Matemática Contemporânea, Vol. 44, 1–8
http://doi.org/10.21711/231766362015/rmc4421

Complexity of the oriented coloring in planar, cubic oriented graphs

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Abstract

An *oriented k-coloring* of an oriented graph $\vec{G} = (V, \vec{E})$ is a partition of V into k subsets such that there are no two adjacent vertices belonging to the same subset and all the arcs between a pair of subsets have the same orientation. The decision problem k-ORIENTED CHROMATIC NUMBER (OCN$_k$) consists of an oriented graph $\vec{G}$ and an integer $k > 0$, plus the question if there exists an oriented k-coloring of $\vec{G}$. Many papers have presented NP-completeness proofs for OCN$_k$ (e.g., see [BJHM88, CFGK13, CD06, GH10, KM04]). We noticed that it was not known the complexity status of OCN$_k$ when the input graph $\vec{G}$ satisfies that the underlying graph G is cubic.

In this work we prove that OCN$_4$ remains NP-complete even when restricted to a connected, planar and cubic oriented graph $\vec{G}$.

1 Introduction

We use standard notation and terminology used in graph theory to omit repetition. An oriented graph $\vec{G} = (V, \vec{E})$ is obtained from a simple graph

2000 AMS Subject Classification: 05C15 and 05C20.
Key Words and Phrases: oriented coloring, cubic graph, planar graph, NP-complete.
Partially suported by CNPq, CAPES and FAPERJ.

G by arbitrarily giving one of two possible orientations to each edge of G, we say that G is the underlying graph of $\vec{G}$. If G is connected we say that $\vec{G}$ is connected (the same for planar and cubic). The maximum degree of G is denoted by $\Delta(G)$ and we define $\Delta(\vec{G}) = \Delta(G)$. An oriented k-coloring of an oriented graph is a partition $(V_1, V_2, V_3, \ldots, V_k)$ of V into k subsets such that there are no two adjacent vertices belonging to the same subset, and all the arcs between a pair of subsets have the same orientation. The *oriented chromatic number* $\chi_o(\vec{G})$ is the smallest k such that $\vec{G}$ admits an oriented k-coloring.

The k-ORIENTED CHROMATIC NUMBER (OCN_k) was introduced by Courcelle [Cou94] and then studied by Raspaud and Sopena [RS94].

OCN_k - k-ORIENTED CHROMATIC NUMBER
INSTANCE: Oriented graph $\vec{G} = (V, \vec{E})$ and a positive integer k.
QUESTION: Is there an oriented k-coloring of $\vec{G}$?

Let $\vec{G_1}$ and $\vec{G_2}$ be two oriented graphs, a *homomorphism* of $\vec{G_1}$ to $\vec{G_2}$ is a mapping f: $V(G_1) \to V(G_2)$ such that $f(u)f(v) \in \vec{E}(\vec{G_2})$ whenever $uv \in \vec{E}(\vec{G_1})$. In this case, we say that $\vec{G_1}$ is $\vec{G_2}$-*colorable*, that the vertices of $\vec{G_2}$ are the *colors* assigned to the vertices of $\vec{G_1}$, and that $\vec{G_2}$ is the *color digraph* of $\vec{G_1}$. Clearly, an oriented graph $\vec{G}$ has an oriented k-coloring if and only if there is a tournament $\vec{T_k}$ with k vertices, such that $\vec{G}$ has a homomorphism to $\vec{T_k}$.

Many papers have presented NP-completeness proofs for OCN_k, see Table 1. In the present work, we prove that OCN_k remains NP-complete even when restricted to a connected, planar and cubic oriented graph $\vec{G}$. This NP-completeness result is obtained using the reduction in [CFGK13], and the NP-complete problem [BKS03, CFF$^+$08]:

$\text{P3SAT}_{\overline{3}}$ - PLANAR 3SAT WITH AT MOST 3 OCCURRENCES PER VARIABLE
INSTANCE: Set U of variables and collection C of clauses over U, $|U| = n$ and $|C| = m$, such that: (i) each clause $c \in C$ satisfies $|c| = 2$ or $|c| = 3$; (ii) each variable has 2 or 3 occurrences and each negative literal occurs once in C; (iii) the bipartite graph $G = (V, E)$ is planar and connected,

where $V = U \cup C$ and E contains the pairs (u, c) if and only if either u or $\overline{u}$ belongs to clause c.

QUESTION: Is there a satisfying truth assignment for U satisfying all clauses of C?

2 NP-completeness reduction

In [CFGK13] it was shown, using the component design technique, that OCN$_4$ is NP-complete even for connected, planar, bipartite and acyclic oriented graph $\vec{G}$ with $\Delta(G) = 3$. For this purpose, from an instance $I = (U, C)$ of P3SAT$_{\overline{3}}$, it was constructed for each variable u_i of U a *truth setting* $\vec{T_i}$ and for each clause c_j of C a *satisfaction testing* $\vec{S_j}$, see Figure 1. Next, it was considered the planar bipartite graph $\Big((U, C), \vec{E}(U \cup C)\Big)$, and it was obtained a planar drawing for $\vec{G}$ by suitably locating the corresponding graphs $\vec{T_i}$ and the corresponding graphs $\vec{S_j}$.

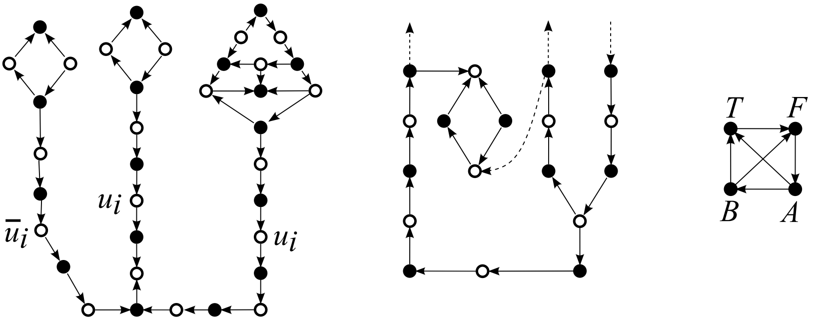

Figure 1: Graphs used in [CFGK13]: Graph $\vec{T_i}$ in the left, graph $\vec{S_j}$ in the middle, and color digraph in the right.

Theorem 2.1. OCN$_4$ is NP-complete even for connected, planar and cubic graphs.

Proof. Now we construct another instance connected, planar and cubic $\vec{G'}$ from $\vec{G}$ built in [CFGK13], such that $\vec{G'}$ has a 4-oriented coloring if and only if $\vec{G}$ has a 4-oriented coloring. For this, note that the special oriented graph $\vec{G}$ has only vertices with degree 2 and 3. To construct $\vec{G}'$ we consider for each vertex v of degree two of $\vec{G}$ the aditional gadget $G_d(v)$ in Figure 2, where $G_d(v) = (\{v'_1, v'_2, v'_3, v'_4, v'_5\}, \{v'_2v'_1, v'_2v'_3, v'_4v'_1, v'_4v'_3, v'_3v'_5, v'_5v'_2, v'_5v'_4\})$ in Figure 2(a), adding the edge v'_1v. Observe that whatever color $\{A, B, F, T\}$ assumed by vertex v, there is a corresponding coloring in one of the Figures 2(b), 2(c) or 2(d) which can be extended to gadget $G_d(v)$. Now the graph G' is cubic and planar. And the instance $I = (U, C)$ is satisfiable if and only if $\vec{G}'$ has a 4-oriented coloring. Note that $\vec{G}'$ is neither acyclic nor bipartite as $\vec{G}$ because $G_d(v)$ has directed cycles of length three. ■

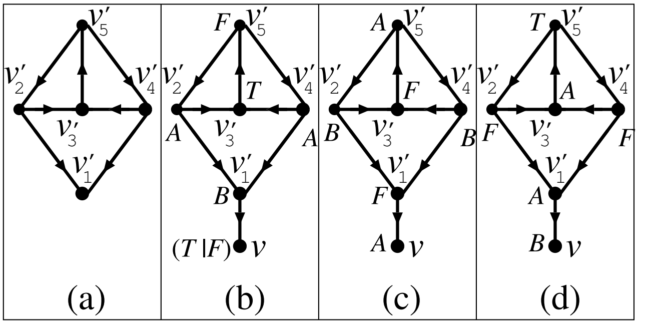

Figure 2: (a) Gadget $G_d(v)$. (b), (c), (d) all possible color assignments to the vertex v using color digraph in Figure 1.

3 Conclusion

In this work, we have established the NP-completeness of OCN_4 on planar and cubic oriented graphs, it is an extension of results obtained in [CFGK13]. The results in [Sop97] and [CFGK13] implies a P versus NP-complete dichotomy: OCN_k, $k \geq 4$ is in P if $\Delta(\vec{G}) \leq 2$ and OCN_k, $k \geq 4$ is NP-complete if $\Delta(\vec{G}) \geq 3$. Table 1 summarizes the state of the art of OCN_k, $k \geq 4$ NP-completeness on the listed special graph classes. Sopena in [Sop97] conjectured that for oriented graphs with maximum degree 3 there exists no such connected graph with oriented chromatic number greater than 7, and Sopena and Vignal [SV96] provided a polynomial-time algorithm to yield an oriented coloring of $\vec{G}$ with maximum degree 3 using 11 colors. We continue with the same open problem posed in [CFGK13], of determining the minimum number $4 < h \leq 11$, such that it is a polynomial-time problem to yield an oriented coloring for an oriented graph $\vec{G}$ with maximum degree 3 using h colors.

Table 1: [X] - Result in this paper.

[BJHM88]	Deciding whether a digraph has a homomorphism to a tournament $\vec{T}$ with at least two directed cycles, is NP-complete.
[KM04]	OCN_4 is NP-complete.
[CD06]	OCN_4 is NP-complete on acyclic oriented graphs with Δ =max(p+3,6). OCN_4 is NP-complete on bipartite oriented graphs with Δ =max(p+3,7).
[GH10]	OCN_4 is NP-complete on acyclic oriented graphs with Δ =max(p+2,4).
[CFGK13]	OCN_4 is NP-complete on connected, planar, bipartite and acyclic oriented graphs with $\Delta = 3$.
[X]	OCN_4 is NP-complete on connected, planar, cubic oriented graph

References

[BJHM88] J. Bang-Jensen, P. Hell, and G. MacGillivray, *The complexity of colouring by semicomplete digraphs*, SIAM J. Discrete Math. **1** (1988), no. 3, 281–298. MR 955645

[BKS03] P. Berman, M. Karpinski, and A. D. Scott, *Approximation hardness and satisfiability of bounded occurrence instances of SAT*, Electronic Colloquium on Computational Complexity, Report 22, 2003.

[CD06] Jean-François Culus and Marc Demange, *Oriented coloring: Complexity and approximation*, SOFSEM 2006: Theory and Practice of

Computer Science (Jiří Wiedermann, Gerard Tel, Jaroslav Pokorný, Mária Bieliková, and Július Štuller, eds.), Lecture Notes in Computer Science, vol. 3831, Springer Berlin Heidelberg, 2006, pp. 226–236.

[CFF+08] M. R. Cerioli, L. Faria, T. O. Ferreira, C. A. J. Martinhon, F. Protti, and B. Reed, *Partition into cliques for cubic graphs: planar case, complexity and approximation*, Discrete Appl. Math. **156** (2008), no. 12, 2270–2278. MR 2433583

[CFGK13] Hebert Coelho, Luerbio Faria, Sylvain Gravier, and Sulamita Klein, *Oriented coloring in planar, bipartite, bounded degree 3 acyclic oriented graphs*, Electronic Notes in Discrete Mathematics **44** (2013), 195 – 200.

[Cou94] B. Courcelle, *The monadic second order logic of graphs. VI. On several representations of graphs by relational structures*, Discrete Appl. Math. **54** (1994), no. 2-3, 117–149, Efficient algorithms and partial k-trees. MR 1300243

[GH10] Robert Ganian and Petr Hliněný, *New results on the complexity of oriented colouring on restricted digraph classes*, SOFSEM 2010: Theory and Practice of Computer Science (Jan van Leeuwen, Anca Muscholl, David Peleg, Jaroslav Pokorný, and Bernhard Rumpe, eds.), Lecture Notes in Computer Science, vol. 5901, Springer Berlin Heidelberg, 2010, pp. 428–439.

[KM04] W. F. Klostermeyer and G. MacGillivray, *Homomorphisms and oriented colorings of equivalence classes of oriented graphs*, Discrete Math. **274** (2004), no. 1-3, 161–172. MR 2025082

[RS94] A. Raspaud and E. Sopena, *Good and semi-strong colorings of oriented planar graphs*, Inform. Process. Lett. **51** (1994), no. 4, 171–174. MR 1294309

[Sop97] E. Sopena, *The chromatic number of oriented graphs*, J. Graph Theory **25** (1997), no. 3, 191–205. MR 1451297

[SV96] E. Sopena and L. Vignal, *A note on the oriented chromatic number of graphs with maximum degree three*, LaBRI Research Report, 1996.

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