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Descent theory of simple sheaves on C_1-fields

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Abstract

Let K be a C_1-field of any characteristic and X a projective variety over K. In this article we prove that for a finite Galois extension L of K, a simple sheaf with covering datum on $X \times_K L$ descends to a simple sheaf on X. As a consequence, we show that there is a $1-1$ correspondence between the set of geometrically stable sheaves on X with fixed Hibert polynomial P and the set of K-rational points of the corresponding moduli space.

1 Introduction

Let $f : Y \to X$ be a morphism of schemes and $\mathrm{pr}_i : Y \times_X Y \to Y$ the natural projection morphisms. For any quasi-coherent sheaf $\mathcal{F}$ on Y,

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a *covering datum of* $\mathcal{F}$ is an isomorphism $\phi : \mathrm{pr}_1^* \mathcal{F} \to \mathrm{pr}_2^* \mathcal{F}$. If f is faithfully flat and quasi-compact then the functor from the category of quasi-coherent sheaves on X to the category of quasi-coherent sheaves on Y with covering datum, induced by pull-back by f, is fully faithful (see [1, §6.1, Proposition 1]). Let us consider the simple case when X is a projective variety over a field K, L a finite Galois extension of K and $Y := X \times_K \mathrm{Spec}(L)$ is the base change of X by the field extension L of K. For a coherent sheaf $\mathcal{E}_L$ on Y, descent theory gives a criterion for when there exists a coherent sheaf $\mathcal{E}$ on X such that $f^*\mathcal{E} \cong \mathcal{E}_L$. The descent datum associated to $\mathcal{E}_L$ consists of a covering datum satisfying a cocycle condition. In general, it is not possible to associate to $\mathcal{E}_L$ a descent datum, for example, if $\mathcal{E}_L$ is the structure sheaf of an L-point on Y that does not come from a K-point of X, then there does not exist a descent datum associated to $\mathcal{E}_L$.

Recall that a field is called C_1 (pseudo-algebraically closed) if any degree d polynomial in n variables with coefficients in the field and $n > d$ has a non-trivial solution. These fields have been studied by Tsen [19], Lang [13], Chevalley [2], Greenberg [6], Grabber-Harris-Starr [5] and many others (see [9, §2.1] for a short introduction to C_1 fields). In this article, we consider the case when X is a projective variety over a C_1-field K, L is a finite Galois extension of K and $Y := X \times_K \mathrm{Spec}(L)$. We call a simple (resp. geometrically stable) sheaf $\mathcal{E}_L$ on Y, K-*simple* (resp. K-*geometrically stable*) if one can associate a covering datum to $\mathcal{E}_L$ (see Definition 2.7).

We prove that:

Theorem 1.1. The natural functor from the category $\mathfrak{S}_X$ of simple sheaves on X to the category $\mathfrak{S}\mathfrak{K}_Y$ of K-simple sheaves on Y, defined by pull-back of sheaves via f, is an equivalence of categories.

Clearly, any coherent sheaf $\mathcal{E}_L$ with covering datum on Y descends to X if the covering datum satisfies the cocycle condition. However, we show that the cocycle condition is indeed satisfied if $\mathcal{E}_L$ is K-simple and

$H^2(K, \mathbb{G}_m)$ vanishes (see proof of Proposition 2.8). Since K is a C_1-field, the cohomology group $H^2(K, \mathbb{G}_m)$ is trivial (see [4, Proposition 6.2.3]). We then give an application of Theorem 1.1. Fix P the Hilbert polynomial of a coherent sheaf on X with rank coprime to degree. Denote by $M_X(P)$ the moduli scheme of geometrically stable sheaves on X with Hilbert polynomial P. Recall, a point $x \in X$ is called a K-*rational point* if the corresponding residue field is contained in K. We show:

Theorem 1.2. There is a $1-1$ correspondence between the set of K-rational points of $M_X(P)$ and the set of geometrically stable sheaves on X with Hilbert polynomial P.

We note that a similar result to Theorem 1.2 for stacks has been proven by Kraschen and Lieblich in [12, Proposition 1.1.5]. However their proof uses the theory of gerbes. In contrast our proof uses only basic algebraic geometry.

In §4 we give possible applications of the above theorems to the existence of rational points (C_1-conjecture due to Lang, Manin and Kollár) and index of varieties.

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2 Galois action on simple sheaves

We begin by recalling the basic definitions and results we need.

Definition 2.1. Let $f : S' \to S$ be a morphism of schemes. Set $S'' := S' \times_S S', S''' := S' \times_S S' \times_S S'$, $\mathrm{pr}_i : S'' \to S'$ and $\mathrm{pr}_{ij} : S''' \to S''$ the natural projections onto the factors with indices i and j, for $i < j$, $i, j \in \{1, 2, 3\}$. A *descent datum* on a quasi-coherent sheaf $\mathcal{F}$ on S' is a covering datum $\phi : \mathrm{pr}_1^* \mathcal{F} \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{F}$ on $\mathcal{F}$ which satisfies the cocycle condition $\mathrm{pr}_{13}^* \phi = \mathrm{pr}_{23}^* \phi \circ \mathrm{pr}_{12}^* \phi$ i.e., $\mathrm{pr}_{13}^* \phi$ coincides with the composition:

$$\mathrm{pr}_{13}^* \mathrm{pr}_1^* \mathcal{F} \cong \mathrm{pr}_{12}^* \mathrm{pr}_1^* \mathcal{F} \xrightarrow{\mathrm{pr}_{12}^* \phi} \mathrm{pr}_{12}^* \mathrm{pr}_2^* \cong \mathrm{pr}_{23}^* \mathrm{pr}_1^* \mathcal{F} \xrightarrow{\mathrm{pr}_{23}^* \phi} \mathrm{pr}_{23}^* \mathrm{pr}_2^* \mathcal{F} \cong \mathrm{pr}_{13}^* \mathrm{pr}_2^* \mathcal{F},$$

where the three isomorphisms follow from $\mathrm{pr}_1 \circ \mathrm{pr}_{13} = \mathrm{pr}_1 \circ \mathrm{pr}_{12}$, $\mathrm{pr}_1 \circ \mathrm{pr}_{23} = \mathrm{pr}_2 \circ \mathrm{pr}_{12}$ and $\mathrm{pr}_2 \circ \mathrm{pr}_{23} = \mathrm{pr}_2 \circ \mathrm{pr}_{13}$, respectively.

Proposition 2.2 ([1, §6.1, Proposition 1])**.** Notations as in Definition 2.1. Let $f : S' \to S$ be a faithfully flat and quasi-compact morphism of schemes and $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent S-modules and set $q := f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$. Then, identifying $q^*\mathcal{F}$ (resp. $q^*\mathcal{G}$) canonically with $\mathrm{pr}_i^*(f^*\mathcal{F})$ (resp. $\mathrm{pr}_i^*(f^*\mathcal{G})$) for $i = 1, 2$, the sequence

$$\mathrm{Hom}_S(\mathcal{F}, \mathcal{G}) \xrightarrow{f^*} \mathrm{Hom}_{S'}(f^*\mathcal{F}, f^*\mathcal{G}) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_2^*}{\rightrightarrows}} \mathrm{Hom}_{S''}(q^*\mathcal{F}, q^*\mathcal{G})$$

is exact. In other words, the functor $\mathcal{F} \mapsto f^*\mathcal{F}$ from quasi-coherent S-modules to quasi-coherent S'-modules with covering datum is fully faithful.

We now recall the definition of semi-stable sheaves and simple sheaves.

Definition 2.3. Let $\mathcal{E}$ be a coherent sheaf with support of dimension d. The Hilbert polynomial $P(\mathcal{E})(t)$ of $\mathcal{E}$ can be expressed as (see [7, Lemma 1.2.1])

$$P(\mathcal{E})(t) := \chi(\mathcal{E} \otimes \mathcal{O}_{X_k}(t)) = \sum_{i=0}^{d} \alpha_i(\mathcal{E}) \frac{t^i}{i!} \text{ for } t >> 0.$$

The reduced Hilbert polynomial is defined as $P_{\mathrm{red}}(\mathcal{E})(t) := \dfrac{P(\mathcal{E})(t)}{\alpha_d(\mathcal{E})}$. The sheaf $\mathcal{E}$ is called *Gieseker (semi)stable* if for any proper subsheaf $\mathcal{F} \subset \mathcal{E}$, $P_{\mathrm{red}}(\mathcal{F})(t)(\leq) < P_{\mathrm{red}}(\mathcal{E})(t)$ for all t large enough. In other words, $\mathcal{E}$ is (semi)stable if properly included subsheaves have (strictly) smaller reduced Hilbert polynomials.

Definition 2.4. A sheaf $\mathcal{E}$ defined on a projective variety defined over a field k is called *simple* if $\mathrm{End}(E) \simeq k$.

Lemma 2.5 ([8, Corollary 1.2.8])**.** If $\mathcal{E}$ is a stable sheaf on a projective variety defined over an algebraically closed field, say k, then $\mathrm{End}(E) \simeq k$.

Notation 2.6. Let K be a C_1 field of any characteristic, X a projective variety over K. Let $K \subset L$ be an algebraic field extension of K (not necessarily finite). There exist natural morphisms

$$\mathrm{pr}_{1,L} : L \to L \otimes_K L, \, \mathrm{pr}_{2,L} : L \to L \otimes_K L$$

where

$$\mathrm{pr}_{1,L}(a) = a \otimes 1 \text{ and } \mathrm{pr}_{2,L}(a) = 1 \otimes a.$$

This induces morphisms

$$\mathrm{pr}_{i,L} : X_{L \otimes_K L} \to X_L \text{ for } i = 1, 2.$$

Denote by $G_L := \mathrm{Gal}(L/K)$ the Galois group. For any $\sigma \in G_L$, we denote by

$$\sigma : X_L \to X_L$$

the induced natural morphism. Moreover, for $\sigma, \tau \in G_L$, we have $\sigma\tau : L \xrightarrow{\tau} L \xrightarrow{\sigma} L$. As taking spectrum is contravariant, this induces $(\sigma\tau) : X_L \xrightarrow{\sigma} X_L \xrightarrow{\tau} X_L$. Thus, for any coherent sheaf $\mathcal{E}$ on X_L, the pull-back $(\sigma\tau)^*\mathcal{E} = (\sigma^* \circ \tau^*)\mathcal{E}$. In the case $L = \overline{K}$, the algebraic closure, denote by $\mathrm{pr}_i := \mathrm{pr}_{i,\overline{K}}$ and $G := G_{\overline{K}}$.

Definition 2.7. Recall, a sheaf $\mathcal{E}$ on X_L is called *geometrically stable* if for any field extension L' of L, the sheaf $\mathcal{E} \otimes_L L'$ is stable over $X_L \times_L \mathrm{Spec}(L')$. We call a simple (resp. geometrically stable) sheaf $\mathcal{E}$ on X_L, K-*simple* (resp. K-*geometrically stable*) if there exists an isomorphism $\psi : \mathrm{pr}_{1,L}^* \mathcal{E} \to \mathrm{pr}_{2,L}^* \mathcal{E}$. In other words, a K-simple (resp. K-geometrically stable) sheaf is a simple (resp. geometrically stable) sheaf on X_L such that one can associate to it a covering datum.

We now study the action of the Galois group G_L on a simple sheaf $\mathcal{E}$ on X_L. We observe that in the case L is a finite Galois extension of K, $\mathcal{E}$ descends to X if and only if it is K-simple (Theorem 2.9).

Proposition 2.8. Let $\mathcal{E}$ be a K-simple sheaf on X_L. Then, there exists a collection $(\lambda_\sigma)_{\sigma \in G_L}$ of isomorphisms $\lambda_\sigma : \mathcal{E} \to \sigma^* \mathcal{E}$ satisfying the cocycle condition:

$$(\sigma^* \lambda_\tau) \circ \lambda_\sigma = \lambda_{\sigma\tau} \text{ for any pair } \sigma, \tau \in G_L.$$

Proof. Fix $\sigma \in G_L$. Consider the homomorphism $L \otimes_K L \to L$ defined by $a \otimes b$ maps to $a\sigma(b)$. This induces a natural morphism

$$p_\sigma : X_L \to X_{L \otimes_K L}.$$

Observe that the morphism p_σ has the property that its composition with $\mathrm{pr}_{1,L}$

$$X_L \xrightarrow{p_\sigma} X_{L \otimes_K L} \xrightarrow{\mathrm{pr}_{1,L}} X_L$$

is simply the identity map and with $\mathrm{pr}_{2,L}$,

$$X_L \xrightarrow{p_\sigma} X_{L \otimes_K L} \xrightarrow{\mathrm{pr}_{2,L}} X_L$$

is the morphism $\sigma : X_L \to X_L$. Then, $\mathrm{pr}_{1,L}^* \mathcal{E} \cong \mathrm{pr}_{2,L}^* \mathcal{E}$ implies that

$$\mathcal{E} \cong p_\sigma^* \mathrm{pr}_{1,L}^* \mathcal{E} \cong p_\sigma^* \mathrm{pr}_{2,L}^* \mathcal{E} \cong \sigma^* \mathcal{E}.$$

Therefore, for any $\sigma \in G_L$, there exists an isomorphism $\lambda'_\sigma : \mathcal{E} \to \sigma^* \mathcal{E}$. Let $\tau \in G_L$. Since $\mathcal{E}$ is simple,

$$\mathrm{End}(\sigma^* \tau^* \mathcal{E}) \cong \sigma^* \tau^* \mathrm{End}(\mathcal{E}) \cong L.$$

Hence, there exists $a_{\sigma,\tau} \in L^\times = \mathrm{Aut}(\sigma^* \tau^* \mathcal{E})$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\lambda'_\sigma} & \sigma^* \mathcal{E} \\ \lambda'_{\sigma\tau} \downarrow & \circlearrowleft & \downarrow \sigma^* \lambda'_\tau \\ \sigma^* \tau^* \mathcal{E} & \xrightarrow{a_{\sigma,\tau}} & \sigma^* \tau^* \mathcal{E} \end{array}$$

This directly implies the following equalities: Given $g_1, g_2, g_3 \in G_L$, we have

$$a_{g_1,(g_2g_3)} \circ \lambda'_{g_1(g_2g_3)} = (g_1^*\lambda'_{g_2g_3}) \circ \lambda'_{g_1} \tag{2.1}$$
$$g_1^* a_{g_2g_3} \circ g_1^*\lambda'_{g_2g_3} = (g_1^* g_2^* \lambda'_{g_3}) \circ g_1^*\lambda'_{g_2} \tag{2.2}$$
$$a_{g_1,g_2} \circ \lambda'_{g_1g_2} = (g_1^*\lambda'_{g_2}) \circ \lambda'_{g_1} \tag{2.3}$$
$$((g_1g_2)^*\lambda'_{g_3}) \circ \lambda'_{g_1g_2} = a_{(g_1g_2),g_3} \circ \lambda'_{(g_1g_2)g_3} \tag{2.4}$$
$$\tag{2.5}$$

Applying $g_1^* a_{g_2,g_3} \circ -$ to both sides of (2.1), we get,

$$\begin{aligned} g_1^* a_{g_2,g_3} \circ a_{g_1,(g_2g_3)} \circ \lambda'_{g_1(g_2g_3)} &= g_1^* a_{g_2,g_3} \circ (g_1^*\lambda'_{g_2g_3}) \circ \lambda'_{g_1} \\ &= (g_1^* g_2^* \lambda'_{g_3}) \circ g_1^*\lambda'_{g_2} \circ \lambda'_{g_1} \quad \text{by (2.2)} \\ &= (g_1^* g_2^* \lambda'_{g_3}) \circ a_{g_1,g_2} \circ \lambda'_{g_1g_2} \quad \text{by (2.3)} \\ &= a_{g_1,g_2} \circ a_{(g_1g_2),g_3} \circ \lambda'_{g_1g_2g_3} \quad \text{by (2.4)} \end{aligned}$$

where the last equality follows from the fact that multiplication by a scalar a_{g_1,g_2} commutes with $(g_1^* g_2^* \lambda'_{g_3})$. Since $\lambda'_{g_1g_2g_3}$ is an isomorphism, we have the 2-cocycle condition:

$$g_1^* a_{g_2,g_3} \circ a_{g_1,(g_2g_3)} = a_{g_1,g_2} \circ a_{(g_1g_2),g_3}.$$

Since K is a C_1 field, $H^2(K, \mathbb{G}_m) = 0$ (see [16, p. 161, Proposition 10]). This means that for any sequence $(a_{\sigma,\tau})_{\sigma,\tau \in G_L}$ satisfying the 2-cocycle condition there exists a continuous morphism $\phi : G_L \to L^\times$ such that

$$a_{\sigma,\tau} = \sigma\phi(\tau)\phi(\sigma\tau)^{-1}\phi(\sigma).$$

Consider now the isomorphism given by

$$\lambda_\sigma := \phi(\sigma)^{-1}\lambda'_\sigma : \mathcal{E} \xrightarrow{\lambda'_\sigma} \sigma^*\mathcal{E} \xrightarrow{\phi(\sigma)^{-1}} \sigma^*\mathcal{E}, \text{ for all } \sigma \in G_L.$$

Since $\phi(\sigma)$ is scalar, it commutes with $\lambda'_{\sigma\tau}$ i.e., we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\lambda'_{\sigma\tau}} & (\sigma\tau)^*\mathcal{E} \\ \phi(\sigma)\downarrow & \circlearrowleft & \downarrow\phi(\sigma) \\ \mathcal{E} & \xrightarrow{\lambda'_{\sigma\tau}} & (\sigma\tau)^*\mathcal{E} \end{array}$$

Using (2.3) and substituting for $a_{\sigma,\tau}$, we conclude that $\sigma^*\lambda'_\tau \circ \lambda'_\sigma$ equals

$$a_{\sigma,\tau} \circ \lambda'_{\sigma\tau} = \sigma\phi(\tau)\phi(\sigma\tau)^{-1}\phi(\sigma)\lambda'_{\sigma\tau} = \sigma\phi(\tau)\phi(\sigma\tau)^{-1}\lambda'_{\sigma\tau}\phi(\sigma).$$

Therefore,

$$(\sigma^*\lambda_\tau) = (\sigma\phi(\tau))^{-1}\sigma^*\lambda'_\tau = (\phi(\sigma\tau)^{-1}\lambda'_{\sigma\tau}) \circ (\phi(\sigma)^{-1}\lambda'_\sigma)^{-1} = \lambda_{\sigma\tau} \circ \lambda_\sigma^{-1}.$$

Hence, $(\sigma^*\lambda_\tau) \circ \lambda_\sigma = \lambda_{\sigma\tau}$. This proves the proposition. □

Theorem 2.9. Let L be a finite Galois extension of K and $f : X_L \to X$ the natural morphism. The natural functor from the category $\mathfrak{S}_X$ of simple sheaves on X to the category $\mathfrak{S}\mathfrak{K}_{X_L}$ of K-simple sheaves on X_L, defined by pull-back of sheaves via f, is an equivalence of categories.

Proof. It suffices to show that any simple sheaf $\mathcal{E}_L$ on X_L is K-simple if and only if there exists a simple sheaf $\mathcal{E}$ on X such that $f^*\mathcal{E} \cong \mathcal{E}_L$. By Proposition 2.2, for any simple sheaf $\mathcal{E}$ on X, we have $f^*\mathcal{E}$ is K-simple on X_L. We now prove the converse. Let $\mathcal{E}_L$ be K-simple on X_L. Proposition 2.8 implies that there exists a collection $(\lambda_\sigma)_{\sigma\in G_L}$ of isomorphisms $\lambda_\sigma : \mathcal{E}_L \to \sigma^*\mathcal{E}_L$ such that $(\sigma^*\lambda_\tau) \circ \lambda_\sigma = \lambda_{\sigma\tau}$ for any pair $\sigma, \tau \in G_L$. By Galois descent, this implies that there exists a coherent sheaf $\mathcal{E}$ on X such that $\mathcal{E}_L \cong f^*\mathcal{E}$. Since f is flat,

$$f^*(\mathrm{End}(\mathcal{E})) \cong \mathrm{End}(\mathcal{E}_L) \cong L.$$

As $\mathrm{End}(\mathcal{E})$ is a K-vector space, this directly implies that $\mathrm{End}(\mathcal{E}) \cong K$ i.e., $\mathcal{E}$ is simple. This proves the theorem. □

3 Moduli of K-geometrically stable sheaves

Keep Notations 2.6. In this section we use Theorem 2.9 to prove that the set of K-rational points of the moduli space of geometrically stable sheaves on X is in $1-1$ correspondence with the set of geometrically stable sheaves on X.

Recall, the definition of the moduli functor of semi-stable sheaves over a projective variety X.

Definition 3.1. Fix P the Hilbert polynomial of a coherent sheaf on X with rank coprime to degree. Denote by $\mathcal{M}_X(P)$ the moduli functor:

$$\mathcal{M}_X(P) : \{\mathrm{Sch}/K\}^{\circ} \to \mathrm{Sets}$$

such that for a K-scheme T,

$$\mathcal{M}_X(P)(T) := \left\{ \begin{array}{l} \text{isomorphism classes of pure sheaves } \mathcal{F} \text{ on } X \times T \text{ flat} \\ \text{over } T \text{ and for every geometric point } t \in T,\ \mathcal{F}|_{X_t} \\ \text{is a stable sheaf with Hilbert polynomial } P \text{ on } X_t \end{array} \right\} / \sim$$

where $\mathcal{F} \sim \mathcal{G}$ if there exists an invertible sheaf $\mathcal{L}$ on T such that $\mathcal{F} \cong \mathcal{G} \otimes p_T^*\mathcal{L}$, where $p_T : X_T \to T$ is the natural projection map.

Theorem 3.2 ([14, Theorem 4.1])**.** Let R be a universally Japanese ring and $f : X \to S$ a projective morphism of R-schemes of finite type with geometrically connected fibres, and let $\mathcal{O}_X(1)$ be an f-ample line bundle. Then, for a fixed polynomial P, there exists a projective S-scheme $M_{X/S}(P)$ of finite type over S which uniformly corepresents the functor

$$\mathcal{M}_{X/S}(P) : \{\mathrm{Sch}/S\}^{\circ} \to \mathrm{Sets}$$

such that for a S-scheme T,

$$\mathcal{M}_{X/S}(P)(T) := \left\{ \begin{array}{c} S \text{ equivalence classes of pure sheaves } \mathcal{F} \text{ on } X \times_S T \text{ flat} \\ \text{over } T \text{ such that for every geometric point } t \in T,\ \mathcal{F}|_{X_t} \\ \text{is a semi-stable sheaf with Hilbert polynomial } P \text{ on } X_t \end{array} \right\} / \sim$$

where $\mathcal{F} \sim \mathcal{G}$ if there exists an invertible sheaf $\mathcal{L}$ on T such that $\mathcal{F} \cong \mathcal{G} \otimes p_T^*\mathcal{L}$, where $p_T : X \times_S T \to T$ is the natural projection map.

Moreover, there is an open subscheme $M^s_{X/S}(P)$ of $M_{X/S}(P)$ which universally corepresents the subfunctor of families of geometrically stable sheaves.

Remark 3.3. Note that a C_1-field K is a universally Japanese ring. If the rank and degree of coherent sheaves with Hilbert polynomial P, are coprime, then stability and semi-stability coincide (see [8, Lemmas 1.2.13 and 1.2.14]). Then by Theorem 3.2, there exists a projective K-scheme of finite type $M_X(P)$, universally corepresenting the functor $\mathcal{M}_X(P)$.

We now review briefly the construction of the moduli scheme $M_X(P)$. By [14, Theorem 4.2], there exists an integer e such that any semi-stable sheaf on X with Hilbert polynomial P is e-regular (in the sense of Castelnuovo-Mumford regularity). Fix such an integer e. Denote by $\mathcal{H} := \mathcal{O}_X(-e)^{\oplus P(e)}$ and by $\mathrm{Quot}_{\mathcal{H}/X/P}$ the Quot scheme parametrizing all quotients of the form $\mathcal{H} \twoheadrightarrow \mathcal{Q}_0$, where $\mathcal{Q}_0$ has Hilbert polynomial P (see [15, §4.4] for more details).

Let $\mathcal{R}$ be the subset of $\mathrm{Quot}_{\mathcal{H}/X/P}$ consisting of all points which parametrize quotients of the form $\mathcal{H} \twoheadrightarrow \mathcal{Q}_0$ such that $\mathcal{Q}_0$ is semi-stable and $H^0(\mathcal{Q}_0(e))$ is (non-canonically) isomorphic to $k^{\oplus P(e)}$. Now, semi-stability is an open condition (see [8, Proposition 2.3.1]). Therefore, $\mathcal{R}$ is an open subscheme in $\mathrm{Quot}_{\mathcal{H}/X/P}$. The group $\mathrm{GL}(P(e)) = \mathrm{Aut}(\mathcal{H})$ acts on $\mathrm{Quot}_{\mathcal{H}/X/P}$ from the right by the composition $[\rho] \circ g = [\rho \circ g]$, where $[\rho : \mathcal{H} \to \mathcal{F}] \in \mathrm{Quot}_{\mathcal{H}/X/P}$ and $g \in \mathrm{GL}(P(e))$. By [14, Theorem 4.3], $\mathcal{R}$ is the set of semi-stable points of $\mathrm{Quot}_{\mathcal{H}/X/P}$ under this group action. The moduli scheme $M_X(P)$ of semi-stable sheaves on X with Hilbert polynomial P is the geometric quotient of $\mathcal{R}$ under this action (see [14, pp. 582, after Theorem 4.3]). Denote by

$$\pi : \mathcal{R} \to M_X(P) \tag{3.1}$$

the corresponding quotient morphism. The quotient exists due to Seshadri's result [17, Theorem 4].

Theorem 3.4. There is a $1-1$ correspondence between the set of K-rational points of $M_X(P)$ and the set of isomorphism classes of geometrically stable sheaves on X with Hilbert polynomial P.

Proof. Let $x : \mathrm{Spec}(K) \to M_X(P)$ be a K-rational point of $M_X(P)$. Denote by $\mathcal{R}_K$ the base change of the morphism π in (3.1) by the morphism x. Let $y \in \mathcal{R}_K$ be a closed point. Then y is a L-rational point on $\mathcal{R}_K$ for some finite extension L of K. Without loss of generality assume that L is a finite Galois extension of K. Since the Quot functor is representable, the point y corresponds to a quotient $\phi_y : \mathcal{H}_L \twoheadrightarrow \mathcal{E}_L$ on $X_L := X \times_K \mathrm{Spec}(L)$, where $\mathcal{H}_L \cong \mathcal{H} \otimes_K L$ and $\mathcal{E}_L$ is geometrically stable with Hilbert polynomial P on X_L. For any $\sigma \in \mathrm{Gal}(L/K)$ and the induced morphism $f_\sigma : X_L \xrightarrow{\mathrm{id}_X \times \sigma} X_L$, denote by $f_\sigma^* \phi_y$ the quotient morphism,

$$\mathcal{H}_L \cong f_\sigma^* \mathcal{H}_L \twoheadrightarrow f_\sigma^* \mathcal{E}_L.$$

By the universal property of Quot scheme, $f_\sigma^* \phi_y$ corresponds to the composed morphism:

$$\mathrm{Spec}(L) \xrightarrow{\sigma} \mathrm{Spec}(L) \xrightarrow{y} \mathcal{R}_K.$$

Since $\mathcal{R}_K$ is a fiber to the morphism π, this implies $f_\sigma^* \mathcal{E}_L \cong \mathcal{E}_L$. As a result we obtain a set $(\lambda_\sigma)_{\sigma \in \mathrm{Gal}(L/K)}$ of isomorphisms $\lambda_\sigma : \mathcal{E}_L \to \sigma^* \mathcal{E}_L$.

Consider now the morphism

$$\Phi : L \otimes_K L \to \coprod_{\sigma \in \mathrm{Gal}(L/K)} L, \text{ defined by } a \otimes b \mapsto (a\sigma(b))_{\sigma \in \mathrm{Gal}(L/K)}.$$

It is easy to check that Φ is an isomorphism. Consider now the induced morphisms,

$$\Phi_i : \coprod_{\sigma \in \mathrm{Gal}(L/K)} X_L \xrightarrow{\Phi} X_{L \otimes_K L} \xrightarrow{\mathrm{pr}_{i,L}} X_L \text{ for } i = 1, 2.$$

Observe that

$$\Phi_1 = \coprod \mathrm{id} \text{ and } \Phi_2 = \coprod_{\sigma \in \mathrm{Gal}(L/K)} f_\sigma,$$

where $f_\sigma : X_L \xrightarrow{\mathrm{id}_X \times \sigma} X_L$. The set of isomorphims $(\lambda_\sigma)_{\sigma \in \mathrm{Gal}(L/K)}$ then induce an isomorphism $\psi : \Phi_1^* \mathcal{E}_L \xrightarrow{\sim} \Phi_2^* \mathcal{E}_L$. Since Φ is an isomorphism, ψ induces an isomorphism $\mathrm{pr}_{1,L}^* \mathcal{E}_L \xrightarrow{\sim} \mathrm{pr}_{2,L}^* \mathcal{E}_L$ i.e., $\mathcal{E}_L$ is K-geometrically stable.

Since $\mathcal{E}_L$ is geometrically stable, $\mathcal{E}_{\overline{K}} := \mathcal{E}_L \otimes_L \overline{K}$ is stable. By Lemma 2.5, $\mathcal{E}_{\overline{K}}$ is simple. Since $\mathrm{End}(\mathcal{E}_L)$ is an L-vector space and

$$\mathrm{End}(\mathcal{E}_L) \otimes_L \overline{K} \cong \mathrm{End}(\mathcal{E}_{\overline{K}}) \cong \overline{K}$$

we conclude that $\mathrm{End}(\mathcal{E}_L) \cong L$ i.e., $\mathcal{E}_L$ is simple. In particular, $\mathcal{E}_L$ is K-simple. Using Theorem 2.9, there exists a simple sheaf $\mathcal{E}$ on X such that $\mathcal{E} \otimes_K L \cong \mathcal{E}_L$. By [8, Theorem 1.3.7], it follows that $\mathcal{E}$ is geometrically stable.

Conversely, by Theorem 3.2, any geometrically stable sheaf on X with Hilbert polynomial P corresponds to a K-rational point on $M_X(P)$. It is easy to check that the geometrically stable sheaf $\mathcal{E}$ corresponds to the K-rational point x of $M_X(P)$. This gives us a $1-1$ correspondence between the K-rational points of $M_X(P)$ and the set of geometrically stable sheaves on X with Hilbert polynomial P. This proves the theorem. □

4 Applications of descent theory

In this section, we mention the application of descent theory studied before. Recall, the definition of a C_1 field given in the introduction.

Example 4.1. We state without proof some examples of C_1 fields:

1. An algebraically closed field is trivially C_1.

2. Finite fields are C_1 (see [2]).

3. The function field of an irreducible curve defined over an algebraically closed field is C_1(see [19]).

4. Let R be a Henselian discrete valuation ring of characteristic 0 with residue field denoted k, of characteristic p and fraction field denoted K. If k is algebraically closed, then K is C_1 (see [13, Theorem 14]).

Definition 4.2. A variety Y over an algebraically closed field $\overline{K}$ is *separably rationally connected* if there exists a morphism $f : \mathbf{P}^1 \to Y$ such that $f^*(T_Y)$ is ample.

Remark 4.3. Note that over an algebraically closed field $\overline{K}$ of characteristic 0, rationally connected is equivalent to separably rationally connected (see [11, Proposition IV.3.3.1]).

The C_1 conjecture (Lang-Manin-Kollár): A smooth, proper, separably rationally connected variety over a C_1 field always has a rational point.

The conjecture has already been proven for various C_1 fields (see [9, Chapter 2] for a complete list).

Remark 4.4. The conjecture remains open in the case when the C_1 field is the fraction field of a maximal unramified discrete valuation ring with algebraically closed residue field of mixed characteristic.

Recently, the conjecture was shown to hold trivially for certain rationally connected varieties over such fields (see [10]). Let $M^s_{X_K,\mathcal{L}_K}(r,d)$ be the moduli space of geometrically stable locally free sheaves of rank r and determinant $\mathcal{L}_K$. Denote by $M^s_{X_{\overline{K}},\mathcal{L}_{\overline{K}}}(r,d)$ the moduli space of geometrically stable locally free sheaves of rank r and determinant $\mathcal{L}_{\overline{K}} := \mathcal{L}_K \otimes_K \overline{K}$ over the curve $X_{\overline{K}} := X_K \times_K \operatorname{Spec}(\overline{K})$. By [18], $M^s_{X_{\overline{K}},\mathcal{L}_{\overline{K}}}(r,d)$ is a unirational variety and therefore rationally connected. Since the moduli space $M^s_{X_{\overline{K}},\mathcal{L}_{\overline{K}}}(r,d)$ is the base change $M^s_{X_K,\mathcal{L}_K}(r,d) \times_K \operatorname{Spec}(\overline{K})$, this implies $M^s_{X_K,\mathcal{L}_K}(r,d)$ is rationally connected. Suppose that K is the fraction field of a Henselian discrete valuation ring with algebraically closed residue field. Then, $M^s_{X_K,\mathcal{L}_K}(r,d)$ has a K-rational point. This is shown using descent theory of stable sheaves on smooth, projective curves ([10, Theorem 1.2]). It is natural to ask how the descent theory of simple sheaves over C_1 fields can be used to prove the existence of rational points, in the higher dimension case. We cite some recent results in this direction.

Definition 4.5. Recall, the *index* of X, denoted $\operatorname{ind}(X)$, is the gcd of the set of degrees of zero dimensional cycles on X.

Remark 4.6. Clearly, the notion of index generalizes the idea of having a rational point. Applying Theorem 2.9 to the case of invertible sheaves on varieties, one can obtain a sufficient criterion for any projective variety defined over a C_1 field to have index 1.

Definition 4.7. Denote by G the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. An invertible sheaf $\mathcal{L}_{\overline{K}}$ on $X_{\overline{K}} := X \times_K \mathrm{Spec}(\overline{K})$ is called G-*invariant* if for any $\sigma \in G$ and the corresponding morphism $\sigma : X_{\overline{K}} \to X_{\overline{K}}$, we have $\sigma^*\mathcal{L}_{\overline{K}} \cong \mathcal{L}_{\overline{K}}$. Denote by $\Lambda \subset \mathrm{Pic}(X_{\overline{K}})$ the subgroup of $\mathrm{Pic}(X_{\overline{K}})$ consisting of all G-invariant invertible sheaves on $X_{\overline{K}}$. Denote by $e := \gcd\{\chi(\mathcal{L}_{\overline{K}})|\mathcal{L}_{\overline{K}} \in \Lambda\}$. We call e the *linear index of* X, denoted $\mathrm{lin}-\mathrm{ind}(X)$.

Theorem 4.8. Suppose that $H^1(\mathcal{O}_X) = 0$, $\mathrm{Pic}(X_{\overline{K}})$ is of rank r, generated by $\mathcal{L}_1, ..., \mathcal{L}_{r-1}$ and $\mathcal{L}_r := H_{\overline{K}} = H \otimes_K \overline{K}$ satisfying the following conditions:

1. the ideal $(\deg(\mathcal{L}_1), \deg(\mathcal{L}_2), ..., \deg(\mathcal{L}_r))$ in $\mathbb{Z}$ generated by $\deg(\mathcal{L}_i)$ for $i = 1, ..., r$ coincides with the ideal (1),

2. for any $r \times r$-matrix $A = (a_{i,j})$ with integral entries $a_{i,j}$, $a_{r,k} = 0$ for all $k < r$, $a_{r,r} = 1$, $A \neq \mathrm{Id}$ and $A^t = \mathrm{Id}$ for some $t > 0$, we have $\sum_j a_{ij} \deg(\mathcal{L}_j) \neq \deg(\mathcal{L}_i)$ for some $i > 0$.

Then, each $\mathcal{L}_i$ is G-invariant,

$$\mathrm{lin}-\mathrm{ind}(X) = \gcd\{\chi(\mathcal{L}_i(n))|i = 1, ..., r \text{ and } n \in \mathbb{Z}\} = 1$$

and

$$\mathrm{ind}(X) = 1 \text{ if } \mathrm{char}(k) = 0$$

and

$$\text{prime-to-p part of } \mathrm{ind}(X) \text{ equals 1 if } \mathrm{char}(k) = p > 0.$$

By prime-to-p part of N we mean the largest divisor of N which is prime to p.

Proof. See [3] for the proof. □

As a consequence of Theorem 4.8, we obtain numerous examples of smooth, projective varieties on C_1-fields with index 1.

Example 4.9. Let X be a smooth, projective variety with $\deg(H_{\overline{K}}) > 2$, $H^1(\mathcal{O}_X) = 0$, $\mathrm{Pic}(X_{\overline{K}})$ is of rank 2 and there exists an invertible sheaf $\mathcal{L}_0$ of degree coprime to $\deg(H_{\overline{K}})$ (for example, a smooth surface X in $\mathbb{P}^3_K$ of degree at least 3 with $\mathrm{rk}(\mathrm{Pic}(X_{\overline{K}})) = 2$ and $X_{\overline{K}}$ contains a curve of degree coprime to $\deg(X)$). Theorem 4.8 implies that every invertible sheaf on $X_{\overline{K}}$ is G-invariant and $\mathrm{ind}(X) = \mathrm{lin}-\mathrm{ind}(X) = 1$.

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