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On the l-neighborhood convexity

Carmen C. Centeno **Erika M. M. Coelho**
Mitre C. Dourado **Jayme L. Szwarcfiter**

Abstract

Let G be a finite, simple, and undirected graph and let S be a set of vertices of G. We say that a set $S \subseteq V(G)$ is a convex set, in the l-neighborhood convexity, if every vertex in $V(G) \setminus S$ has less than l neighbors in S. The *convex hull* $H_G(S)$ of S is the smallest convex set containing S. A *hull set* of G is a set of vertices whose convex hull equals the whole vertex set of G, and the minimum cardinality of a hull set of G is the *hull number* $h(G)$ of G. Finally, the *Carathéodory number* of G is the smallest integer c that for every set S and every vertex u in $H_G(S)$, there is a set $F \subseteq S$ with $|F| \leq c$ and $u \in H_G(F)$. In this work, we study the hull number and the Carathéodory number for the l-neighborhood convexity of graphs considering $l > 2$. We determine the hull number for cographs in the 3-neighborhood convexity and determine the Carathéodory number of trees for l-neighborhood convexity, where $l > 2$.

1 Introduction

One simple combinatorial puzzle, studied by Balogh and Pete [BP98], is called Disease Process and can be modeled in terms of graph conve-

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xity. Let us initially define the problem considering a graph G, a set U of vertices of G that initially possess a property, and an iterative process whereby new vertices u enter U whenever sufficiently many neighbors of u are already in U. The simplest choice for "sufficiently many" that results in interesting effects is 2. This choice leads to the *irreversible* 2*-threshold processes* considered by Dreyer and Roberts [DR09] and also is studied in the P_3*-convexity* [BCD+12, CDS09, CDD+10, CDP+11, CPRS12, EFHM72, PWW08].

An arbitrary fixed number of l-neighbors is studied by Pete [Pet97], who determines some results for a grid and a cube. We continue this study modeling the problem as the l-neighborhood convexity in finite, simple, and undirected graphs where for a graph G, the vertex set is denoted $V(G)$ and the edge set is denoted $E(G)$.

A set $\mathcal{C}$ of subsets of $V(G)$ is a *convexity* on $V(G)$ if

- $\emptyset, V(G) \in \mathcal{C}$ and
- $\mathcal{C}$ is closed under intersections.

The elements of $\mathcal{C}$ are called *convex sets*. Several well known such convexities $\mathcal{C}$ are defined using a set $\mathcal{P}$ of paths of the underlying graph G. In this case, a set C of vertices of G is convex, that is, it belongs to $\mathcal{C}$, if and only if for every path $P : v_0v_1 \ldots v_l$ in $\mathcal{P}$ such that v_0 and v_l belong to C, all vertices of P belong to C. When $\mathcal{P}$ is the set of all shortest paths in G, then $\mathcal{C}$ is the *geodetic convexity* of G [CHM+06, DPRS10, ES85, FJ87] and when $\mathcal{P}$ is the set of all induced paths of G, then $\mathcal{C}$ is the *monophonic convexity* of G [DPS10, Duc88, FJ86]. Similarly, the *triangle path convexity* of G is defined by considering as $\mathcal{P}$ the set of all triangle paths of G [CM99]. Lastly, when $\mathcal{P}$ is the set of all paths of G with 3 vertices, we have the P_3 *convexity* of G [BCD+12, CDS09, CDD+10, CDP+11, CPRS12, EFHM72, PWW08].

In this paper we study problems related to the l*-neighborhood convexity* of G which is defined by the number of neighbors of a vertex. In this case, we say that a set $S \subseteq V(G)$ is a convex set if every vertex in $V(G) \setminus S$ has less than l neighbors in S. The P_3 convexity is an example of the

2-neighborhood convexity, for this matter, we study the l-neighborhood convexity considering $l > 2$.

Let G be a graph, $\mathcal{C}$ a convexity on $V(G)$ and the *convex hull* in $\mathcal{C}$ of a set S of vertices of G the smallest set $H_{\mathcal{C}}(S)$ in $\mathcal{C}$ containing S. We say that a subset of vertices X is *contaminated* by the vertices of the set S if $X \subseteq H_G(S)$. A *hull set* of G is a set of vertices whose convex hull equals the whole vertex set of G, and the minimum cardinality of a hull set of G is the *hull number* $h_{\mathcal{C}}(G)$ of G.

Also, a fundamental result about convex sets in $\mathbb{R}^d$ known as *Carathéodory's theorem* [Car11, Eck93] states that every point u in the convex hull of a set $S \subseteq \mathbb{R}^d$ lies in the convex hull of a subset F of S of order at most $d+1$.

The *Carathéodory number* of $\mathcal{C}$ is the smallest integer c such that for every set S of vertices of G and every vertex u in $H_{\mathcal{C}}(S)$, there is a set $F \subseteq S$ with $|F| \leq c$ and $u \in H_{\mathcal{C}}(F)$. A set S of vertices of G is a *Carathéodory set* of $\mathcal{C}$ if the set $\partial H_{\mathcal{C}}(S)$ defined as $H_{\mathcal{C}}(S) \setminus \bigcup_{u \in S} H_{\mathcal{C}}(S \setminus \{u\})$ is not empty. This notion allows an alternative definition of the *Carathéodory number* of $\mathcal{C}$ as the largest cardinality of a Carathéodory set of $\mathcal{C}$.

The Carathéodory number has been determined for some graph convexities and graph classes. In [DRdS$^+$13] was proved that the Carathéodory number of the geodetic convexity of split graphs is at most 3, but was showed that it is algorithmically hard to determine the Carathéodory number for this convexity. The Carathéodory number of the monophonic convexity of a graph G is 1 if G is complete and 2 otherwise [Duc88]. The Carathéodory number of the triangle path convexity of G is 2 whenever G has at least one edge [CM99]. Concerning the P_3 convexity in directed graphs, it has been shown that the maximum Carathéodory number of a multipartite tournament is 3 [PWW08]. The Carathéodory number of the P_3-convexity of a graph G is unlimited and in [BCD$^+$12] was shown characterizations and polynomial algorithms for the class of trees and block graphs, was established a best possible upper bound on the Carathéodory number of general graphs and a upper bound for claw-free graphs, finally, was demonstrated that it is algorithmically hard to determine the Carathéodory number for P_3-convexity even for bipartite graphs. In [CDRS14] was shown characterizations and polynomial algorithms for the chordal graphs.

Since a graph G uniquely determines its convexity $\mathcal{C}$, we may write $H_G(S)$ and $\partial H_G(S)$ instead of $H_{\mathcal{C}}(S)$ and $\partial H_{\mathcal{C}}(S)$, respectively. We denote by $h_l(G)$ the hull number and $c_l(G)$ the Carathéodory number of the graph G for the l-neighborhood convexity.

In this paper we study the hull number and the Carathéodory number for the l-neighborhood convexity of graphs. We characterized the class of cographs considering the 3-neighborhood convexity. Relative to the Carathéodory number of G, we prove that $c_l(T)$, for any tree T, is the number of leaves of the largest strictly l-ary subtree of T.

2 Results

A *cograph* is a graph G which has no induced P_4. An important property of a cograph that will be considered here is: G is connected if and only if $\overline{G}$ is disconnected. Let G be a connected cograph, denote by u the number of universal vertices in G, in other words, a vertex adjacent to every other vertex except itself. Denote by $\overline{G_1}, \dots, \overline{G_u}, \dots, \overline{G_t}$ the connected components of $\overline{G}$ and by $G_1, \dots, G_u, \dots, G_t$ the subgraphs of G induced by the vertices of the respective components of $\overline{G}$, where $|V(G_i)| \geq 2$ when $i > u$. Note that the complement of G has t connected components.

The following considerations can be made: the components $\overline{G_1}, \dots, \overline{G_u}$ are isolated vertices in $\overline{G}$ and $G_1, \dots G_u$ are universal vertices in G; the vertices of a subgraph G_{u+i} such that $i > 0$ are adjacent to every vertex of $G - G_{u+i}$. Still, a subgraph G_k of G can have many connected components and even isolated vertices. Denote by C_i^j a connected component j that belongs to the subgraph G_i of G, and by $C(G_k)$ the number of connected components of G_k.

Considering cographs and $l = 3$ we can state the following theorem:

Theorem 2.1. Let G be a cograph, then:

$$h_3(G) = \begin{cases} 4 \; if \; G = K_{2,2} \\ \max\{3, C(G_3)\} \; if \; (u = 2 \; and \; t = 3) \\ \max\{3, C(G_2)\} \; if \; (u = 0 \; and \; t = 2) \; and \; |V(G_1)| = 2 \\ \sum_{j=1}^{C(G_i)} \; h_2(C_2^j) \; if \; t = 2 \; and \; u = 1, \\ 3, \; otherwise. \end{cases}$$

Proof. Three vertices of a subgraph G_i will contaminate $G - G_i$, which will in the next step contaminate G_i, except in the following cases:

Case 1. The graph is a complete bipartite graph $K_{2,2}$. In this case is easy to check that all four vertices are needed.

Case 2. The graph G has exactly three subgraphs, say G_1, G_2 and G_3. Here a note is worth to be made. If G has more than three subgraphs it is easy to see that three vertices are enough to contaminate G. Also, if G has three subgraphs and every subgraph have at least two vertices, whithout loss of generality, we can choose two vertices of G_1 and one vertex of G_2. They will contaminate G_3 which will contaminate all G_2, that garantee that G_1 will be contaminated. If G_1 is a universal vertex and G_2 and G_3 have at least two vertices, without loss of generality, choose two vertices of G_2 and one vertex of G_3. In the worst case, G_2 and G_3 will contaminate only G_1. Next, G_1 and G_2 contaminate G_3 and since G_3 has at least two vertices, G_3 and G_1 contaminate the remaining vertices of G_2. Now, consider that G_1 and G_2 are universal vertices. If G_3 has exactly two vertices, choose the vertices of G_3 and G_2, and these vertices will contaminate G_1. If G_3 has more than two vertices and at most three components, choose three vertices, if possible, of different connected components. They will contaminate G_1 and G_2, which will contaminate the remaining vertices of G_3. Then, consider G_3 has more than three connected components. So, in this case is needed to choose one vertice of each connected component of G_3, these vertices will contaminate G_1 and G_2. Since G_1 and G_2 contributes with two neighboors for the vertices in the components of G_3, it is enough only one vertex of each component for the whole G_3 be contaminated.

Case 3. Let consider now that the graph G has exactly two subgraphs. If each subgraph has exactly two vertices, the only possible graph is the $K_{2,2}$ that was mentioned previously. If each subgraph has more than two vertices, it is sufficient three vertices of G_1 or G_2. Without loss of generality, suppose that G_1 has two vertices and G_2 has more than two vertices. The same idea of case 2, taking one vertex of each component of G_2, can be applied. If G_1 is a single vertex, than it is not enough three vertices of G_2, since G_2 can have many connected components as possible. Also it is not enough one vertice of each component of G_2, since G_1 will contribute with only one neighboor, and one vertex of each component of G_2 do not guarantee the contamination of the whole G_2. Thus we should find the necessary vertices to contaminate each component of G_2 considering the rule of the 2 neighboors contamination, denoted by h_2. Note that if G_2 has only one component and $h_2(G_2)$ is 2, it is clear that three vertices of G_2 will be necessary instead.

■

Our first result on Carathéodory number collects several elementary properties of Carathéodory sets.

Proposition 2.2. Let G be a graph and let S be a Carathéodory set of G.

a) If G has order at least 2 and is either a path, or a cycle, then $c(G) = 1$.

b) If G has order at least 2 and is a complete graph, then $c(G) = l$;

c) The convex hull $H_G(S)$ of S induces a connected subgraph *of* G.

Proof. a) Every $S \subseteq V(G)$ satisfy $H_G(S) = S$. So $c(G) = 1$.

b) Every $S \subseteq V(G)$ such that $|S| = l$ of a complete graph G satisfy $H_G(S) = V(G)$. So, $c(G) = l$.

c) Let $u \in \partial H_G(S)$. By contradiction, suppose that $H_G(S)$ does not induce a connected subgraph of G. Every connected component is induced by a proper subset of S. Then, $u \in H_G(S')$, where $S' \subset S$, contradicting the fact that S is a Carathéodory set. ■

Our next result characterizes the Carathéodory sets and Carathéodory number of trees. A rooted tree in which every internal vertex has exactly q children is a *strictly q-ary tree.* Strictly l-ary (sub)trees play a central role for the Carathéodory number of l-neighborhood convexity.

Theorem 2.3. Let T be a tree.

a) If T is a strictly l-ary tree with root r, then the set L of leaves of T satisfies $H_T(L) = V(T)$ and $\{r\} = \partial H_T(L)$, i.e. L is a Carathéodory set of T.

b) If r is a vertex of T and S is a set of vertices of T with $r \in \partial H_T(S)$, then S is the set of leaves of a strictly l-ary subtree of T rooted in r.

c) $c(T)$ equals the largest number of leaves of a strictly binary subtree of T.

Proof. a) We prove the statement by induction on the order n of T. If $n = 1$, then the statement is trivial. Hence let $n \geq 2$. Let $r_1, r_2, \cdots, r_q$ be the children of r in T. For $i \in \{1, 2, \cdots q\}$, let T_i be the strictly l-ary subtree of T containing r_i and all descendants of r_i and let L_i be the set of its leaves. By induction, $H_{T_i}(L_i) = V(T_i)$ and $\{r_i\} = \partial H_{T_i}(L_i)$. Since $L = \bigcup_{i=1}^{l} L_i$ and r has exactly the l neighbours $r_1, r_2, \cdots, r_l$, this implies $H_T(L) = V(T)$ and $\{r\} = \partial H_T(L)$.

b) Let r be a vertex of T and let S be a set of vertices of T with $r \in \partial H_T(S)$. By Proposition 2.2 c), the subgraph T' of T induced by $H_T(S)$ is a subtree of T. We consider T' as rooted in r. If some internal vertex v of T' belongs to S, then some descendant w of v in T' belongs to S and $r \in H_T(S \setminus \{w\})$, which is a contradiction. Hence all elements of S are leaves of T'. Since every vertex in $H_T(S) \setminus S$ has l neighbours in $H_T(S)$, all leaves of T' belong to S, that is, S equals the set of leaves of T'. If some internal vertex v of T' does not have exactly l children, then let w be a leaf of T' that is a descendant of v. It follows easily that $r \in H_T(S \setminus \{w\})$, which is a contradiction. Hence T' is a strictly l-ary tree rooted in r and S is the set of leaves of T'.

Since c) follows easily from b), the proof is complete. ∎

Theorem 2.3 implies that for every integer $k \geq 1$, there exists a tree T with $c(T) = k$. Based on Theorem 2.3, we describe an efficient algorithm for determining the Carathéodory number of a given tree T. According to Theorem 2.3 it is sufficient to find the largest strictly l-ary subtree of T and count the number of its leaves. The idea of the algorithm is to select an arbitrary vertex v as the root and to consider the rooted version T_v of T with root v. For a vertex w of T_v, let $l_v(w)$ denote the maximum number of leaves of a strictly l-ary subtree $T_v(w)$ of T_v with root w such that $T_v(w)$ contains no ancestor of w with respect to T_v. If w has at most $l-1$ children in T_v, then $l_v(w) = 1$. If w has at least d children $w_1, w_2, \ldots, w_d$ in T_v with $l_v(w_1) \geq l_v(w_2) \geq \ldots \geq l_v(w_d)$, then $l_v(w) = l_v(w_1) + l_v(w_2) + \ldots + l_v(w_q)$. Processing the vertices of T_v in an order of non-increasing distance to v, one can determine $l_v(v)$ in linear time. By Theorem 2.3, we have $c(T) = \max\{l_v(v) \mid v \in V(T)\}$, which allows to determine $c(T)$ in $O(n^2)$ time for trees of order n.

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Carmen C. Centeno
Departamento de Computação - PUC-GO
Goiânia - Goiás, Brazil
cecilia@pucgoias.edu.br

Erika M. M. Coelho
Instituto de Matemática - UFG
Goiânia - Goiás, Brazil
erikamorais@inf.ufg.br

Mitre C. Dourado
Instituto de Matemática - UFRJ
Rio de Janeiro, Brazil
mitre@dcc.ufrj.br

Jayme L. Szwarcfiter
PESC, COPPE - UFRJ and INMETRO
Rio de Janeiro, Brazil
jayme@nce.ufrj.br