of pseudo-symmetric manifolds (in the sense of R. Deszcz) and as a direct generalization of semi-parallel immersions. We will prove some basic results and discuss some examples.

1. Introduction

A Riemannian manifold M is *locally symmetric* if its Riemannian curvature tensor R is parallel, i.e. $\nabla R = 0$, where ∇ is the Levi-Civita connection extended to act on tensors. A natural generalization of these manifolds are *semi-symmetric* manifolds, i.e. Riemannian manifolds which satisfy $R(X,Y)\cdot R = 0$, for all vectors X and Y tangent to M, where the curvature operator $R(X,Y)$ acts as a derivation on R. Semi-symmetric manifolds were introduced by E. Cartan in the twenties and classified by Szabó only in the early eighties (see [S_1], [S_2]). The study of totally umbilical submanifolds of semi-symmetric manifolds, leads to the concept of *pseudo-symmetric* manifolds, i.e. Riemannian manifolds M which satisfy the condition $R(X,Y)\cdot R = \phi X \wedge Y \cdot R$, for all vectors X and Y tangent to M and some smooth function ϕ of M, where the endomorphism $X \wedge Y : TM \to TM$, defined by $X \wedge Y(Z) = \langle Y,Z\rangle X - \langle X,Z\rangle Y$, is extended to act on tensors (see [AD]). The class of pseudo-symmetric manifolds naturally generalizes the class of semi-symmetric manifolds, but there exist many examples of pseudo-symmetric manifolds which are not semi-symmetric (see e.g. [D] and references therein). In the last decade, several studies involving this class of manifolds have been published, but a full classification is not yet available.

In the theory of isometric immersions in space forms, conditions analogous to local symmetry and semi-symmetry have been introduced and studied in the last two decades. Firstly Ferus and others introduced the concept of *parallel* immersions, i.e. immersions whose second fundamental forms are parallel (see (1) below), and classified such immersions (see [F], [BR], [Ta]). Afterwards, Deprez and others introduced and studied the concept of *semi-parallel* immersions, i.e. immersions whose second fundamental forms are anihilated by the curvature tensor of the ambient space, extended to act on tensors (see (4) below). A full classification of these immersions is not available yet, but several partial results are known (see e.g. [D_1], [D_2], [L], [Di], [DN], [AM]).

In this paper we will give the definition of *pseudo-parallel* immersions (see (5) below), as an extrinsic analogue of pseudo-symmetric manifolds and a natural generalization of semi-parallel immersions. We will give some examples of pseudo-parallel hypersurfaces which are not semi-parallel and of pseudo-symmetric hypersurfaces which are not pseudo-parallel. We will give a proof of the following result:

Theorem 1.1. *Let $f : M^n \to Q^N(c)$ be a pseudo-parallel immersion. If $H(p) = 0$ and $\phi(p) \geq c$, then p is a geodesic point, i.e. the second fundamental form vanishes at p.*

As a consequence, we get:

Corollary 1.2. *Let $f : M^n \to Q^{n+2}(c)$ be a pseudo-parallel immersion. If $\phi(p) \geq c$ or $H(p) \neq 0$, then $R^\perp(p) = 0$.*

Finally we will make some remarks on work on progress on the subject.

2. Definitions and Examples

Let M^n be a connected n-dimensional Riemannian manifold and let $Q^N(c)$ be an N-dimensional manifold with constant sectional curvature c. Given an isometric immersion $f : M^n \to Q^N(c)$, we will denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $Q^N(c)$ respectively and $\nu(f)$ will denote the normal bundle of the immersion. Then the second fundamental form $\alpha : TM \otimes TM \to \nu(f)$ is given by $\alpha(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where X, Y are tangent vectors. As usual, $\nabla^\perp$ and $R^\perp$ will denote the normal connection and respective curvature tensor and, if $\xi \in \nu(f)$, $A_\xi : TM \to TM$ will denote the Weingarten operator in the ξ direction, $A_\xi X = \nabla^\perp_X \xi - \tilde{\nabla}_X \xi$. A_ξ and α are symmetric and related by $\langle A_\xi X, Y\rangle = \langle \alpha(X,Y), \xi\rangle$. The local geometry of f is described by the above data and the basic equations:

$$\begin{aligned}\langle R(X,Y)Z, W\rangle &= c\langle X \wedge Y(Z), W\rangle + \langle \alpha(X,W), \alpha(Y,Z)\rangle \quad \text{(Gauss)} \\ &\quad - \langle \alpha(X,Z), \alpha(Y,W)\rangle,\end{aligned}$$

$$\left(\overline{\nabla}\alpha\right)(X,Y,Z) = \left(\overline{\nabla}\alpha\right)(X,Z,Y), \qquad \text{(Codazzi-Mainardi)}$$

$$\left\langle R^\perp(X,Y)\xi, \eta\right\rangle = \langle [A_\xi, A_\eta] X, Y\rangle, \qquad \text{(Ricci)}$$

where $X \wedge Y : TM \to TM$ is the endomorphism defined by $X \wedge Y(Z) = \langle Y, Z\rangle X - \langle X, Z\rangle Y$ and $\overline{\nabla}\alpha : TM \otimes TM \otimes TM \to \nu(f)$ is the covariant derivative of α defined by

$$\begin{aligned}\left(\overline{\nabla}\alpha\right)(X,Y,Z) &= \left(\overline{\nabla}_Z\alpha\right)(X,Y) \\ &= \nabla^\perp_Z[\alpha(Y,Z)] - \alpha(\nabla_Z Y, Z) - \alpha(Y, \nabla_Z Z). \qquad (1)\end{aligned}$$

The second covariant derivative $\overline{\nabla}^2\alpha : TM \otimes TM \otimes TM \otimes TM \to \nu(f)$ is defined by

$$\begin{aligned}\left(\overline{\nabla}^2\alpha\right)(X,Y,Z,W) &= \left(\overline{\nabla}_W\overline{\nabla}_Z\alpha\right)(X,Y) \\ &= \nabla^\perp_W\left[\left(\overline{\nabla}_Z\alpha\right)(X,Y)\right] - \left(\overline{\nabla}_Z\alpha\right)(\nabla_W X, Y) \qquad (2) \\ &\quad - \left(\overline{\nabla}_Z\alpha\right)(X, \nabla_W Y) - \left(\overline{\nabla}_{\nabla_W Z}\alpha\right)(X,Y).\end{aligned}$$

Then we have:

$$\left(\overline{\nabla}_X\overline{\nabla}_Y\alpha\right)(Z,W) - \left(\overline{\nabla}_Y\overline{\nabla}_X\alpha\right)(Z,W)$$

$$= R^{\perp}(X,Y)[\alpha(Z,W)] - \alpha(R(X,Y)Z,W) - \alpha(Z,R(X,Y)W). \quad (3)$$

The immersion $f : M^n \to Q^N(c)$ is *parallel* if $\overline{\nabla}\alpha = 0$ and is *semi-parallel* if

$$\overline{R}(X,Y)\cdot\alpha := \left(\overline{\nabla}_X\overline{\nabla}_Y\alpha\right) - \left(\overline{\nabla}_Y\overline{\nabla}_X\alpha\right) = 0, \quad (4)$$

for all $X, Y \in TM$. Clearly parallelism implies semi-parallelism, but the converse is not true in general, as it is shown, for example, in the classification of semi-parallel hypersurfaces in $Q^{n+1}(c)$, given by Deprez for $c = 0$ in [D_2] and by Dillen for $c \neq 0$ in [Di]. In those papers it has also been proved that if $f : M^n \to Q^N(c)$ is a parallel or a semi-parallel immersion, then M is a locally symmetric or a semi-symmetric manifold, respectively.

Definition 2.1. *An isometric immersion $f : M^n \to Q^N(c)$ is pseudo-parallel if there exists a smooth function $\phi : M \to \mathbb{R}$ such that*

$$\overline{R}(X,Y)\cdot\alpha = \phi(p)\, X\wedge Y\cdot\alpha, \quad (5)$$

fo all $p \in M$ and $X, Y \in T_pM$, where the linear endomorphism $X \wedge Y : T_pM \to T_pM$ is extended to act on α as follows:

$$[X\wedge Y\cdot\alpha](Z,W) := -\alpha(X\wedge Y(Z),W) - \alpha(Z, X\wedge Y(W)), \quad (6)$$

for all $Z, W \in T_pM$.

It follows easily from the equations of Gauss and Ricci that the condition of pseudo-parallelism can be rewritten as:

$$\begin{aligned} R^{\perp}(X,Y)[\alpha(Z,W)] &= \alpha(R(X,Y)Z,W) + \alpha(Z,R(X,Y)W) \quad (7)\\ &\quad -\phi(p)\langle Y,Z\rangle\alpha(X,W) + \phi(p)\langle X,Z\rangle\alpha(Y,W)\\ &\quad -\phi(p)\langle Y,W\rangle\alpha(Z,X) + \phi(p)\langle X,W\rangle\alpha(Z,Y). \end{aligned}$$

Every semi-parallel immersion is a pseudo-parallel immersion with $\phi = 0$. The converse is false in general. In fact, we will show that there exist *proper* pseudo-parallel immersions, i.e. not semi-parallel. First we need Proposition

2.2, which follows easily from the definitions and the Gauss equation, to the Weingarten operator of a pseudo-parallel hypersurface:

Proposition 2.2. *A hypersurface $f : M^n \to Q^{n+1}(c)$ is a pseudo-parallel immersion with $\overline{R}(X,Y) \cdot \alpha = \phi(X \wedge Y) \cdot \alpha$ if and only if the Weingarten operator has at most two different eigenvalues λ, μ. If $\lambda \neq \mu$, then $\phi = \lambda\mu + c$.*

Example 1. A hypersurface of revolution in $Q^{n+1}(c)$ is locally a warped product $I \times_h M^{n-1}(\delta)$ of an open interval I and a space of constant curvature $M^{n-1}(\delta)$ with warping function $h > 0$, where δ is $1, 0$ or -1 depending on the sign of c. The principal curvatures are given by

$$\lambda = -\frac{\sqrt{\delta - ch^2 - \dot{h}^2}}{h} \qquad \text{and} \qquad \mu = \frac{\ddot{h} + ch}{\sqrt{\delta - ch^2 - \dot{h}^2}}, \tag{8}$$

where λ has multiplicity at least $n-1$ (see [CD]). Then it follows from Proposition 2.2 that every rotation hypersurface is pseudo-parallel. Moreover, from (8), such a hypersurface is not semi-parallel if the function h has nonvanishing second derivative.

The notion of pseudo-parallelism is an extrinsic analogue of pseudo-symmetric manifolds in the following sense:

Proposition 2.3. *Let $f : M^n \to Q^N(c)$ be a ϕ-pseudo-parallel immersion. Then M^n is a ϕ-pseudo-symmetric manifold.*

Proof. First we observe that the definition of $R(X,Y) \cdot R$ can be rewritten as

$$\begin{aligned} [R(U,V) \cdot R](X,Y,Z) &= R(U,V)[R(X,Y)Z] - R(R(U,V)X,Y)Z \\ &\quad - R(X,R(U,V)Y)Z - R(X,Y)[R(U,V)Z]. \end{aligned}$$

By the Gauss equation we obtain

$$\begin{aligned} &\langle [R(U,V) \cdot R](X,Y,Z), W \rangle \\ = &\ -c\{\langle (X \wedge Y)(Z), R(U,V)W \rangle + \langle (X \wedge Y)(R(U,V)Z), W \rangle \end{aligned} \tag{9}$$

$$
\begin{aligned}
&+\langle(\left[R(U,V)X\right]\wedge Y)(Z),W\rangle+\langle(X\wedge\left[R(U,V)Y\right])(Z),W\rangle\}\\
&+\langle\alpha(X,Z),\alpha(R(U,V)W,Y)+\alpha(W,R(U,V)Y)\rangle\\
&-\langle\alpha(Y,Z),\alpha(R(U,V)X,W)+\alpha(X,R(U,V)W)\rangle\\
&+\langle\alpha(W,Y),\alpha(R(U,V)X,Z)+\alpha(X,R(U,V)Z)\rangle\\
&-\langle\alpha(X,W),\alpha(R(U,V)Y,Z)+\alpha(Y,R(U,V)Z)\rangle\,.
\end{aligned}
$$

Finally we use (7) and again the equation of Gauss, on the right side of (9) to obtain that

$$
\begin{aligned}
&\langle[R(U,V)\cdot R](X,Y,Z),W\rangle\\
=\;&\phi(p)\{\langle\alpha(X,Z),\alpha(Y,(U\wedge V)(W))\rangle-\langle\alpha(X,(U\wedge V)(W)),\alpha(Y,Z)\rangle\\
&+\langle\alpha((U\wedge V)(X),Z),\alpha(Y,W)\rangle-\langle\alpha((U\wedge V)(X),W),\alpha(Y,Z)\rangle\\
&+\langle\alpha(X,Z),\alpha((U\wedge V)(Y),W)\rangle-\langle\alpha(X,W),\alpha((U\wedge V)(Y),Z)\rangle\\
&+\langle\alpha(X,(U\wedge V)(Z)),\alpha(Y,W)\rangle-\langle\alpha(X,W),\alpha(Y,(U\wedge V)(Z))\rangle\}\\
=\;&\phi(p)\left[(U\wedge V)\cdot R\right](X,Y,Z)\,.
\end{aligned}
$$

□

Example 2. The converse of Proposition 2.3 is false in general. In fact, let $f: M^3 \to Q^4(c)$ be a hypersurface with principal curvatures equal to 0 and $\pm\lambda \neq 0$. Then it follows easily that $R(X,Y)\cdot R = c(X\wedge Y)\cdot R$ in p, i.e. M^3 is pseudo-symmetric in p with $\phi(p) = c$. In particular, if $c > 0$ and $f: M^3 \to S^4(c)$ is a Cartan hypersurface, that is, M^3 is a compact minimal hypersurface in the sphere $S^4(c)$ with constant principal curvature 0 and $\pm\sqrt{3c}$, then M^3 is *proper pseudo-symmetric* (not semi-symmetric) with $\phi(p) = c$, for every p in M (see [?], Examples 1,2). It follows from Proposition 2.2 that the Cartan hypersurface $f: M^3 \to S^4(c)$ is not pseudo-parallel. Moreover, if $c = 0$, it is well-known that the cone over a Clifford torus of dimension two, $g: \mathbb{R}_+ \times S^1(1) \times S^1(1) \to \mathbb{R}^4$, is a minimal hypersurface which has three different principal curvatures 0, $\pm\lambda \neq 0$. It follows that g is a semi-symmetric hypersurface, but by the Proposition 2.2 cannot be a pseudo-parallel hypersurface. This shows that pseudo-symmetry and semi-symmetry do not imply pseudo-parallelism, in general. However:

Proposition 2.4. *If $n \geq 3$ and $f : M^n \to I\!R^{n+1}$ is an isometric immersion, then f is a proper pseudo-parallel immersion if and only if M^n is a proper pseudo-symmetric manifold.*

Proof. It follows directly from Proposition 2.2 and Theorem 1 given in [DDV].

We observe explicitly that, in the example above, the cone over the Clifford torus is not proper pseudo-symmetric.

3. Proof of 1.1 and 1.2

Let $f : M^n \to Q^N(c)$ be an isometric immersion. We choose a local orthonormal frame field $\{e_1, ..., e_n, e_{n+1}, ..., e_N\}$ adapted to f. Then the components of the second fundamental form α are given by $h_{ij}^\sigma = \langle \alpha(e_i, e_j), e_\sigma \rangle$, $i, j \in \{1, \dots n\}$, $\sigma \in \{n+1, \dots N\}$. The components of the covariant derivative of α are given by $h_{ijk}^\sigma = \left\langle \left(\overline{\nabla}_{e_k} \alpha\right)(e_i, e_j), e_\sigma \right\rangle = \overline{\nabla}_{e_k} h_{ij}^\sigma$, and the components of the second covariant derivative of α are given by $h_{ijkl}^\sigma = \left\langle \left(\overline{\nabla}_{e_l} \overline{\nabla}_{e_k} \alpha\right)(e_i, e_j), e_\sigma \right\rangle = \overline{\nabla}_{e_l} h_{ijk}^\sigma = \overline{\nabla}_{e_l} \overline{\nabla}_{e_k} h_{ij}^\sigma$. We have that f is pseudo-parallel if and only if

$$h_{ijkl}^\sigma = h_{ijlk}^\sigma - \phi \left\{ \delta_{ki} h_{lj}^\sigma - \delta_{li} h_{kj}^\sigma + \delta_{kj} h_{il}^\sigma - \delta_{lj} h_{ik}^\sigma \right\}, \tag{10}$$

where $i, j, k, l = 1, ..., n$ and $\sigma = n+1, ..., N$.

Recall that the Laplacian $\triangle h_{ij}^\sigma$ of h_{ij}^σ is defined by $\triangle h_{ij}^\sigma = \sum_{k=1}^{n} h_{ijkk}^\sigma$. Then

$$\frac{1}{2} \triangle \left(\|\alpha\|^2 \right) = \sum_{i,j,k=1}^{n} \sum_{\sigma=n+1}^{N} h_{ij}^\sigma h_{ijkk}^\sigma + \left\| \overline{\nabla} \alpha \right\|^2, \tag{11}$$

where $\|\alpha\|^2 = \sum_{i,j=1}^{n} \sum_{\sigma=n+1}^{N} \left(h_{ij}^\sigma\right)^2$ is the square of the length of the second fundamental form and $\left\|\overline{\nabla}\alpha\right\|^2 = \sum_{i,j,k=1}^{n} \sum_{\sigma=n+1}^{N} \left(h_{ijk}^\sigma\right)^2$. Then we have (see [Ch], pp. 90-91):

$$\frac{1}{2}\triangle\left(\|\alpha\|^2\right)=\sum_{i,j=1}^{n}\sum_{\sigma=n+1}^{N}h_{ij}^{\sigma}\left(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}H^{\sigma}\right)+nc\left(\|\alpha\|^2-n\|H\|^2\right) \tag{12}$$

$$-\sum_{\sigma,\tau=n+1}^{N}[(trace(A_\sigma\circ A_\tau))^2+\|[A_\sigma,A_\tau]\|^2]+H^{\tau}trace(A_\sigma\circ A_\tau\circ A_\sigma)]+\left\|\overline{\nabla}\alpha\right\|^2,$$

where $H^\sigma=\sum\limits_{k=1}^{n}h_{kk}^{\sigma}$, $\|H\|^2=\frac{1}{n^2}\sum\limits_{\sigma=n+1}^{N}(H^\sigma)^2$. Now, by (10) and the Codazzi equation $h_{ijk}^{\sigma}=h_{ikj}^{\sigma}$ on the right side of (11), we get:

$$\frac{1}{2}\triangle\left(\|\alpha\|^2\right)=\sum_{i,j=1}^{n}\sum_{\sigma=n+1}^{N}h_{ij}^{\sigma}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}H^{\sigma})+n\phi(\|\alpha\|^2-n\|H\|^2)+\|\overline{\nabla}\alpha\|^2. \tag{13}$$

Then, in general, for a pseudo-parallel immersion, we have:

$$\begin{aligned} n(\phi-c)(\|\alpha\|^2-n\|H\|^2)+\sum_{\sigma,\tau=n+1}^{N}[(trace(A_\sigma\circ A_\tau))^2 \\ +\|[A_\sigma,A_\tau]\|^2-H^{\tau}trace(A_\sigma\circ A_\tau\circ A_\sigma)]=0. \end{aligned} \tag{14}$$

Now, if $H(p)=0$ then $H^\sigma=0$, $\forall\sigma$, and we have (at p):

$$n(\phi(p)-c)\|\alpha\|^2+\sum_{\sigma,\tau=n+1}^{N}[(trace(A_\sigma\circ A_\tau))^2+\|[A_\sigma,A_\tau]\|^2]=0.$$

If $\phi(p)\geq c$, then $trace(A_\sigma\circ A_\tau)=0,\ \forall\,\sigma,\tau$. In particular $\|A_\sigma\|^2=trace(A_\sigma\circ A_\sigma)=0$, hence $\alpha(p)=0$.

□

We will prove now Corollary 1.2:

First observe that $R^\perp(X,Y)H=0$ for any pseudo-parallel immersion. In fact, let ξ be a normal vector and $\{e_1,\dots,e_n\}$ an orthonormal basis of eigenvectors of A_ξ. Then:

$$\langle R^\perp(X,Y)H,\xi\rangle=\frac{1}{n}\sum_{i=1}^{n}\langle R^\perp(X,Y)[\alpha(e_i,e_i)],\xi\rangle=\frac{1}{n}\sum_{i=1}^{n}\langle\alpha(R(X,Y)e_i,e_i)$$

$$+\alpha(e_i,R(X,Y)e_i)+\phi(p)[\langle Y,e_i\rangle\alpha(X,e_i)+\langle X,e_i\rangle\alpha(Y,e_i)$$

$$- \langle Y, e_i \rangle \alpha(X, e_i) + \langle X, e_i \rangle \alpha(e_i, Y)], \xi \rangle = \frac{2}{n} \sum_{i=1}^{n} \langle \alpha(R(X,Y)e_i, e_i), \xi \rangle$$

$$= \frac{2}{n} \sum_{i=1}^{n} \langle A_\xi e_i, R(X,Y)e_i \rangle = 0.$$

Now, if $H(p) = 0$, by Theorem 1.1, $\alpha = 0$ and consequently $R^\perp(X,Y) = 0$. If $H(p) \neq 0$, $R^\perp(X,Y)$ is an antisymmetric operator on a 2-dimensional space with non trivial kernel, hence is zero.

□

4. Final Remarks

Similarly to the case of parallel and semi-parallel immersions it is possible to associate to a pseudo-parallel immersion a Jordan triple system at every point of the manifold. The study of the algebraic properties of such a system leads to many results in the theory of semi-parallel immersions. Research in progress (that we hope will appear shortly) indicates that the same should occur also for the pseudo-parallel case. For example we can give a pure linear algebraic proof of 1.1 and 1.2. We hope that such a point of view will help in better understanding the connection between parallel, semi-parallel and pseudo-parallel immersions. Also this point of view gives a better understanding of the curvature tensor of (a class of) pseudo-symmetric submanifolds which leads to results on the topology of the manifold like the following (see [ALM]):

Theorem 1.1. *Let $f : M^n \to Q^N(c)$ be a pseudo-parallel immersion, M compact and simply connected, with $\phi \geq 0$. Then M is a Riemannian product of manifolds of the following type:*

(a) Manifolds homeomorphic to spheres;

(b) Manifolds biholomorphic to complex projective spaces;

(c) Symmetric spaces of compact type.

Moreover, if ϕ is not identically zero, then M is of the first type.

The above fact, together with corollary 1.2. implies that in the codimension two case and $c = 0$, the manifold is homeomorphic to a sphere or the immersion is the product of two convex embeddings.

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