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Two Families of Cayley Graph Interconnection Networks

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Abstract

The family $H_{l,p}$ has been defined in the context of edge partitions, and subsequently shown to be composed by Hamiltonian Cayley graphs. The p^{l-1} vertices of the graph $H_{l,p}$ are the l-tuples with values between 0 and $p-1$, such that the sum of the l values is congruent to 0 mod p, and there is an edge between two vertices having two corresponding pairs of entries whose values differ by one unit. In the sparser graph $H_{l,p}$ there is an edge when only one pair of entries differs by 1. In this work, we establish that the graph $H'_{l,p}$ has degree $(2l-2)$ and diameter $(\lfloor \frac{p}{2} \rfloor (l-1))$ and the graph $H_{l,p}$ has degree (l^2-l) and the diameter can be calculated by an algorithm of time $O(l)$. These properties support the graphs $H_{l,p}$ and $H'_{l,p}$ to be good schemes of interconnection networks.

1 Introduction

In this work, our interests are in the design and analysis of static networks. Static networks can be modeled using tools from Graph Theory.

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A graph represents an interconnection network, where the processors are the vertices and the communication links between processors are the edges connecting the vertices. There are several parameters of interest to specify a network: low degree, low diameter, and the distribution of the node disjoint paths between a pair of vertices in the graph. The degree relates to the port capacity of the processors and hence to the hardware cost of the network. The maximum communication delay between a pair of processors in a network is measured by the diameter of the graph. Thus, the diameter is a measure of the running cost. The number of the parallel paths between a pair of nodes is limited by the degree of the underlying graph, the knowledge of this distribution is helpful in the evaluation of the fault-tolerance of the network [2, 5].

Our goal is to propose two new families of Cayley graphs that can be used to design interconnection networks. The definition of Cayley graphs was introduced to explain the concept of abstract groups which are described by a generating set. The Cayley graphs are regular, may have logarithmic diameter and may be used to design interconnection networks [3, 5]. One graph usually used to design interconnection networks is the "hypercube graph", that is isomorphic to $H'_{n+1,2}$.

We have established that the classes $H_{l,p}$ and $H'_{l,p}$ not only are Cayley graphs but also are Hamiltonian [4]. The present work shows that the Cayley graphs $H'_{l,p}$ and $H_{l,p}$ have logarithmic diameter and low degree.

2 Cayley graphs

In this section, we define two new families of Cayley graphs, denoted by graph $H_{l,p}$ and graph $H'_{l,p}$. These families are Cayley graphs of the same Abelian group. First, we give some definitions about Cayley graphs.

Definition 2.1. A subset C of the group G is a *generating set* if every element of G can be expressed as a finite product of elements in C. We also say that G is generated by C.

Let C be a generating set for a group $(G, +)$. Now, we define a Cayley graph.

Definition 2.2. We say that a directed graph $\Gamma = (V, E)$ is a *Cayley graph* associated to a group $(G, +)$ with a generating set C, if there is a bijection mapping every $v \in V$ to an element $g_v \in G$, such that (v, w) is a directed edge of E if and only if there exists $c \in C$ such that $g_w = c + g_v$.

If the identity element $\iota \notin C$, then there are no loops in Γ, which we define as the *identity free* property. If $c \in C$ implies $-c \in C$, then for every edge from g to $g+c$, there is also an edge from $g+c$ to $(g+c)+(-c) = g$, which we define as the *symmetry condition.* A Cayley graph with identity free property and symmetry condition is an undirected graph. In this paper, we only consider these graphs.

Lemma 2.1. [2] *Cayley graphs are vertex transitive.*

2.1 Graph $H_{l,p}$

In this section, we define of the group $(V_{l,p}, +)$ and generating set $C_{l,p}$. Following, we show that the graph $H_{l,p}$ is the Cayley graph associated to the group $(V_{l,p}, +)$ with generating set $C_{l,p}$.

For each $l \geq 3$ and $p \geq 3$, Holyer [1] defined a graph $H_{l,p} = (V_{l,p}, E_{l,p})$ by

$V_{l,p} = \{x = (x_1, ..., x_l) \in Z_p^l \text{ with } \sum_{i=1}^{l} x_i = 0 \bmod p\}$,

$E_{l,p} = \{(x, y) :$ there are distinct i, j such that $y_k \equiv_p x_k$ for $k \neq \{i, j\}$

and $y_i \equiv_p x_i + 1,\ \ y_j \equiv_p x_j - 1$ or $y_i \equiv_p x_i - 1,\ \ y_j \equiv_p x_j + 1\}$.

The graph $H_{l,p}$ consists of vertices that contain l components with values between 0 and $p-1$, such that, the sum of its components is equivalent to $0 \mod p$, with $p \in Z_+$. Figure 1 shows a representation of the graph $H_{3,4}$ with repetition of vertices. This graph is not a planar graph because the graph $H_{3,4}$ is regular of degree 6, but the drawing presented is a representation on the torus without crossings of edges.

The vertices of the graph $H_{l,p}$ are the elements of a finite group $(V_{l,p}, +)$, where the operation $+$ is $(f_{a_1}, \ldots, f_{a_l}) + (f_{b_1}, \ldots, f_{b_l}) = (f_{a_1} + f_{b_1}, \ldots,$

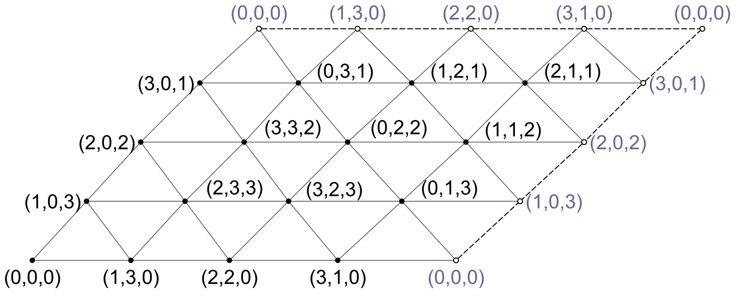

Figure 1: Graph $H_{3,4}$ with repetition of vertices

$f_{a_l} + f_{b_l})$, with f_{a_i} and $f_{b_i} \in (Z_p, +)$. The pair $(V_{l,p}, +)$ is a finite group. The element $(0, 0, \ldots, 0, 0) \in V_{l,p}$ is the identity element.

Definition 2.3. We define the set $C_{l,p} = \{(c_1, \ldots, c_l) \in V_{l,p}$: there are $i, j \in \{1, \ldots, l\}, i \neq j$, such that $c_i = 1, c_j = p - 1$ and, for all $k \in \{1, \ldots, l\} - \{i, j\}$ $c_k = 0\}$.

Theorem 2.1. The graph $H_{l,p} = (V_{l,p}, E_{l,p})$ is the Cayley graph of the group $(V_{l,p}, +)$ with the generating set $C_{l,p}$.

Proof (Sketch). To prove that the graph $H_{l,p}(V_{l,p}, E_{l,p})$ is a Cayley graph we have to show that: 1)the identity element $\iota \notin C_{l,p}$; 2) the set $C_{l,p}$ has the symmetry condition; 3) $C_{l,p}$ is a generating set of the group $(V_{l,p}, +)$. □

Corollary 2.1. The Cayley graph $(V_{l,p}, E_{l,p})$ is isomorphic to the graph $H_{l,p}$.

Lemma 2.2. [1] *The graph $H_{l,p}$ has the following properties: a) the number of vertices is p^{l-1}; b) the degree of each vertex is $2\binom{l}{2} = l(l-1)$; c) if $\log_2 p > l$, then degree is less than $\log_2 n$.*

2.2 Graph $H'_{l,p}$

In this section, we show that the graph $H'_{l,p}$ is the Cayley graph of the same group $(V_{l,p}, +)$ but associated with the generating set $C'_{l,p}$.

Definition 2.4. We define the *set* $C'_{l,p} = \{(c'_1, \dots, c'_l) \in V_{l,p}$: there is $i \in \{1, \dots, l-1\}$ such that $c'_i = 1, c'_l = -1$ or $c'_i = -1, c'_l = 1$ and, for all $k \in \{1, \dots, l-1\} - \{i\}$, we have that $c'_k = 0\}$.

Theorem 2.2. The graph $H'_{l,p}(V_{l,p}, E'_{l,p})$ is the Cayley graph associated to the group $(V_{l,p}, +)$ with generating set $C'_{l,p}$.

Proof (Sketch). It follows the sketch for Theorem 2.1. □

Lemma 2.3. *The graph* $H'_{l,p}$ *has the following properties: a) the degree of each vertex is* $2(l-1)$*; b) if* $p > 4$*, then degree is less than* $\log_2 n$*.*

Proof (Sketch). Property a) follows from $|C'_{l,p}| = 2(l-1)$ and b) if $p > 4$, then $\log_2 p > 2$, hence $\log_2 p^{l-1} > 2(l-1)$, and so $\log_2 n > d$. □

3 Diameter

In this section, we study the diameter of the graphs $H_{l,p}$ and $H'_{l,p}$. The distance $d(u, v)$ between the vertices u and v in a graph is the number of edges in a shortest path connecting them. The diameter D is the largest distance among all pairs of vertices.

Lemma 2.1 shows that Cayley graphs are vertex-transitive, then the problem to find the distance between two vertices can be reduced to find the distance between a vertex v and the identity vertex. We can define $d(v)$ as the minimum length generating sequence for the element v [3]. So, in order to find the diameter is sufficient to calculate the greatest distance between the identity vertex and all other vertices. The vertices that hold this distance are called *diametral.*

3.1 Graph $H_{l,p}$

For graph $H_{l,p}$, we use the following results to find the diameter.

Lemma 3.1. *The distance satisfies* $d(x) = d(x')$*, where* x' *is a permutation of the components of* x*, such that, the components in* x' *are in ascending order, i.e.,* $x'_i \le x'_{i+1}$*.*

Proof (Sketch). Let $x = \{x_1, \ldots, x_l\} \in H_{l,p}$. Any vertex associated with a permutation of the components of x has the same distance to identity vertex as x, i.e., the set of these permutations forms an equivalence class with respect to distance. The vertex x' has ascending order of the components of x. □

Now, we know that two vertices x and x', with the same components, have identical distance and we can always compute the distance between the vertex x' to the identity. Considering that each step in the graph increases a component by one unit and decrease other component by one unit of a vertex, i.e., the vertex is divided in two parts. The first part of the vertex contains l_- components and these components are subtracted of 1 until each l_- component gets value 0. The second part of the vertex contains l_+ components and these components are added of 1 until each l_+ component gets value p. Let s_- be the sum of the first part and s_+ be the sum of the second part. By definition, we have that $s_- - s_+ = 0$. We denote by $s_t = \sum_{i=1}^{l} x_i$.

Lemma 3.2. *The distance of the vertex x' of $H_{l,p}$ is $d(x') = s_-$, where $s_- = \sum_{i=1}^{l_-} x'_i = \sum_{j=l_-+1}^{l} p - x'_j$ and $l_- = l - \frac{1}{p} \sum_{i=1}^{l} x'_i$.*

Proof (Sketch). As $l = l_- + l_+$, $s_- = s_+$ and $d(x', \iota) = s_-$, it follows $s_- - s_+ = 0 \Rightarrow \sum_{i=1}^{l_-} x'_i - \sum_{i=l_-+1}^{l} (p - x'_i) = 0$, $\sum_{i=1}^{l_-} x'_i + \sum_{i=l_-+1}^{l} x'_i - \sum_{i=l_-+1}^{l} p = 0 \Rightarrow s_t - p(l - l_-) = 0$. $s_t = p(l - l_-) \Rightarrow l_- = l - \frac{1}{p} \sum_{i=1}^{l} x'_i$.
□

The distance is an integer number for which there is a $q \in Z$ such that $s_t = p * q$. The problem is that we do not have the value of l_- as a function only of l and p. If we want to find the diametral vertex of graph $H_{l,p}$, then we need to find the best value for l_-. We have the maximum value of the sum s_t.

Definition 3.1. We define $y = (y_a, \ldots, y_a, y_b, \ldots, y_b) \in H_{l,p}$ for $l_a = (l_- * p) \mod l$ and $l_b = l - l_a$ as the vertex y, where $y_b = y_a + 1$ or $y_b \equiv_p 0$

and l_a is the number of components y_a and l_b is the number of components y_b.

For example, the graph $H_{5,7}$ for $l_- = 2$ has $l_a = 7 * 2 \mod 5 = 4$ and for $l_- = 3$ has $l_a = 7 * 3 \mod 5 = 1$.

Lemma 3.3. *There is a vertex y in graph $H_{l,p}$ for each value l_- that has the maximum value of the sum s_t.*

Lemma 3.4. *Let y be a vertex by definition 3.1, then the value of the component y_a is*

$$y_a = \begin{cases} (l_+ * (p-1) + (l_a - l_-))/l & \textit{if } l_a \geq l_-, \\ (((p * l_+) + l_a)/l) - 1 & \textit{if } l_a < l_-. \end{cases}$$

Lemma 3.5. *Let y be a vertex by definition 3.1, then the distance $d(y)$ is*

$$d(y) = \begin{cases} l_- * y_a & \textit{if } l_a \geq l_-, \\ (l_- * (y_a + 1)) - l_a & \textit{if } l_a < l_-. \end{cases}$$

So, one can obtain the diameter with the following result.

Theorem 3.1. Let y be a vertex with maximum value of the sum s_t for each value of the l_-. The diameter of the graph $H_{l,p}$ is $D = max(d(y))$ for $1 \leq l_- < l$.

Proof (Sketch). Since l_- has maximum value of the sum s_t among the others, then we have the diameter. □

Theorem 3.1 yields an algorithm to find the diameter of the graph $H_{l,p}$ with time complexity $O(l)$.

3.2 Graph $H'_{l,p}$

For the graph $H'_{l,p}$, one can obtain the distance with the following result.

Lemma 3.6. *The distance of the vertex x of $H'_{l,p}$ is $d(x) = \sum_{i=1}^{l-1} min(x_i, p - x_i)$.*

Next, in Theorem 3.2, we determine the diameter of the graphs $H'_{l,p}$, that is obtained as consequence of Lemma 3.6.

Theorem 3.2. *The diameter of the graph $H'_{l,p}$ is $D = (\lfloor \frac{p}{2} \rfloor (l-1))$.*

Proof. By Lemma 3.6 we have that the diametral vertex satisfies $(l - 1) * min(x_i, p - x_i)$. If $x_i = \lfloor \frac{p}{2} \rfloor$ we have that $x_i = p - x_i$, then the maximum value of x_i is $\lfloor \frac{p}{2} \rfloor$. Therefore, the diameter of the graph $H'_{l,p}$ is $D = (\lfloor \frac{p}{2} \rfloor (l-1))$. ■

4 Conclusion

Several authors observed that Cayley graphs provide a useful and unified framework for the design of interconnection networks for parallel computers. We presented and analyzed two new families of Cayley graphs for interconnection networks called graphs $H_{l,p}$ and $H'_{l,p}$.

The graph $H'_{l,p}$ has degree $2(l-1)$ and diameter $D = (\lfloor \frac{p}{2} \rfloor (l-1))$. The graph $H_{l,p}$ has degree $l(l-1)$ and your diameter is less than $(\lfloor \frac{p}{2} \rfloor (l-1))$, and can be found in linear time over parameter l. How the number of vertices of both graphs is p^{l-1}, yours degrees is logarithmic.

Considering these established properties, we propose that the Cayley graphs $H_{l,p}$ and $H'_{l,p}$ are good schemes of interconnection networks.

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