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Characterizing Clique Graphs of Chordal Comparability Graphs

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Abstract

The *clique graph* $K(G)$ of a graph G is the intersection graph of the maximal cliques of G. A well-known characterization of clique graphs is that by Roberts and Spencer (1971). In addition there are characterizations for clique graphs of several graph classes. In this work, we add new classes to this list, by describing the clique graphs of chordal comparability graphs and of split comparability graphs. The proofs are based on some properties of the maximal cliques of chordal comparability graphs. We recall that clique graphs of chordal graphs have been already characterized. As for comparability graphs, only partial characterizations of subclasses, such as cographs, are known. The problem of characterizing the clique graphs of general comparability graphs remains open.

1 Introduction

The study of intersection graphs plays an important role in graph theory. Many classes have been defined in terms of the intersection graph of sets

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Chordal	Dually Chordal
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Interval	Proper Interval
Split	Star
Forest	Block

Table 1: Some answers for Question 1.1.

satisfying a certain property and for some others classes it has been found that they are exactly the intersection graphs of a given model.

Clique graphs are the graphs obtained by taking the maximal cliques of a given graph as vertices and connecting two vertices whenever their corresponding cliques intersect, i.e., they are the intersection graphs of the maximal cliques of a graph. Several properties of clique graphs have been considered over the years and, although partial [5] and total [7] characterizations have been known since the early studies of this class, it has been shown that its recognition is NP-complete [1]. One of the questions concerning clique graphs that has been studied extensively is the following.

Question 1.1. Given a graph class $\mathcal{C}$, what structure does the clique graph of G have, if G belongs to $\mathcal{C}$?

Stating in other way, by answering Question 1.1, we are trying to understand what structure arises as a result of taking the clique graph of a graph satisfying a given property. Table 1 presents some known results of this nature. For a comprehensive list, please check the survey [8].

In this paper we give a partial answer to the Problem (3) posed in [8], which asked for a characterization of the intersection graphs of the chains of a partial order (the clique graphs of comparability graph). We present characterizations for the clique graphs of chordal comparability and split comparability graphs. The clique graphs of chordal and split graphs are well understood and their structure can be considered fairly general, considering that any graph with a universal vertex is the clique graph of a split graph (and hence of a chordal graph). We recall that clique graphs of chordal graphs have already been characterized, e.g. [2], [3], [4]. As for comparability graphs, only partial characterizations of subclasses, such as cographs [6], are known.

The rest of this paper is organized as follows. Section 2 presents the definitions and notation used in the rest of this paper. In Section 3 we describe the key observations on the structure of chordal comparability graphs and use them to prove our main result. We conclude presenting some final remarks on Section 4.

2 Preliminaries

We consider finite graphs. For a graph G we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. We denote the *neighborhood* of a vertex v as $N(v)$ and the *closed neighborhood* $N(v) \cup \{v\}$ as $N[v]$. If u and v are vertices satisfying $N[u] = N[v]$ then we say u and v are *twins.* A *clique* C is a set of pairwise adjacent vertices. An *independent set* is a set of vertices pairwise non adjacent. The *intersection graph* of a family $\mathcal{F}$ of sets is the graph $G = (\mathcal{F}, E)$ where for $R, S \in \mathcal{F}$ the edge $e = \{R, S\}$ belongs to E whenever $R \cap S \neq \emptyset$. The *clique graph* $K(G)$ of a graph G is the intersection graph of the maximal cliques of G.

A *chordal graph* is a graph where every cycle with at least 4 vertices has a chord, i.e., an edge between non consecutive vertices of the cycle.

A *transitive orientation* of the edges of a graph G is an assignment of direction to $E(G)$ such that, for every $u, v, w \in V(G)$ whenever there exist the directed edges (u, v) and (v, w) then the directed edge (u, w) must also exist. A graph is a *comparability graph* if it admits a transitive orientation. The *line graph* $L(G)$ of a graph G is the intersection graph of the edges of G.

In Figure 1 (a), G is a chordal graph and also a comparability graph, in

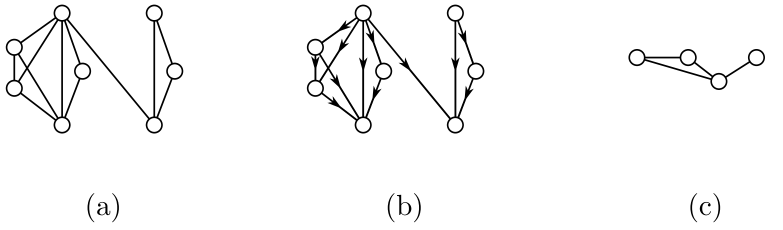

Figure 1: A graph G, a transitive orientation of G and $K(G)$.

(b) we show a transitive orientation of the edges of G, and in (c) we show the clique graph $K(G)$.

3 The characterizations

We start by presenting an auxiliary result concerning chordal comparability graphs.

Lemma 3.1. *Let G be a chordal comparability graph. Then for any transitive orientation of G the following properties hold.*

(P_1) *For any maximal cliques C_1 and C_2 and $u \in C_1 \cap C_2$, if u is a source (sink) of C_1, then u is also a source (sink) of C_2.*

(P_2) *For any maximal cliques C_1 and C_2 from G, $C_1 \setminus C_2$ corresponds to consecutive vertices of the linear order induced by C_1.*

Proof. Suppose (P_1) is false. Without loss of generality, let s be a vertex in $C_1 \cap C_2$, such that s is a source of C_1 but not of C_2. Let r be the source of C_2. Since s belongs to C_2, (r, s) is an oriented edge in G. Since s is a source, for every other vertex v in C_1, (s, v) is an oriented edge in G. Hence, by transitivity, (r, v) must be an oriented edge for every vertex in C_1, contradicting the maximality of C_1.

Now suppose that (P_2) is false. This implies that there are vertices $\{v_1, v_2, v_3,\ w_1, w_3\}$ such that $v_1, v_3 \in C_1 \setminus C_2$, $w_1, w_3 \in C_2 \setminus C_1$, and

$v_2 \in C_1 \cap C_2$, and that (v_1, v_2), (v_1, v_3), (v_2, v_3), (w_1, v_2), (w_1, w_3) and (v_2, w_3) are directed edges in the orientation. The set $\{v_1, v_3, w_1, w_3\}$ induces a cycle with 4 vertices and without chord, a contradiction. ■

Corollary 3.1. Let G be a chordal comparability graph. If C_1 and C_2 are cliques of G with $C_1 \cap C_2 \neq \emptyset$, then for any transitive orientation of G, $C_1 \cap C_2$ contains a source or a sink.

Corollary 3.1 plays a central role in the proof of the following theorem.

Theorem 3.1. The clique graphs of chordal comparability graphs are line graphs of acyclic multigraphs.

Proof. Initially, we create a multigraph H with vertex set consisting of the sources and sinks of a transitive orientation of G. Then for each maximal clique C with source s and sink t, we add a new edge st in $E(H)$. We claim that H is an acyclic multigraph and $K(G)$ is isomorphic to $L(H)$. First note that H is bipartite, since there are no edges between two sources or two sinks. Also note that, ignoring the multi-edges, H is an induced subgraph of G, which is chordal, so H is also chordal. Every induced cycle in a bipartite graph has size at least four, and cannot occur in a chordal graph, so bipartite chordal graphs are acyclic. It remains to show that $K(G)$ is isomorphic to $L(H)$. From Corollary 3.1 it follows that, given a transitive orientation of $E(G)$, two cliques intersect if and only if they share a source or a sink. Hence two cliques intersect in G if and only if the corresponding edges intersect in H, which completes the proof. ■

Note that the line graphs of forests coincide with clique graphs of forests which are block graphs. Adding multi-edges in a graph adds a corresponding twin vertex to the line graph. This implies the following consequence.

Corollary 3.2. The clique graph of a chordal comparability graphs can be obtained by a sequence of insertions of twin vertices on a block graph.

We now restrict our result to split comparability graphs.

Theorem 3.2. Let G be a split comparability graph. Then $K(G)$ is the union of three cliques K_1, K_2, K_3 with all possible edges between K_2 and K_i, $i \in \{1, 3\}$ and no edge between K_1 and K_3.

Proof. Let C, I be a partition of $V(G)$ such that C induces a clique and I induces an independent set. Note that the maximal cliques of G are exactly the clique C and the cliques $N[v]$ for every $v \in I$. For an orientation of G, let s and t be the source and sink of C with respect to this orientation. We define K_1, K_2, and K_3 as follows.

- $K_1 = \{S \subseteq V(G) \mid S \text{ is a clique and } s \in S, t \notin S\}$
- $K_2 = \{S \subseteq V(G) \mid S \text{ is a clique and } s, t \in S\}$
- $K_3 = \{S \subseteq V(G) \mid S \text{ is a clique and } t \in S, s \notin S\}$

Note that it follows from Corollary 3.1 that K_1, K_2, K_3 is a partition of the cliques of G. Also note that each K_i will be a clique in $K(G)$, since its elements intersect each other, and each element in K_2 intersects every element of K_1 and K_3. It remains to show that there is no edge in $K(G)$ between the vertices corresponding to the cliques of K_1 and the vertices corresponding to the cliques of K_3.

Suppose, for contradiction, that there is an edge between two vertices corresponding to elements of K_1 and K_3. Then there are cliques $C_1 \in K_1$ and $C_3 \in K_3$ such that $C_1 \cap C_3 \neq \emptyset$. Since $s \in C_1$, by Lemma 3.1, we have that s is a source of C_1, but not of C_3, otherwise we would have $C_3 \in K_2$. Similarly, t is a sink of C_3, but not of C_1, but by Corollary 3.1, C_1 and C_3 should have a source or a sink in common, a contradiction. ■

4 Conclusion

The problem of characterizing the clique graphs of comparability graphs remains as a main open problem. As it can be noticed by the reader, the absence of induced cycles of order 4 was the key property that made the problem easier. A main step in the direction of understanding the clique

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Forest	Block
Chordal ∩ Comparability	Theorem 3.1
Split ∩ Comparability	Theorem 3.2
Cograph	Open
Comparability	Open

Table 2: Known results, contributions of this paper and open problems.

graphs of comparability graphs would be to determine the consequences of the presence of an induced cycle of order 4 on a comparability graph.

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