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Multicolored Ramsey Numbers in Multipartite Graphs

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Abstract

The bi-colored set multipartite Ramsey numbers were introduced by Burger et al. in 2004. In this work we extend the set multipartite Ramsey numbers to an arbitrary number of colors. Growth properties, connections with classical Ramsey numbers, lower and upper bounds are obtained, including some improvements of bounds given by Burger et al.

1 Introduction

Let K_r denote the complete graph on r vertices. Given positive integers $n_1 \geq 2$ and $n_2 \geq 2$, recall that the celebrated Ramsey number $r(n_1, n_2)$ denotes the smallest natural number r such that every red-blue coloring of the edges of K_r yields a red copy of K_{n_1} or a blue copy of K_{n_2}. Determining Ramsey numbers has been a great challenge in graph theory. Indeed, the only known exact values are: $r(2,2) = 2$, $r(3,3) = 6$, and $r(4,4) = 18$, but $r(5,5)$ still remains an open problem. Since the seminal paper by Ramsey

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[6] in 1930, some contributions have focused on upper and lower bounds, for instance, $43 \leq r(5,5) \leq 49$. Up-to-date tables on bounds are available in [7].

Many concepts, variants and extensions have been introduced in order to shed light on the computation of Ramsey numbers: bipartite Ramsey numbers, induced Ramsey numbers, geometric Ramsey numbers etc (see the book [5] for an overview on Ramsey theory). For our purpose, we mention the following extension: the multicolored Ramsey number $r(n_1, \ldots, n_k)$ denotes the smallest natural r such that every coloring of the edges of K_r with k colors produces at least one copy of K_{n_i} colored i, for $1 \leq i \leq k$. If $n_i = n$ for every i, this number is also denoted by $r(n;k)$. For instance: $r(3;2) = r(3,3) = 6$, the only known value of a multicolored classical Ramsey number is $r(3;3) = r(3,3,3) = 17$, and $51 \leq r(3;4) \leq 62$, according to [7].

In particular, Burger et al. [2, 3] introduced the following Ramsey-type problem. Let $K_{c\times s}$ denote the balanced, complete multipartite graph having c classes with s vertices per each class. Given positive integers s, $n_1 \geq 2$, m_1, $n_2 \geq 2$, and m_2, the *set multipartite Ramsey number* $M_s(K_{n_1\times m_1}, K_{n_2\times m_2})$ is the smallest natural number c such that every red-blue coloring of the edges of $K_{c\times s}$ yields either a red $K_{n_1\times m_1}$ or a blue $K_{n_2\times m_2}$. These numbers can be regarded as an extension of the classical Ramsey numbers. Indeed, note that $M_1(K_{n_1\times 1}, K_{n_2\times 1}) = r(K_{n_1}, K_{n_2}) = r(n_1, n_2)$, since $K_{n\times 1}$ is isomorphic to K_n for every n. Several results on these numbers are presented in [3]; some of them in connection with the classical numbers $r(n_1, n_2)$. Moreover, general bounds are obtained, including a general lower bound by using the probabilistic method.

In this work we extend the set multipartite Ramsey numbers to an arbitrary number of colors, as described in the next section. We discuss the connections with the multicolored Ramsey numbers. Several results in [3] are extended, including general lower and upper bounds. Moreover, we also prove sharper upper bounds for certain classes of parameters, improving previous bounds.

2 Definitions and existence

The graph $K_{c\times s}$ denotes the multipartite graph formed by c classes, where each class has s vertices.

$$K_{c\times s} = K_{s,s,\dots,s} \ \ (c \text{ times}).$$

Note that $K_{n\times 1} = K_n$, but $K_{1\times n}$ denotes the complement of K_n (n isolated vertices).

Given positive integers s, $k \geq 2$, $n_i \geq 2$, and m_i, $1 \leq i \leq k$, the *set multipartite Ramsey number* $M_s(K_{n_1\times m_1}, \dots, K_{n_k\times m_k})$ denotes the smallest c such that every k-coloring of the edges of $K_{c\times s}$ yields a monochromatic copy of $K_{n_i\times m_i}$ for some color i, $1 \leq i \leq k$. The "diagonal" case: $n_i = n$ and $m_i = m$, $1 \leq 1 \leq k$, is simply denoted by $M_s(K_{n\times m}; k)$.

In particular, note that $M_1(K_{n_1\times 1}, \dots, K_{n_k\times 1}) = r(n_1, \dots, n_k)$. In this work we consider mainly the "diagonal" case.

Theorem 2.1. The number $M_s(K_{n\times m}; k)$ is well defined, and

$$M_s(K_{n\times m}; k) \leq r(nm; k).$$

Sketch of proof: Let $c = r(nm; k)$. Given an arbitrary coloring of $K_{c\times s}$ with k colors, choose one vertex of each class. Let V be the set formed by these c chosen vertices. The graph induced by V is isomorphic to K_c. Since $c = r(nm; k)$, there is a monochromatic copy of K_{nm}. Because $K_{n\times m}$ is a subgraph of K_{nm}, a monochromatic copy of $K_{n\times m}$ appears. ■

3 Expansive coloring

It seems to be important to explore relationships between multipartite Ramsey numbers and classical Ramsey numbers in order to get sharper bounds. One of these connections is based on expansive coloring, described below.

A coloring (of edges) in $K_{c\times s}$ is called an *expansive coloring* if it satisfies the property: all edges induced by each pair of classes in $K_{c\times s}$ have the same color, see [2].

The expansive coloring of $K_{3\times 3}$ in Figure 1 induces the coloring of K_3, and vice versa.

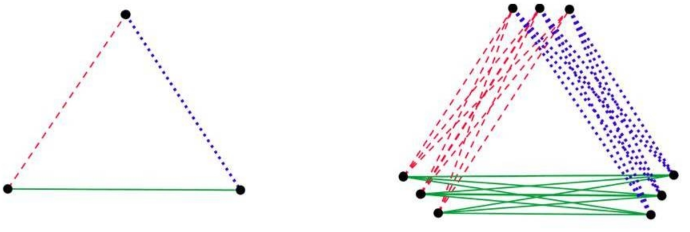

Figure 1: an expansive coloring of $K_{3\times 3}$

Theorem 3.1. The relation below holds

$$r(n;k) \leq M_s(K_{n\times m};k).$$

Sketch of proof: Let $r = r(n;k)$. The definition of $r(n;k)$ implies that there is a k-coloring of K_{r-1} without monochromatic K_n. Consider the expansive coloring of $H = K_{(r-1)\times s}$ induced by the coloring of K_{r-1} above. Note that this expensive coloring of H does not contain any monochromatic K_n. Since K_n is isomorphic to $K_{n\times 1}$ and $K_{n\times 1}$ is a subgraph of $K_{n\times m}$, thus there is not a monochromatic $K_{n\times m}$ in the coloring of H. ■

Example 3.1. We illustrate the proof above with an example. The construction associated to $M_4(K_{3\times 1};2) > 5$ is given next. Because $r(3,3) = 6$, there is a 2-coloring of $G = K_5$ that does not contain any monochromatic copy of K_3, for instance see the coloring of G in Figure 2. Let $H = K_{5\times 4}$ be the expansive coloring induced by this coloring of G. We claim that H does not contain any copy of $K_{3\times 1} = K_3$. Indeed, if H contains a monochromatic copy of K_3, the vertices of this K_3 must be in three distinct classes, thus there would be a monochromatic copy of K_3 in G, a contradiction.

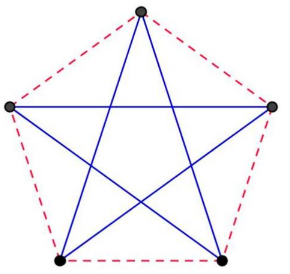

Figure 2: coloring of G

4 Some relationships

The next result presents a growth property, which generalizes a result in [3].

Theorem 4.1. Given positive integers s, $n_i \geq 2$, $m_i \geq 1$, $1 \leq i \leq k$, the following inequality holds

$$\max\{r(n_1, \ldots, n_k), \min\{\left\lceil \frac{m_i}{s} \right\rceil n_i : 1 \leq i \leq k\}\} \leq M_s(K_{n_1 \times m_1}, \ldots, K_{n_k \times m_k}).$$

A few classes of exact values are established by applications of the previous result, more specifically.

Example 4.1. For positive integers $s \geq n \geq 2$, $m \geq 1$ and $n_i \geq 2$, $1 \leq i \leq k$,

(1) $M_s(K_{2\times 1}, \ldots, K_{2\times 1}, K_{n\times m}) = \lceil \frac{m}{s} \rceil n$.

(2) $M_s(K_{n_1\times 1}, \ldots, K_{n_k\times 1}) = r(n_1, \ldots, n_k)$.

5 A lower bound from the probabilistic method

Erdős proved an exponential lower bound for the classical Ramsey numbers in 1947, by using probabilistic arguments (see [1], for instance). Nowadays this method is a powerful tool to estimate bounds on extremal problems in combinatorics, see [1]. By using this method, Burger and van

Vuuren [3] presented a lower bound on the multipartite Ramsey numbers for two colors. The method can be extended to an arbitrary number of colors.

Theorem 5.1. The lower bound holds

$$M_s(K_{n\times m;k}) > \frac{1}{s}\left(n!(m!)^n k^{m^2\binom{n}{2}-1}\right)^{\frac{1}{nm}}.$$

6 Some upper bounds

The upper bounds given by Theorem 2.1 can be improved for suitable classes of multipartite graphs. The next upper bound focuses on a copy of the bipartite graph $K_{2\times m} = K_{m,m}$, more specifically.

Theorem 6.1.

$$M_s(K_{2\times m}; k) \leq \left\lceil \frac{k(m-1)+1}{s} \right\rceil + \left\lceil \frac{k(m-1)\binom{k(m-1)+1}{m}+1}{k} \right\rceil.$$

The particular case where $k = 2$ was proved in [2]. Moreover, we establish sharper upper bounds for the special case $K_{2\times 3}$, according to the next statement.

Theorem 6.2. For every s,

$$M_s(K_{2\times 3}; 2) \leq \left\lceil \frac{28}{s} \right\rceil + 3.$$

The proof is an adaptation of the techniques used in [4]. The main tools are density arguments and a variant of Turán numbers.

Example 6.1. A simple application of this result yields $M_2(K_{2\times 3}; 2) \leq 17$, improving the bound $M_2(K_{2\times 3}; 2) \leq 24$, by Burger and van Vuuren *[3]*.

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