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Enumeration of hypersurfaces with prescribed non-isolated singular subschemes

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Abstract

Let $\mathbb{W}$ be an irreducible subvariety of a Hilbert scheme $\mathrm{Hilb}_{P_W(t)}(\mathbb{P}^n)$. We show under mild hypothesis that there are polynomial formulas for the degrees of the loci of hypersurfaces in $\mathbb{P}^n$ with singular subschemes containing some member of the family $\mathbb{W}$. The formulas are made explicit in a number of cases.

1 Introduction

The enumeration of singular hypersurfaces has a rich history. We refer the reader to Kleiman [20], [21] for a guide to the classical sources. Recent work has centered on generalizations of Göttsche's pioneering article [12]. Polynomial formulas have been shown to exist for the counting of any type of specified *isolated* singularities for hypersurfaces in higher dimensions, cf. Rennemo [27]; alas, his method is nonconstructive and doesn't lead to formulas. A different approach for the *existence* of universal polynomials

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enumerating singular subvarieties is offered by Tzeng [31], via cobordism theory of bundles and divisors. A few special cases of explicit polynomial formulas, still for isolated singularities in higher dimensions, can be found in [32].

The purpose of this work is to investigate the loci of hypersurfaces with possibly *nonisolated* singularities. More precisely, given a closed, irreducible subvariety of a Hilbert scheme, $\mathbb{W} \subset \mathrm{Hilb}_{P_W(t)}(\mathbb{P}^n)$, we define a *generalized discriminant subvariety* $\Sigma(\mathbb{W}, d) \subset \mathbb{P}^{N_d} = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d)))$, the points of which correspond to the hypersurfaces of degree d in $\mathbb{P}^n$ singular along some (variable) member $W \in \mathbb{W}$. Assuming that a general member $W \in \mathbb{W}$ is smooth and of pure dimension $\leq n-2$, we show that the degree of $\Sigma(\mathbb{W}, d)$ is expressed by a polynomial $p^{\mathbb{W}}(d)$ for all $d \gg 0$. Our argument uses Grothendieck-Riemann-Roch. The degree of $p^{\mathbb{W}}(d)$ is shown to be bounded *a priori* by $n \dim \mathbb{W}$. The polynomial is made explicit for a few families $\mathbb{W}$, distinguished by an adequate description in the literature. Notably, we study the cases

$$\begin{array}{lcl} \bullet\, \mathbb{W}_{(k,n)} & := & \{\, \mathbb{P}^k - \text{ linear in } \mathbb{P}^n \,\}, \\ \bullet\, \mathbb{W}_m & := & \{ \text{ plane curves of degree } m \text{ in } \mathbb{P}^3\}, \\ \bullet\, \mathbb{W}_{twc} & := & \{ \text{ twisted cubic curves in } \mathbb{P}^3 \}, \\ \bullet\, \mathbb{W}_{rc} & := & \{ \text{ ruled cubic surfaces in } \mathbb{P}^4 \}, \\ \bullet\, \mathbb{W}_{seg} & := & \{ \text{ Segre cubic 3-folds in } \mathbb{P}^5 \} \text{ and} \\ \bullet\, \mathbb{W}_{eqc} & := & \{ \text{ elliptic quartic curves in } \mathbb{P}^3 \}. \end{array} \tag{1}$$

In all examples we actually find that the degree of our polynomial is

$$\deg p^{\mathbb{W}}(d) = (k+1) \dim \mathbb{W}, \text{ where } k = \dim W (\leq n-2),\ W \in \mathbb{W}. \tag{2}$$

We conjecture this is always the case. Our main result is the following

Theorem 1. *Notation and hypotheses as above, set* $\mathcal{F}_d := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. *There exists a desingularization* $\widetilde{\mathbb{W}} \to \mathbb{W}$ *such that*

(**i**) *for* $d \gg 0$ *there exists a vector subbunble* $\mathcal{E}_d \subset \widetilde{\mathbb{W}} \times \mathcal{F}_d$ *whose fiber over a general* $W \in \widetilde{\mathbb{W}}$ *is the subspace* $H^0(\mathbb{P}^n, (\mathcal{I}_W)^2(d)) \subset \mathcal{F}_d$ *formed by homogeneous polynomials of degree* d *with gradient null along* W*;*
(**ii**) *the map* $\mathbb{P}(\mathcal{E}_d) \longrightarrow \mathbb{P}^{N_d} = \mathbb{P}(\mathcal{F}_d)$ *induced by projection is generically injective and its image,* $\Sigma(\mathbb{W}, d) \subset \mathbb{P}(\mathcal{F}_d)$*, has degree*

$$\deg \Sigma(\mathbb{W}, d) = \int Segre(w, \mathcal{E}_d) \cap [\widetilde{\mathbb{W}}],$$

where $w = \dim \widetilde{\mathbb{W}} = \dim \mathbb{W}$*.*
(**iii**) $\deg(\Sigma(\mathbb{W}, d))$ *is a polynomial in* d *of degree* $\leq nw$ *for all* $d \gg 0$*.*

Let us summarize the contents. §2 contains the proof of the theorem. The first step is to associate to a family $\mathbb{W}$ as above a family $\mathbb{W}'$ of thickenings, cf. Def. 4. The general member $W' \in \mathbb{W}'$ has ideal $\mathcal{I}_{W'} = (\mathcal{I}_W)^2$ for $W \in \mathbb{W}$ a smooth member. A hypersurface of degree $d >> 0$ is singular along W if and only if its equation F lies in $H^0(\mathbb{P}^n, \mathcal{I}_{W'}(d)) \subset \mathcal{F}_d$ (Lemma 5). The set of pairs (W', F) such that $F \supset W'$ is a vector subbundle $\mathcal{E}_d$ of $\mathbb{W}' \times \mathcal{F}_d$. Our generalized discriminant $\Sigma(\mathbb{W}, d) \subset \mathbb{P}^{N_d}$ is the image of the projectivization $\mathbb{P}(\mathcal{E}_d)$, cf. (10). Standard techniques of intersection theory enable us to express $\deg \Sigma(\mathbb{W}, d)$ as a top Chern class of the quotient bundle $\mathcal{D}_d := \mathcal{F}_d / \mathcal{E}_d$, cf. (12),(13). The latter bundle is a direct image, (7). Now GRR applies (14) to ensure that the desired top Chern class is a polynomial in d of degree $\leq n \dim \mathbb{W}$.

Polynomial formulas for the families envisaged in (1) are derived via Bott's localization at fixed points (15), as we learn from Ellingsrud and Strømme [9] and Meurer [24]. The fixed points of $\mathbb{W}_{twc}$ are available in op.cit. Additional work is required since our parameter space $\mathbb{W}'_{twc}$ is in fact a blowup of $\mathbb{W}_{twc}$, cf. Prop. 14, Remark 15. Ditto for the families $\mathbb{W}_{rc}$ (§3.3.2), $\mathbb{W}_{seg}$ (§3.3.3) and $\mathbb{W}_{eqc}$ (§3.4). We work over $\mathbb{C}$.

2 There are a vector bundle and a polynomial formula

Let $\mathbb{W}$ be a closed, irreducible subvariety of a Hilbert scheme $\mathrm{Hilb}_{P_{\mathbb{W}}(t)}(\mathbb{P}^n)$. We assume the general member $W \in \mathbb{W}$ is smooth and of pure dimension $\leq n-2$. Let $W' \subset \mathbb{P}^n$ be the subscheme with ideal sheaf $\mathcal{I}_{W'} = (\mathcal{I}_W)^2$.

Lemma 2. *Notation as just above, set $\mathcal{N} = \mathcal{I}_W/(\mathcal{I}_W)^2$. We have the formula for the Hilbert polynomials*

$$P_{W'}(d) := \chi\left(\mathcal{O}_{W'}(d)\right) = \chi\left(\mathcal{O}_W(d)\right) + \chi\left(\mathcal{N}(d)\right). \tag{3}$$

Proof. The assertion follows from the exact sequence

$$0 \to \mathcal{N} \longrightarrow \mathcal{O}_{\mathbb{P}^n}/(\mathcal{I}_W)^2 = \mathcal{O}_{W'} \longrightarrow \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_W = \mathcal{O}_W \to 0.$$

□

Lemma 3. *Notation as just above, we have a generically injective rational map*

$$\begin{array}{ccc} \mathbb{W} & \dashrightarrow & \mathrm{Hilb}_{P_{W'}(t)}(\mathbb{P}^n) \\ W & \mapsto & W',\ \mathcal{I}_{W'} = (\mathcal{I}_W)^2, \end{array} \tag{4}$$

which is a morphism on the open subset of $\mathbb{W}$ consisting of smooth members.

Proof. By Hirzebruch-Riemann-Roch [11, Cor.15.2.1,p. 288] and Conservation of Number [11, 10.21,p. 180], the r.h.s. in (3) is independent of the particular (smooth) W. By the universal property of Hilb [15], (see [8] for a wonderful introduction or [29] for the state of the art) the map exists over the open subset where flatness is ensured. Finally, if $Z, W \in \mathbb{W}$ are smooth members such that $(\mathcal{I}_Z)^2 = (\mathcal{I}_W)^2$ it follows that $\mathcal{I}_Z = \mathcal{I}_W$. □

Definition 4. *Denote by $\mathbb{W}'$ the closure of the image of the map* (4)*.*

Note the occurrence of a new Hilbert polynomial, $P_{\mathbb{W}'}(t)$. For instance, if we take $\mathbb{W}$ as the family of lines in $\mathbb{P}^3$, we have $P_{\mathbb{W}}(t) = t+1$ whereas presently $P_{\mathbb{W}'}(t) = 3t+1$. The latter is the Hilbert polynomial of the subscheme defined by the ideal $\langle x_0^2, x_0x_1, x_1^2\rangle = \langle x_0, x_1\rangle^2$. Our starting point is the elementary fact that a surface in $\mathbb{P}^3$ is singular along the line $\langle x_0, x_1\rangle$ if and only if its defining homogeneous polynomial lies in $\langle x_0, x_1\rangle^2$. Quite generally, to ask a hypersurface F of degree d to be singular along a general member $W \in \mathbb{W}$ is equivalent to requiring F to be an element of $H^0((\mathcal{I}_W)^2(d))$. This approach was probably inaugurated by Harris and Pandharipande [17] and followed by Göttsche and Rennemo for isolated singularities.

We write $\mathrm{Sing}(F)$ for the singular locus of F. The next lemma is a main step towards the proof of Theorem.1**(ii)**.

Lemma 5. *Suppose $\mathcal{J}_d := (\mathcal{I}_W)^2(d)$ globally generated. Let F be a general element of $H^0(\mathcal{J}_d)$. Then Sing$(F) = W$ set-theoretically.*

Proof. The hypothesis that $\mathcal{J}_d$ be globally generated implies by Bertini (cf. [18, 10.9.2]) that $\mathrm{Sing}(F) \subseteq W$. The inclusion $W \subseteq \mathrm{Sing}(F)$ is evident: if F lies in $H^0(\mathcal{J}_d)$ then its gradient is zero all along W. □

Next we borrow from [1] the technical construction of the correspondence

$$\widetilde{\Sigma}(\mathbb{W}', d) := \{(Z, F) \in \mathbb{W}' \times \mathbb{P}^{N_d} |\, Z \subset F\}. \tag{5}$$

Lemma 6. *Notation as in* Definition 4*, consider the projection maps*

$$\begin{array}{ccc} & \mathbb{W}' \times \mathbb{P}^{N_d} & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathbb{W}' & & \mathbb{P}^{N_d}. \end{array} \tag{6}$$

Then for all $d \gg 0$, the correspondence $\widetilde{\Sigma}(\mathbb{W}', d)$ (see (5)) *is a projective bundle over $\mathbb{W}'$ via the first projection p_1.*

Proof. Let $\widetilde{Z} \subset \mathbb{W}' \times \mathbb{P}^n$ be the universal subscheme and similarly $\widetilde{F} \subset \mathbb{P}^{N_d} \times \mathbb{P}^n$ the universal hypersurface of degree d. Let us denote $\widehat{Z}, \widehat{F}$ their pullbacks to $\mathbb{W}' \times \mathbb{P}^{N_d} \times \mathbb{P}^n$. We have the following diagram of sheaves

$$\begin{array}{ccccc} \mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y(\widehat{F}) & \longrightarrow & \mathcal{O}_{\widehat{F}}(\widehat{F}) \\ & \overset{\rho}{\searrow} & \downarrow & & \\ & & \mathcal{O}_{\widehat{Z}}(\widehat{F}) & & \end{array}$$

over $Y := \mathbb{W}' \times \mathbb{P}^{N_d} \times \mathbb{P}^n$. By construction, the oblique arrow ρ vanishes at a point $(Z, F, x) \in Y$ if and only if $x \in F \cap Z$. So the inclusion $Z \subset F$ holds when the previous condition occurs for all $x \in Z$. Thus, $\widetilde{\Sigma}(\mathbb{W}', d)$ is equal to the scheme of zeros of ρ along the fibers of the projection $p_{12} : \widehat{Z} \to \mathbb{W}' \times \mathbb{P}^{N_d}$. Recalling Altman & Kleiman [1, (2.1) p. 14], this is equal to the scheme of zeros of the adjoint section of the direct image vector bundle $p_{12*}(\mathcal{O}_{\widehat{Z}}(\widehat{F}))$. Look at the projection maps

$$\begin{array}{ccc} \mathbb{W}' \times \mathbb{P}^n & \xrightarrow{q_2} & \mathbb{P}^n \\ {\scriptstyle q_1}\downarrow & & \\ \mathbb{W}'. & & \end{array}$$

Since $\mathcal{O}(\widetilde{F}) = \mathcal{O}_{\mathbb{P}^{N_d}}(1) \otimes \mathcal{O}_{\mathbb{P}^n}(d)$, by the projection formula we have produced a section of $\mathcal{O}_{\mathbb{P}^{N_d}}(1) \otimes \mathcal{D}_d$, where $\mathcal{D}_d = q_{1*}(\mathcal{O}_{\widetilde{Z}}(d))$. By Castelnuovo-Mumford and base change theory, there is an integer d_0 (= regularity) such that $\mathcal{D}_d$ is a vector bundle of rank $P_{\mathbb{W}'}(d)$ for all $d \geq d_0$, where $P_{\mathbb{W}'}(t)$ denotes the Hilbert polynomial of the members of $\mathbb{W}'$. In fact, $\mathcal{D}_d$ fits into the exact sequence of vector bundles over $\mathbb{W}'$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & q_{1*}(\mathcal{I}_{\widetilde{Z}} \otimes \mathcal{O}_{\mathbb{P}^n}(d)) & \longrightarrow & q_{1*}(q_2{}^*\mathcal{O}_{\mathbb{P}^n}(d)) & \longrightarrow & q_{1*}(\mathcal{O}_{\widetilde{Z}}(d)) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathcal{E}_d & \rightarrowtail & \mathcal{F}_d & \twoheadrightarrow & \mathcal{D}_d\,. & & \end{array} \tag{7}$$

Taking the projectivization and pulling back to $\mathbb{W}' \times \mathbb{P}^{N_d}$, we get (omitting pullbacks):

$$
\begin{array}{ccccc}
 & & \mathcal{O}_{\mathbb{P}^{N_d}}(-1) & & \\
 & & \big\downarrow & \overset{\overline{\rho}}{\searrow} & \\
\mathcal{E}_d & \rightarrowtail & \mathcal{F}_d & \twoheadrightarrow & \mathcal{D}_d\,.
\end{array}
\tag{8}
$$

By construction, $\overline{\rho}$ vanishes precisely over $\widetilde{\Sigma}(\mathbb{W}', d)$. And this tells us that

$$\widetilde{\Sigma}(\mathbb{W}', d) = \mathbb{P}(\mathcal{E}_d). \tag{9}$$

□

Lemma 7. *Notation as in* (7),(9) *we have that* $\widetilde{\Sigma}(\mathbb{W}', d)$ *represents the top Chern class of* $\mathcal{O}_{\mathbb{P}^{N_d}}(1) \otimes \mathcal{D}_d$.

Proof. As $\mathrm{codim}_{\mathbb{W}' \times \mathbb{P}^{N_d}} \widetilde{\Sigma}(\mathbb{W}', d) = P_{\mathbb{W}'}(d)$, which coincides with the rank of $\mathcal{D}_d$ due to (7), the assertion follows from Fulton [11, 3.2.16, p. 61]. □

Definition 8. *We call the* $\mathbb{W}$*-discriminant, denoted by* $\Sigma(\mathbb{W}, d)$*, the subvariety of* $\mathbb{P}^{N_d}$ *corresponding to the hypersurfaces which contain some member of* $\mathbb{W}'$.

For a general point $W' \in \mathbb{W}'$ and a hypersurface F, asking that $F \supset W'$ as schemes is equivalent to requiring that the hypersurface be singular along the reduced scheme $W = W'_{red} \in \mathbb{W}$. With the notation of (6), we have

$$\Sigma(\mathbb{W}, d) = p_2(\widetilde{\Sigma}(\mathbb{W}', d)). \tag{10}$$

The usual discriminant hypersurface corresponds to the choice $\mathbb{W} = \mathbb{P}^n$.

Lemma 9. *Notation as above, the map*

$$
\begin{array}{rcl}
\mathbb{W}' \times \mathbb{P}^{N_d} \supset \widetilde{\Sigma}(\mathbb{W}', d) & \xrightarrow{p_2} & \Sigma(\mathbb{W}, d) \subset \mathbb{P}^{N_d} \\
(Z, F) & \mapsto & F
\end{array}
\tag{11}
$$

is generically injective for all $d >> 0$.

Proof. We must show that for a general $F \in \Sigma(\mathbb{W}, d)$, the fiber $p_2^{-1}(F) \subset \widetilde{\Sigma}(\mathbb{W}', d)$ consists of a single element. In view of Lemma 3, there is an open subset $\mathbb{W}'_0 \subset \mathbb{W}'$ formed by subschemes W' with ideal of the form $\mathcal{I}_{W'} = (\mathcal{I}_W)^2$ with $W \in \mathbb{W}$ smooth. Now it suffices to show that the restriction of p_2 over $\mathbb{W}'_0$ is injective. Let F be a general hypersurface of degree d containing $W' \in \mathbb{W}'_0$. This means that F is general in $H^0(\mathcal{I}_{W'}(d)) = H^0((\mathcal{I}_W)^2(d))$. By Lemma 5 we have that $\mathrm{Sing}(F) = W$ (as sets). Let $Z' \in \mathbb{W}'_0$ be such that $Z' \subset F$. So we have (W', F) and $(Z', F) \in p_2^{-1}(F)$. By construction of $\mathbb{W}'_0$, we have $\mathcal{I}_{Z'} = (\mathcal{I}_Z)^2$ for some smooth $Z = Z'_{red} \in \mathbb{W}$. So $Z \subseteq \mathrm{Sing}(F) = W$. Since the Hilbert polynomials of Z, W are one and the same, therefore $Z = W$ and so $Z' = W'$. This shows that the map in (11) is generically injective as asserted. □

Lemma 10. *Notation as above, we have*

$$\deg \Sigma(\mathbb{W}, d) = \int \mathit{Segre}(w, \mathcal{E}_d) \cap [\mathbb{W}'], \tag{12}$$

where $w := \dim \mathbb{W}' = \dim \mathbb{W}$.

Proof. We have the equality of cycle classes

$$(p_2)_\star[\widetilde{\Sigma}(\mathbb{W}', d)] = [\Sigma(\mathbb{W}, d)].$$

This follows from [11, §1.4, p. 11] since $\widetilde{\Sigma}(\mathbb{W}, d) \xrightarrow{p_2} \Sigma(\mathbb{W}, d)$ is birational, as shown in Lemma 9. Set $\delta := \dim \Sigma(\mathbb{W}, d)$. We have $\delta = w + \epsilon$, with $\epsilon := \mathrm{rk}\mathcal{E}_d - 1$. Set $H = c_1\mathcal{O}_{\mathbb{P}^{N_d}}(1)$, the hyperplane class. By projection formula we may write

$$\begin{aligned} \deg \Sigma(\mathbb{W}, d) &= \int H^\delta \cap [\Sigma(\mathbb{W}, d)] = \int p_2^\star H^\delta \cap [\widetilde{\Sigma}(\mathbb{W}', d)] \\ &= \int (p_1)_\star \left(p_2^\star H^{w+\epsilon} \cap [\widetilde{\Sigma}(\mathbb{W}', d)] \right) = \int \mathrm{Segre}(w, \mathcal{E}_d) \cap [\mathbb{W}'], \end{aligned}$$

using Fulton [11, §3.1, p. 47, Prop.4.4,p. 83 and Ex. 8.3.14, p. 143]. □

Proposition 11. *The degree of the* $\mathbb{W}$*-discriminant,* $\Sigma(\mathbb{W}, d)$*, is a polynomial in* d *of degree* $\leq n \dim(\mathbb{W})$ *for all* $d \gg 0$.

Proof. Let $\widetilde{\mathbb{W}} \to \mathbb{W}'$ be a desingularization (cf. [19]). Pulling back $\mathcal{E}_d, \mathcal{D}_d$ in (7) to $\widetilde{\mathbb{W}}$, we may as well simplify notation and assume $\widetilde{\mathbb{W}} = \mathbb{W}'$ smooth. We now argue as in [6] and [33]. Recall $\mathcal{D}_d$ is a direct image of a sheaf over $\mathbb{W}' \times \mathbb{P}^n$ (cf. 7). The same diagram of sheaves tells us

$$\text{Segre}(w, \mathcal{E}_d) = c_w(\mathcal{D}_d). \tag{13}$$

Now we can apply Grothendieck-Riemann-Roch (cf. [11, Thm.15.2, p. 286]) to express the Chern character of $\mathcal{D}_d$ as

$$ch(\mathcal{D}_d) = ch((q_1)_!(\mathcal{O}_{\widetilde{Z}}(d))) = (q_1)_* \left(ch(\mathcal{O}_{\widetilde{Z}}) \cdot ch(\mathcal{O}_{\mathbb{P}^n}(d)) \cdot \text{todd}(\mathbb{P}^n)\right). \tag{14}$$

Note that the right hand side is a polynomial in d of degree $\leq n$. On the other hand, the Chern class c_w is a weighted polynomial of degree w on the coefficients of the Chern character ([11, 3.2.3,p. 56]). This implies that $c_w(\mathcal{D}_d)$ is a polynomial in d of degree $\leq nw$.

□

Remark 12. In order to get a polynomial formula, it suffices to calculate the degree of $\Sigma(\mathbb{W}, d)$ for $n \dim \mathbb{W} + 1$ values of d. In all cases treated in this work, we find that the degree of the polynomial $p^{\mathbb{W}}(d)$ actually is $(k+1) \times \dim(\mathbb{W})$, where k denotes the dimension of a member of $\mathbb{W}$. The validity for $\mathbb{W}$ arbitrary remains conjectural. [31] and [27] handle the case $k = 0$.

To compute explicitly the integral in (12), we will apply Bott's residues formula in the equivariant flavor of [9] (see also [23], [24]),

$$\int \text{Segre}(w, \mathcal{E}_d) \cap [\mathbb{W}'] = \sum_F \frac{c_w^{\mathbb{T}}(-\mathcal{E}_d) \cap [F]_{\mathbb{T}}}{c_{top}^{\mathbb{T}}(\mathcal{N}_{F|\mathbb{W}'})}, \tag{15}$$

where the sum runs through all fixed components F of a convenient action of the torus $\mathbb{T} := \mathbb{C}^*$ on $\mathbb{W}'$. The $\mathcal{N}_{F|\mathbb{W}'}$ appearing in the denominator denotes the normal bundle of a fixed component F in $\mathbb{W}'$. In all cases treated in this work the set of fixed points is finite. Thus the denominator in (15) is the $\mathbb{T}$-equivariant top Chern class, $c_{top}^{\mathbb{T}}(\mathcal{T}_F\mathbb{W}')$, where $\mathcal{T}_F\mathbb{W}'$ denotes the tangent space at a fixed point F in $\mathbb{W}'$.

Remark 13. Notation as in (1), for $\mathbb{W} = \mathbb{W}_{(k,n)}$ (as well as $\mathbb{W} = \mathbb{W}_m$), the family $\mathbb{W}'$ which parameterizes subschemes of $\mathbb{P}^n$ defined by $(I_W)^2$ with $W \in \mathbb{W}$ is flat. In fact, we have $\mathbb{W} = \mathbb{W}'$: the map (4) is an isomorphism. However, in the other cases dealt with in this work we have only the generic flatness guaranteed over the locus of smooth $W \in \mathbb{W}$. In fact, the Hilbert polynomial for $(I_W)^2$ may jump at special points. A blowup will be required in order to achieve flatness, following Raynaud [26].

3 Enumerative results

"For many problems it would be miraculous and totally unexpected if somebody were to find a precise formula for the solution; most of the time one must settle for a rough estimate instead." [1]

A detailed exposition of the fixed points and the computations of their contributions on Bott's formula (15), including scripts for Macaulay2 [14], Maple [22] and Singular [7] and for the resolution of indeterminacies in the cases of sections 3.3 and 3.4 can be found in Sellin [28].

3.1 Hypersurfaces singular along a linear $\mathbb{P}^k \subset \mathbb{P}^n$

Here, the parameter space $\mathbb{W}_{(k,n)} := \mathbb{G}(k+1, n+1)$, the grassmannian of $k+1$ dimensional vector subspaces of $\mathbb{C}^{n+1}$. Our goal is to determine the degree of the family of hypersurfaces of degree d singular along some $\mathbb{P}^k \subset \mathbb{P}^n$.

For the reader's benefit we will show the calculations for $\deg \Sigma(\mathbb{W}_{(1,3)}, d)$.

Consider the torus $\mathbb{T} = \mathbb{C}^*$ acting diagonally on $\mathcal{F}_1 = (\mathbb{C}^4)^\vee$ via

$$t \circ x_i := t^{w_i} x_i,$$

with appropriate weights, say:

$$w_0 = 4, w_1 = 11, w_2 = 17, w_3 = 32; \tag{16}$$

The requirement is that denominators appearing in (15), which turn out to be polynomials in the weights $w_0, ..., w_3$, do not vanish.

[1] Tim Gowers, *Mathematics: A Very Short Introduction*

We get a natural induced action on $\mathbb{W}_{(1,3)} = \mathbb{G}(2,4)$. The tautological vector bundles

$$\mathcal{S} \rightarrowtail \mathcal{F}_1 \twoheadrightarrow \mathcal{Q}$$

on $\mathbb{W}_{(1,3)}$ are $\mathbb{T}$-equivariant. The fiber of $\mathcal{S}$ over a line $l \in \mathbb{G}(2,4)$ is the two dimensional subspace of $\mathcal{F}_1$ of linear forms vanishing on l. Presently we have six fixed points corresponding to the coordinate axes

$$\langle x_0, x_1\rangle, \langle x_0, x_2\rangle, \ldots, \langle x_2, x_3\rangle.$$

Referring to (15), we have

$$\deg \Sigma(\mathbb{W}_{(1,3)}, d) = \sum_F \frac{c_4^{\mathbb{T}}(-\mathcal{E}_d) \cap [F]_{\mathbb{T}}}{c_4^{\mathbb{T}}(\mathcal{T}\mathbb{W}_{(1,3)})} \tag{17}$$

summing over the six fixed points. The denominator in (17), let's say for $F = \langle x_0, x_1\rangle$, is obtained as follows. First we find the fiber of the tangent

$$\mathcal{T}_F\mathbb{W}_{(1,3)} = \mathrm{Hom}(\mathcal{S}_F, \mathcal{Q}_F) = \langle x_0, x_1\rangle^\vee \otimes \langle x_2, x_3\rangle = \frac{x_2}{x_0} + \frac{x_3}{x_0} + \frac{x_2}{x_1} + \frac{x_3}{x_1},$$

where $\frac{x_i}{x_j}$ denotes the $\mathbb{T}$-space with weight $w_i - w_j$. In this way, we obtain $c_4^{\mathbb{T}}(\mathcal{T}\mathbb{W}_{(1,3)}) \cap [F]_{\mathbb{T}} = (w_2 - w_0)(w_3 - w_0)(w_2 - w_1)(w_3 - w_1)$. With the choice of weights in (16), this gives us the value 45864. Similarly, the numerator requires the weight decomposition of the fiber $(\mathcal{E}_d)_F$. To fix the ideas, take $d = 3$. Now that fiber consists of the cubic forms $f \in H^0(\mathcal{O}_{\mathbb{P}^3}(3))$ with gradient null along the line F. The weight decomposition is given by

$$(\mathcal{E}_3)_F = x_0^3 + x_0^2x_1 + x_0^2x_2 + x_0^2x_3 + x_0x_1^2 + x_0x_1x_2 + x_0x_1x_3 + x_1^3 + x_1^2x_2 + x_1^2x_3.$$

Since we actually need the Segre class, $\mathrm{Chern}(-\mathcal{E}_d) = \mathrm{Chern}(\mathcal{D}_d)$ cf. (7), we find the complementary decomposition

$$(\mathcal{D}_d)_F = x_0x_2^2 + x_1x_2^2 + x_2^3 + x_0x_2x_3 + x_1x_2x_3 + x_2^2x_3 + x_0x_3^2 + x_1x_3^2 + x_2x_3^2 + x_3^3.$$

Here $x_i^\alpha x_j^\beta x_k^\gamma$ denotes the $\mathbb{T}$-space with weight $\alpha w_i + \beta w_j + \gamma w_k$. The corresponding numerical contribution is 3217978137. The fixed point

$F = \langle x_0, x_1\rangle$ contributes the fraction 3217978137/45864. The total contribution of the six fixed points is

$$\frac{3217978137}{45864} - \frac{2152229961}{17640} + \frac{774359841}{28665} + \frac{1227942219}{28665} - \frac{392711889}{17640} + \frac{218302833}{45864} = 504.$$

This is the degree of the subvariety of $|\mathcal{O}^3_{\mathbb{P}}(3)| = \mathbb{P}^{19}$ consisting of the Whitney umbrellas: surfaces of degree 3 in $\mathbb{P}^3$ which are singular along some line (cf. [5]).

Recalling Remark 12, we need the degrees of $\Sigma(\mathbb{W}_{(1,3)}, d)$ for $3 \times 4 + 1$ values of d. Interpolating, we get

$$\deg \Sigma(\mathbb{W}_{(1,3)}, d) = \frac{1}{32}\binom{d}{2}(27d^6-117d^5+269d^4-375d^3+312d^2-132d+48). \quad (18)$$

We list below the results for $(k, n) \in \{(2, 4), (2, 5), (3, 5)\}$:

$$\begin{aligned}\deg \Sigma(\mathbb{W}_{(2,4)}, d) = \frac{1}{2^7 \cdot 3^3}\tbinom{d+2}{4}(9d^{14} - 18d^{13} - 63d^{12}\\ +396d^{11} - 405d^{10} - 1530d^9 + 5328d^8 - 4176d^7 - 9414d^6\\ +27208d^5 - 24347d^4 - 4696d^3 + 36572d^2 - 32544d + 14400).\end{aligned} \quad (19)$$

$$\begin{aligned}\deg \Sigma(\mathbb{W}_{(2,5)}, d) = \frac{1}{(2)^3 3^{11} 5^3}\tbinom{d+2}{4}(12800d^{23} - 25600d^{22}\\ -224000d^{21} + 966400d^{20} + 520800d^{19}\\ -10632000d^{18} + 18128000d^{17} + 35186000d^{16} - 170677265d^{15}\\ +145358830d^{14} + 449576760d^{13} - 1292773830d^{12} + 778144037d^{11}\\ +2164141556d^{10} - 5208921230d^9 + 3728975455d^8 + 3332483181d^7\\ -10452711042d^6 + 10781927010d^5 - 2523245175d^4 - 7609562253d^3\\ +11511503406d^2 - 8323547040d + 3637418400).\end{aligned} \quad (20)$$

$$\begin{aligned}\deg \Sigma(\mathbb{W}_{(3,5)}, d) = \frac{1}{2^{27} \cdot 3^3 \cdot 5^4}\tbinom{d+2}{4}\left(1125d^{28} + 15750d^{27}\right.\\ +86625d^{26} + 168750d^{25} - 187875d^{24} - 38250d^{23}\\ +8824725d^{22} + 23473350d^{21} - 32467725d^{20} - 128183670d^{19}\end{aligned} \quad (21)$$

$$+426415635d^{18} + 1377078570d^{17} - 2137554049d^{16}$$
$$-7117020302d^{15} + 15925316455d^{14} + 37514746370d^{13}$$
$$-82840806388d^{12} - 125157483544d^{11} + 422227932240d^{10}$$
$$+287672117600d^{9} - 1529648949952d^{8} + 207120164224d^{7}$$
$$+4517312266240d^{6} - 3047085731840d^{5} - 6253154779136d^{4}$$
$$+11893749153792d^{3} + 2911913902080d^{2}$$
$$-8455245004800d + 2378170368000\big) .$$

Recalling $\dim \mathbb{W}_{k,n} = (k+1)(n-k)$, we remark that the degrees of the above polynomials are in agreement with the expectation (2), to wit, $(k+1)\dim\mathbb{W}$.

3.2 Surfaces singular along plane curves

The family of plane curves of degree $m > 1$ in $\mathbb{P}^3$ is parameterized by a $\mathbb{P}^{N_m}$-bundle over $\check{\mathbb{P}}^3$

$$\mathbb{W}_m \longrightarrow \check{\mathbb{P}}^3,$$

where $N_m = \binom{m+2}{2} - 1$. We have calculated $\deg\Sigma(\mathbb{W}_m, d)$ for $m = 2, 3$:

$$\deg\Sigma(\mathbb{W}_2, d) = \frac{1}{2^{13}\cdot 3^2\cdot 5\cdot 7}(d-2)(150903d^{15} - 3809754d^{14}$$
$$+44834472d^{13} - 317080224d^{12} + 1422290970d^{11} - 3579080844d^{10}$$
$$-455933988d^{9} + 47928493544d^{8} - 237841700217d^{7} + 712127741206d^{6} \qquad (22)$$
$$-1498533401372d^{5} + 2287674925704d^{4} - 2504345972608d^{3}$$
$$+1873638158208d^{2} - 859900216320d + 182801203200).$$

$$
\begin{aligned}
\deg \Sigma(\mathbb{W}_3, d) = \frac{1}{2^{20}\cdot 3^4\cdot 5\cdot 7\cdot 11} \big(&13286025d^{24} - 1038081420d^{23} \\
&+39146062158d^{22} - 946074434976d^{21} + 16407919974303d^{20} \\
&-216603408547548d^{19} + 2251372103607528d^{18} \\
&-18776305509313968d^{17} + 126579622223230407d^{16} \\
&-686155959955971780d^{15} + 2911999863446866566d^{14} \\
&-8886007643094113376d^{13} + 12799827743693355329d^{12} \\
&+50456388588134712812d^{11} - 483658040042985949724d^{10} \\
&+2229927488252098274992d^{9} - 7358275057877141245584d^{8} \\
&+18804143410678335462720d^{7} - 38007885859704936084800d^{6} \\
&+60658830486712279959808d^{5} - 75133955486596446561280d^{4} + \\
&69793667761693681135616d^{3} - 45744106516543857328128d^{2} \\
&+18819557445986636267520d - 3636764182567924531200\big)\,.
\end{aligned}
\tag{23}
$$

The reader interested in obtaining $\deg \Sigma(\mathbb{W}_m, d)$ for other m, simply plug in the desired value in the script in [28, Appendix E, p.92]. Notice the degrees of the above polynomials in (22) and (23) are $(k+1)\dim \mathbb{W} = 2\left(2+\binom{d+2}{2}\right)$.

3.3 Hypersurfaces singular along base loci of nets of quadrics of determinantal type

In this section we discuss the case of hypersurfaces in $\mathbb{P}^n$ $(n = 3, 4, 5)$ singular along base loci of nets of quadrics of determinantal type. By this we mean the nets generated by 2×2-minors of a 3×2 matrix of linear forms. Specifically, we consider the families

$$
\begin{cases}
\mathbb{W}_{twc} & = \ \{\text{ twisted cubics in } \mathbb{P}^3\}, \\
\mathbb{W}_{rc} & = \ \{\text{ ruled cubics in } \mathbb{P}^4\} \ \text{ and} \\
\mathbb{W}_{seg} & = \ \{\text{ Segre 3-folds in } \mathbb{P}^5\}.
\end{cases}
$$

3.3.1 Surfaces singular along twisted cubics

A twisted cubic is a rational, smooth curve of degree 3 in $\mathbb{P}^3$. Any such is projectively equivalent to the scheme of zeros of the 2×2 minors of the matrix $\left(\begin{smallmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{smallmatrix}\right)$. Its Hilbert polynomial is $3t+1$. Piene & Schlessinger [25] showed that the component $\mathbb{W}_{twc} \subset \mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$ is a smooth projective variety of dimension 12. Subsequently, Ellingsrud, Piene & Strømme [10] proved that the subvariety of the Grassmannian

$$\mathbb{X} \subset \mathbb{G}(3, \mathcal{F}_2) \tag{24}$$

formed by nets of determinantal type is smooth. Moreover the component $\mathbb{W}_{twc}$ is the blowup of $\mathbb{X}$ along the subvariety $\mathbb{G}_\omega$ of nets projectively equivalent to the net

$$\omega := (x_0^2, x_0x_1, x_0x_2).$$

A typical element on the fiber of the exceptional divisor over ω corresponds to an ideal of the form $\mathcal{I}_{\omega,f} := \langle x_0^2, x_0x_1, x_0x_2, f\rangle$, where $x_0 = f(x_1, x_2, x_3) = 0$ is a plane cubic singular at the point $x_0 = x_1 = x_2 = 0$. The square $(\mathcal{I}_{\omega,f})^2$ of any such ideal has Hilbert polynomial $9t - 7$, same as for the square of the ideal of the standard twisted cubic, $\langle x_1x_3 - x_2^2, x_0x_3 - x_1x_2, x_1^2 - x_0x_2\rangle$.

Unlike the cases $\mathbb{W}_{(k,n)}$ and $\mathbb{W}_m$, the family formed by the subschemes of $\mathbb{P}^3$ defined by $(\mathcal{I}_W)^2, W \in \mathbb{W}_{twc}$ is not flat. In fact, the element

$$\mathbf{o} := \langle x_0, x_1\rangle^2 = \langle x_0^2, x_0x_1, x_1^2\rangle \tag{25}$$

is a member of the good component $\mathbb{W}_{twc}$, but its square has "bad" Hilbert polynomial, namely $P_{\mathbb{W}_{twc}}(t) = 10t - 10$, instead of $9t - 7$.

This is remedied by blowing up $\mathbb{W}_{twc}$ along the orbit $\mathbb{G}_{\mathbf{o}}$. Since $\mathbb{G}_{\mathbf{o}} \cap \mathbb{G}_\omega = \emptyset$, it follows that $\mathbb{G}_{\mathbf{o}}$ lifts isomorphically to an orbit in $\mathbb{W}_{twc}$, still denoted by $\mathbb{G}_{\mathbf{o}}$. Let $\mathbb{W}'_{twc}$ denote the blowup of $\mathbb{W}_{twc}$ along $\mathbb{G}_{\mathbf{o}}$. In fact, $\mathbb{X}$ and $\mathbb{W}_{twc}$ are isomorphic over any neighborhood of $\mathbb{G}_{\mathbf{o}}$ disjoint from $\mathbb{G}_\omega$. The restriction $\mathbb{W}'_{twc|\mathbb{X}\setminus\mathbb{G}_\omega}$ is isomorphic to the restriction $\mathbb{X}'_{|\mathbb{X}\setminus\mathbb{G}_\omega}$ of the blowup $\mathbb{X}'$ of $\mathbb{X}$ along $\mathbb{G}_{\mathbf{o}}$.

Let $\mathcal{C}$ be the tautological subbundle of rank 3 over the grassmannian of nets of quadrics $\mathbb{G}(3,\mathcal{F}_2)$. Write $\mathrm{S}_2(\mathcal{C})$ the symmetric power.

Proposition 14. *Let $\mu : \mathrm{S}_2(\mathcal{C})_{|\mathbb{W}_{twc}} \to \mathcal{F}_4$ be the natural map induced by multiplication. Consider the blowing up diagram of $\mathbb{W}_{twc}$ along $\mathbb{G}_{\mathbf{o}}$*

$$\begin{array}{ccc} \mathbb{W}'_{twc} & \supset & \mathbb{E}' \\ \downarrow & & \downarrow \\ \mathbb{W}_{twc} & \supset & \mathbb{G}_{\mathbf{o}}. \end{array} \tag{26}$$

Then **(i)** *$\mathbb{G}_{\mathbf{o}}$ is the scheme of zeros of $\overset{6}{\wedge}\mu$;*
(ii) *$\mathbb{W}'_{twc}$ embeds in $\mathbb{W}_{twc} \times \mathbb{G}(6,\mathcal{F}_4)$ as the closure of the graph of the rational map $\mathbb{W}_{twc} \dashrightarrow \mathbb{G}(6,\mathcal{F}_4)$ induced by μ.*
(iii) *The fiber of the exceptional divisor $\mathbb{E}'$ over $\mathbf{o}$ is the projectivization of the quotient space of quartic forms,*

$$(\langle x_0, x_1\rangle^3)_4 / (\langle x_0, x_1\rangle^4)_4.$$

Proof. The argument is based on local calculations as shown in [28, Appendix F.1]. We just highlight the main steps. Denote by $\mathfrak{Z}$ the scheme of zeros in question; it is invariant under the natural $\mathbb{PGL}_4$ induced action. Recall $\mathbb{X}$ (24) has precisely two closed orbits, represented by the nets

$$\mathbf{o} = (x_0^2, x_0x_1, x_1^2) \text{ and } \omega = (x_0^2, x_0x_1, x_0x_2).$$

Clearly $\mathbf{o} \in \mathfrak{Z} \not\ni \omega$. Consider the list the 10 quadratic monomials,

$$m_1 := x_0^2, m_2 := x_0x_1, m_3 := x_1^2, m_4 := x_0x_2, \ldots, m_{10} := x_3^2.$$

Use the affine coordinates $a_{ij}, 1 \le i \le 3, 1 \le j \le 7$ for the open subset $\mathbb{G}^0 \subset \mathbb{G}(3,10)$ so that the quadrics

$$\begin{cases} q_1 := x_0^2 + \sum a_{1j} m_{3+j} \\ q_2 := x_0x_1 + \sum a_{2j} m_{3+j} \\ q_3 := x_1^2 + \sum a_{3j} m_{3+j} \end{cases}$$

yield a trivialization for the restriction $\mathcal{C}_{|\mathbb{G}^0}$. Over $\mathbb{G}^0$ the multiplication map $\mathcal{C} \otimes \mathcal{F}_1 \to \mathcal{F}_3$ is of generic rank 12; the rank drops to 10 exactly along $\mathbb{X}^0 := \mathbb{X} \cap \mathbb{G}^0 = \mathbb{W}_{twc} \cap \mathbb{G}^0 =: \mathbb{W}^0_{twc}$; the 2nd equality stems from the fact that $\mathbb{G}^0$ is a neighborhood away from the orbit of ω. This yields explicit equations for $\mathbb{X}^0 \subset \mathbb{G}^0$. These equations allow us to express 9 of the coordinates in terms of the 12 remaining ones; these in turn provide affine coordinates for $\mathbb{X}^0$. Working out a matrix representation for $\mu : \mathcal{S}_2\mathcal{C}_{|\mathbb{X}^0} \to \mathcal{F}_4$ we find that the ideal of $6{\times}6$ minors, which defines $\mathfrak{Z}$, is equal to the ideal of $(\mathbb{G}_{\mathbf{o}})^0 := \mathbb{G}_{\mathbf{o}} \cap \mathbb{X}^0 \subset \mathbb{X}^0$. Since $\mathfrak{Z} \supseteq \mathbb{G}_{\mathbf{o}}$ are closed invariant subschemes which agree in a neighborhood of their unique closed orbit, they must be equal. Blowing it up, we get $\mathbb{X}'$ (resp. $\mathbb{W}'_{twc}$) embedded in $\mathbb{X}{\times}\mathbb{G}(6, \mathcal{F}_4)$ (resp. $\mathbb{W}_{twc}{\times}\mathbb{G}(6, \mathcal{F}_4)$) as the closure of the graph of the rational map induced by μ. Likewise, we find that the fiber of $\mathbb{E}'$ over $\mathbf{o}$ is as stated in **(iii)**. □

Remark 15. The previous result implies that the fixed points in $\mathbb{W}'_{twc}$ are obtained from those well known for $\mathbb{W}_{twc}$ (cf. [9]), except for the six ones belonging to $\mathbb{G}_{\mathbf{o}}$. For each of these, say $\mathbf{o} = \langle x_0, x_1\rangle^2$, we form the ideals $\langle x_0, x_1\rangle^4 + \langle Q\rangle, Q \in \{x_0^3x_2, x_0^3x_3, x_0^2x_1x_2, x_0^2x_1x_3, x_0x_1^2x_2, x_0x_1^2x_3, x_1^3x_2, x_1^3x_3\}$. These eight monomials span the exceptional fiber $(\langle x_0, x_1\rangle^3)_4/(\langle x_0, x_1\rangle^4)_4$.

For details about the explicit contribution of each fixed point the reader is again kindly referred to [28, Appendix F.3]. The polynomial that gives us the degree of $\Sigma(\mathbb{W}_{twc}, d)$ is displayed in (27). Note that its degree is equal to $2 \times \dim(\mathbb{W}_{twc}) = 2 \times 12$ in agreement with (2).

$$
\begin{aligned}
\deg\Sigma(\mathbb{W}_{twc}, d) = {} & \tfrac{1095687}{50462720}d^{24} - \tfrac{19230291}{18022400}d^{23} + \tfrac{24114591}{985600}d^{22} \\
& - \tfrac{3932462817}{11468800}d^{21} + \tfrac{73665592101}{22937600}d^{20} - \tfrac{23321377833}{1146880}d^{19} + \tfrac{4087404048523}{51609600}d^{18} \\
& - \tfrac{205245946577}{2457600}d^{17} - \tfrac{79029321809671}{68812800}d^{16} + \tfrac{2854774357217311}{309657600}d^{15} \\
& - \tfrac{6688891988137}{143360}d^{14} + \tfrac{895445339622112187}{3406233600}d^{13} - \tfrac{4177328126526143027}{2270822400}d^{12} \\
& + \tfrac{1134029525022301939}{94617600}d^{11} - \tfrac{29052565860084958379}{464486400}d^{10} + \tfrac{1100107099486708819}{4300800}d^{9} \\
& - \tfrac{3195009799515883111 9}{38707200}d^{8} + \tfrac{365421773568911927}{172800}d^{7} - \tfrac{8318629615873057099}{1935360}d^{6} \\
& + \tfrac{6153959376914270 21}{89600}d^{5} - \tfrac{33777705898251350 8747}{39916800}d^{4} + \tfrac{516778140945191522 3}{665280}d^{3} \\
& - \tfrac{693707469384158233}{138600}d^{2} + \tfrac{466431399017887}{231}d - 383398629664.
\end{aligned}
\tag{27}
$$

Figure 1 shows with the help of Surfer [30] an example of a quartic surface singular along a twisted cubic.

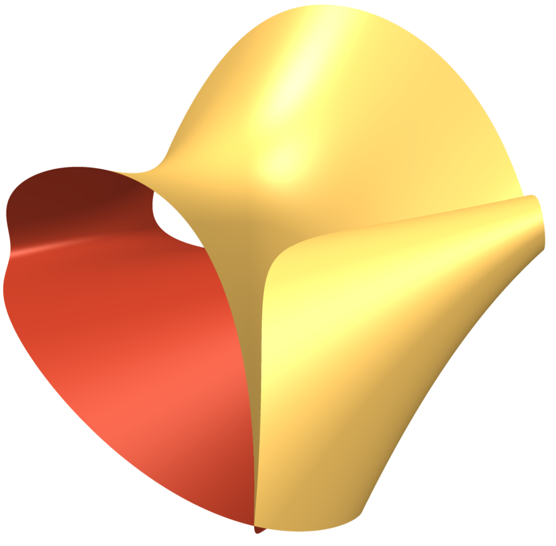


Figure 1: $-8y^4 + 16xy^2z + 8y^3z - 8x^2z^2 - 8xyz^2 - 8y^2z^2 + 2xz^3 + 2yz^3 + 4z^4 - 8xy^2 + 2y^3 + 8x^2z + 10xyz - 2y^2z - 2xz^2 - 8yz^2 - 6x^2 + 2xy + 4y^2 = 0$

3.3.2 Hypersurfaces singular along a ruled cubic surface in $\mathbb{P}^4$

A ruled cubic surface in $\mathbb{P}^4$ is the base locus of a net of quadrics of determinantal type (cf. Beauville [4, Prop. IV.7, p. 44)]; it's projectively equivalent to the subvariety W defined by the ideal $\mathcal{I}_W$ of the 2×2 minors of the matrix $\left(\begin{smallmatrix} x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 \end{smallmatrix}\right)$. It has Hilbert polynomial $P_{rc}(t) := (3/2)t^2 + (5/2)t + 1$. Denote by $\mathbb{W}_{rc}$ the corresponding component in Hilb $\mathbb{P}^4$. The family $\mathbb{W}_{rc}$ has dimension 18. The Hilbert polynomial of the subscheme W'_{rc} defined by $\mathcal{I}_W^2$ is $P_{W'}(t) = (9/2)t^2 - (5/2)t + 2$. The family formed by subschemes of $\mathbb{P}^4$ defined by $\mathcal{I}_W^2$ for some $W \in \mathbb{W}_{rc}$ is not flat. The culprits are again in the orbit of the net $\mathbf{o} = \langle x_0^2, x_0x_1, x_1^2\rangle$, a legitimate member of $\mathbb{W}_{rc}$. Its square has Hilbert polynomial $5t^2 - 5t + 5$ which is different from the expected. Blowing up as before produces a flat family $\mathbb{W}'_{rc}$. Computational details are available in [28, Appendix G.1]. The polynomial that gives the degree of $\Sigma(\mathbb{W}_{rc}, d)$ is described below:

$$\begin{aligned}
\deg \Sigma(\mathbb{W}_{rc}, d) = \frac{1089331}{28207459709485056\,00} d^{54} - \frac{4609327}{138135296519700480} d^{53} \\
+ \frac{17053361977}{12432176686773043200} d^{52} - \frac{44006738257}{1243217668677304320} d^{51} + \frac{43540862009}{68559797904998400} d^{50} \\
- \frac{6776065867607}{822717574859980800} d^{49} + \frac{25203282464989}{329087029943992320} d^{48} - \frac{95461703632727}{205679393714995200} d^{47} \\
+ \frac{3121945759267787}{3290870299439923200} d^{46} + \frac{13975371538743871}{987261089831976960} d^{45} - \frac{176226379304682200 3}{9872610898319769600} d^{44} \\
+ \frac{1571373547792223293}{1645435149719961600} d^{43} - \frac{18657333817850689}{21095322432307200} d^{42} - \frac{21162893089184824063}{822717574859980800} d^{41} \\
+ \frac{88172373953883719 83}{42070785078067200} d^{40} - \frac{7285835577039579827299}{7404458173739827200} d^{39} + \frac{18439965173115436460101}{2278294822689177600} d^{38} \\
- \frac{3062572630275215457014 6789}{251751577907154124800} d^{37} + \frac{28667160534678315148870981 9}{20140126232572329984 0} d^{36} \\
- \frac{595773188957349870818324046 1}{503503155814308249600} d^{35} + \frac{946219385360559194318492423}{130780040471248896 00} d^{34} \\
- \frac{2884364463200375866778580474 1}{888534980848779264 00} d^{33} + \frac{358661230887307084541431663 1}{370222908686991360 0} d^{32} \\
- \frac{27729900578042292117727600 03}{317333921731706880 0} d^{31} - \frac{17323961794405445645822789827 7}{17770699616975585280} d^{30} \\
+ \frac{310736093407096826889145530073 3}{444267490424389632 00} d^{29} - \frac{130277716440587652307279877866 9}{493630544915988480 0} d^{28}
\end{aligned} \tag{28}$$

$$
\begin{aligned}
&+\frac{2175543494720246680252051667789}{374850695045578752 0}d^{27}-\frac{15324266643945858395023213928441}{88853498084877926400}d^{26}\\
&-\frac{58372372369198335073039576886970 7}{13328024712731688960 0}d^{25}+\frac{29500861250635053390086777190928 1}{1480891634747965440 0}d^{24}\\
&-\frac{728822985180459844929716963812 49}{151454826281041920 0}d^{23}+\frac{31794233123655596918816472840075 91}{592356653899186176 00}d^{22}\\
&+\frac{650749157586341489423720909424757 03}{79968148276390133760 0}d^{21}-\frac{176506580277408324467488374190909 39}{3356687705428721664 0}d^{20}\\
&+\frac{431552192876810678973444833623021 09}{354025656431935488 00}d^{19}-\frac{300425317002672898953799977189125 21}{22377918036191477760}d^{18}\\
&-\frac{749894075579299475576086383836784223}{906305680465754849280}d^{17}+\frac{1152884114126290978903651885817821}{176296623184281600}d^{16}\\
&-\frac{679247544279215190070362388445065693}{4998009267274383360 0}d^{15}+\frac{272442096451803568353268951826019 77}{185111454343495680 0}d^{14}\\
&-\frac{141806555225258908786984245739777 69}{833001544545730560 0}d^{13}-\frac{177866738685319493299001739450742 27}{69416795378810880 0}d^{12}\\
&+\frac{81402564808748546820398348272047 17}{14875027581173760 0}d^{11}-\frac{158471934282528921985877223930376 21}{23138931792936960 0}d^{10}\\
&+\frac{51203085967146132778275681925029671}{851933397830860800}d^{9}-\frac{41583309979135814894876041311494 9}{10846374277939200}d^{8}\\
&+\frac{1909222806402780987957309330907 99}{10846374277939200}d^{7}-\frac{478337690398387542649533056 41}{8608233553920}d^{6}\\
&+\frac{2764737243980163013076109790463}{2560949482291200}d^{5}-\frac{155335836443886932189226007 7}{17784371404800}d^{4}\\
&-\frac{1981299728200259795937983}{242514155520}d^{3}+\frac{15743878343562160667}{7623616}d^{2}\\
&-\frac{655521591855018725}{7351344}d+4625512425.
\end{aligned}
$$

Note that the degree in (28) is $54 = (2+1) \times 18$, cf. (2).

3.3.3 Hypersurfaces Singular along a Segre 3-fold in $\mathbb{P}^5$

The Segre variety $\mathbb{S} := \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ has Hilbert polynomial $(1/2)t^3 + 2t^2 + (5/2)t + 1$. It moves in a family $\mathbb{W}_{seg}$ of dimension 24. It is well known (cf. Harris [16, p. 99]) that the homogeneous ideal is spanned by a net of quadrics of determinantal type. Identifying $\mathbb{P}^5 = \mathbb{P}(\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^3))$, $\mathbb{S}$ corresponds to the locus of rank one matrices up to scalar. As in the previous 2 cases, the family formed by the subschemes of $\mathbb{P}^5$ defined by $\mathcal{I}_W^2$ for some $W \in \mathbb{W}_{seg}$ lacks flatness precisely along the nets coming from the Veronese-like embedding $\mathbb{G}(2, \mathcal{F}_1) \cong \mathbb{G}_{\mathbf{o}} \subset \mathbb{G}(3, \mathcal{F}_2), \langle L_0, L_1 \rangle \mapsto$

$\langle L_0^2, L_0L_1, L_1^2 \rangle$. Write $\widehat{\mathbb{X}}$ for the blowing up of $\mathbb{X}$ (= nets of quadrics of determinantal type) along $\mathbb{G}_{\mathbf{o}}$. It embeds in $\mathbb{X} \times \mathbb{G}(6, \mathcal{F}_4)$ and the exceptional divisor $\widehat{\mathbb{E}}$ affords the same description as in Proposition 14**(iii)**. Scripts are available in [28, Appendix H].

Although we have all the information needed to calculate $\deg \Sigma(\mathbb{W}_{seg}, d)$ via Bott's residues formula, computations become prohibitive beyond $d = 28$, last entry in Table 1. So we were not able to perform interpolation, which would require pushing d up to $(3 + 1) \times 24$ (conjecturally).

d	degree
4	4985292672535
5	3808545362392400212560 8
6	75285508677103874434199729447346
7	691992872280130589815255863114100629 7978
8	421819544324666864848023663279460363505636673 73
9	3053853118478213444088322380518816588585076526673097 3
10	4224340951726565859342587822879909669270072209918091111509
11	1584375282811335327343377033109936680842779081038012286193493 18
12	2080035353059957499641534559924163791462457116358313751435919907641
13	1154973599663618994361925498554713929012908746335513407488729946838144 0
14	3129677022760327047365764485946385978830331925722648969765576693528286114 4
15	46218251138854455896028288030807107836206397262026919058025989004860345865068
16	405731780250170532481634557919952531383332488302197496819015246805149206946478 75
17	2269640346038978228291812022009661269306699090248673546303769574845835501210206513 0
18	85600948504320501453886081627643315459749121588267719125341873633046302423781406855 05
19	228021844617928190689443639929953269114706918869529480982537760694675440393202830624412 3
20	4459131223707827852686255332456492502745327413011186069785255174835826716801543373457986 50
21	66136044830890785552763166513088475675562647217232322960605533153943919181299528743949231995
22	764806018274923937995732822272503804438946834144111867803835970803315462229856246176043103198 7
23	7061221238074707907837554402772427735105063360352750265318179595405111259947381992696900028558 31
24	531260493934042664409289468341277144867555477472599610783325148181041857880661751296286740924183 46
25	3315561352388199144671538442416320830215174679718026171794913021436371907104234446224006732128647329
26	174334857471395667347731728239322112964231210603282356701591598514084204408291171080634141000772703155
27	782948298714351394499098694940745547636737774755262570132027875103563806741713959631404094804034485740 0
28	303991364820542511002698414336553281396075120749252336213971319871871164262548779281153647072907136671375

Table 1: $\deg \Sigma(\mathbb{W}_{seg}, d)$

3.4 Surfaces singular along elliptic quartic curves

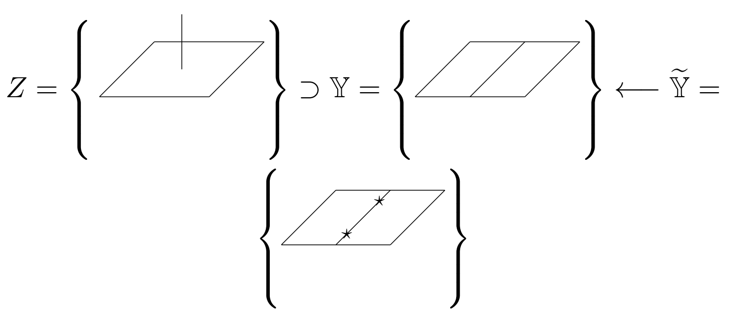
An elliptic quartic curve in $\mathbb{P}^3$ is the complete intersection of a (unique) pencil of quadric surfaces. Avritzer & Vainsencher [34], [3] obtained an explicit description of the component $\mathbb{W}_{eqc}$ of elliptic quartics of the Hilbert scheme $\mathrm{Hilb}_{4t}(\mathbb{P}^3)$. This has been used in [9] for enumerating curves in certain Calabi-Yau 3-folds, and in [6] for studying Noether-Lefschetz loci of systems of surfaces in $\mathbb{P}^3$. G. Gotzmann [13] has shown that $\mathrm{Hilb}_{4t}(\mathbb{P}^3)$ consists of two irreducible components; the second one parameterizes unions of a plane quartic curve and a zero dimensional subcheme of $\mathbb{P}^3$ of length 2.

Put $\mathbb{X} = \mathbb{G}(2, \mathcal{F}_2)$, the grassmannian of pencils of quadrics in $\mathbb{P}^3$. We summarize in the diagram below the construction of $\mathbb{W}_{eqc}$.

$$\begin{array}{ccccccccc}
\mathbb{G}(19,\mathcal{F}_4) \supset \mathbb{W}_{eqc} & = & \widehat{\mathbb{X}} & \longleftarrow & & & \widehat{\mathbb{E}} \\
 & & \downarrow & & & & \downarrow \\
\mathbb{G}(8,\mathcal{F}_3)\times\mathbb{X} & \supset & \widetilde{\mathbb{X}} & \supset & \widetilde{\mathbb{E}} & \supset & \widetilde{\mathbb{Y}} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{G}(2,\mathcal{F}_2) & = & \mathbb{X} & \supset & Z & \supset & \mathbb{Y}
\end{array} \tag{29}$$

where

$$\begin{cases} Z \cong \check{\mathbb{P}}^3 \times \mathbb{G}(2,\mathcal{F}_1) \text{ consists of pencils with a fixed plane;} \\ \mathbb{Y} \cong \{(p,l) \in \check{\mathbb{P}}^3 \times \mathbb{G}(2,\mathcal{F}_1 | p \supset l\} = \text{ closed orbit of } Z; \\ \widetilde{\mathbb{Y}} \to \mathbb{Y} = \mathbb{P}^2\text{-bundle of divisors of degree 2 over a variable line } l \subset p; \\ \widetilde{\mathbb{X}} = \text{ blowup of } \mathbb{X} \text{ along } Z; \\ \widehat{\mathbb{X}} = \text{ blowup of } \widetilde{\mathbb{X}} \text{ along } \widetilde{\mathbb{Y}}. \end{cases}$$



Let

$$\mathcal{A} \subset \mathcal{F}_2 \times \mathbb{X} \tag{30}$$

be the tautological subbundle of rank 2 on our grassmannian of pencils of quadrics. There is a natural map of vector bundles on $\mathbb{X}$ induced by multiplication,

$$\mu_3 : \mathcal{A} \otimes \mathcal{F}_1 \longrightarrow \mathcal{F}_3 \times \mathbb{X},$$

with generic rank 8. The rank drops precisely over Z. Hence we have an induced rational map $\kappa : \mathbb{X} \dashrightarrow \mathbb{G}(8, \mathcal{F}_3)$. Blowing up $\mathbb{X}$ along Z, we find the closure $\widetilde{\mathbb{X}} \subset \mathbb{G}(8, \mathcal{F}_3) \times \mathbb{X}$ of the graph of κ. The fiber

$$\widetilde{\mathbb{E}}_{(p,l)} = \mathbb{P}\left(\mathcal{F}_3^l / p\mathcal{F}_2^l\right)$$

where $\mathcal{F}_d^l$ denotes the space of forms of degree d vanishing on the line l. The fiber of $\widetilde{\mathbb{E}}$ over $\mathbf{y} := (x_0, \langle x_0, x_1 \rangle) \in \mathbb{Y}$ contains the disjoint subspaces

$$\mathbb{M}_{\mathbf{y}} := \mathbb{P}\left(x_1^2 \left(\mathcal{F}_1 / \langle x_0 \rangle\right)\right) \quad \text{and} \quad \mathbb{P}\left(\left(\mathcal{F}_2 / \mathcal{F}_2^l\right)\right) = \widetilde{\mathbb{Y}}_{\mathbf{y}}.$$

The latter embeds into $\widetilde{\mathbb{E}}_{(p,l)}$ via multiplication by $p := x_0$ and coincides with the fiber of $\widetilde{\mathbb{Y}}$. The former is the fiber of a $\mathbb{P}^2$-bundle

$$\mathbb{M} \longrightarrow \mathbb{Y}$$

to be further described in a moment.

Now, over $\widetilde{\mathbb{X}}$ we have a subbundle of cubic forms,

$$\mathcal{B} \subset \mathcal{F}_3 \times \widetilde{\mathbb{X}} \tag{31}$$

of rank 8 obtained by pullback from the tautological subbundle over $\mathbb{G}(8, \mathcal{F}_3)$. Thus we get a map of multiplication

$$\mu_4 : \mathcal{B} \otimes \mathcal{F}_1 \to \mathcal{F}_4 \times \widetilde{\mathbb{X}}$$

with generic rank 19. The scheme of zeros of $\bigwedge^{19} \mu_4$ is equal to $\widetilde{\mathbb{Y}}$ (29). In fact, it can be verified that each fiber of $\mathcal{B}$ is a linear system of cubics such that

- either it has a base locus equal to a curve with "correct" Hilbert polynomial $P_{\mathbb{W}_{eqc}}(t) = 4t$
- or it is of the form $p \cdot \mathcal{F}_2^{**}$, meaning a linear system with fixed component a plane p, and $\mathcal{F}_2^{**}$ denoting an 8-dimensional space of quadrics cutting a subscheme of p of dimension 0 and degree 2.

The exceptional divisor $\widehat{\mathbb{E}}$ is the $\mathbb{P}^8$-bundle over $\widetilde{\mathbb{Y}}$ with fiber

$$\widehat{\mathbb{E}}_{((p,l),y_1+y_2)} = \text{system of quartic curves in the plane } p \text{ which are singular at the ``doublet'' } y_1+y_2.$$

Precisely, assuming the plane $p := x_0$ and the line $l := \langle x_0, x_1\rangle$, a typical doublet has homogeneous ideal of the form $\langle x_0, x_1, f(x_2, x_3)\rangle$, for some binary form f, $\deg f = 2$. Our system of plane quartics lies in the ideal $\langle x_1, f\rangle^2 = \langle x_1^2, x_1 f, f^2\rangle$. Given a non-zero quartic g in this ideal, we may form the ideal $J = \langle x_0^2, x_0x_1, x_0 f, g\rangle$, (*e.g,* $\langle x_0^2, x_0x_1, x_0x_2^2, x_2^4\rangle$). It can be checked that J contains precisely 19 independent quartics and the Hilbert polynomial is correct. Moreover, the subscheme defined by J^2 has the expected Hilbert polynomial $12t-16$. The preceding description suffices to get a hold on the fixed points on $\widehat{\mathbb{X}}$ (29) together with their tangent spaces as explained in Araújo [2] (after [9], [24]). However, as in the case of nets of quadrics, once we pass to the thickenings, one last blowup is required. The new center $\mathbb{M} \subset \widetilde{\mathbb{E}}$ is supported in the locus of $W \in \mathbb{W}_{eqc}$ where the subscheme of $\mathbb{P}^3$ defined by $(\mathcal{I}_W)^2$ has "wrong" Hilbert polynomial: flatness fails. In fact, points corresponding to an ideal like

$$\langle x_0^2, x_0x_1, C\rangle \in \widetilde{\mathbb{E}}_{(x_0,\langle x_0,x_1\rangle)}, \tag{32}$$

where C denotes a cubic form arising from $x_1^2 \cdot (\mathcal{F}_1/\langle x_0\rangle)$, are legitimate members of $\mathbb{W}_{eqc}$, whereas its square has a "bad" Hilbert polynomial (namely $13t-20$). Notation as in (30),(31), let $\nu : \mathcal{A}\otimes\mathcal{B} \to \mathcal{F}_5$ be map of vector bundles over $\widetilde{\mathbb{X}}$ defined by multiplication. The generic rank of ν is 12. Set

$$\mathbb{M} = \text{ scheme of zeros of } \bigwedge^{12} \nu.$$

In a way similar to Prop. 14, local calculations (cf. [28, Appendix I.1, p.177]) show that $\mathbb{M}$ is the indeterminacy locus of the natural rational map

$$\widetilde{\mathbb{X}} \dashrightarrow \mathbb{G}(12, \mathcal{F}_5) \tag{33}$$

induced by ν. One checks that $\mathbb{M}$ is the $\mathbb{P}^2$-bundle over $\mathbb{Y}$ which parameterizes the triples $\langle p, l, C\rangle$, where p denotes a plane, $l = \langle p, p'\rangle$ a line therein and C a class in $\mathbb{P}\left((p')^2 \cdot \mathcal{F}_1/\langle p\rangle\right)$. We have the embedding $\mathbb{M} \subset \widetilde{\mathbb{E}}$ of bundles over $\mathbb{Y}$ such that in the fiber over any $(p,l) \in Y$ the point $\langle p,l,C\rangle$ with $C = (p')^2 p''$ mod $\langle p\rangle$ is mapped to the class $\overline{(p')^2 \cdot p''} \in \mathbb{P}\left(\mathcal{F}_3^l / p\mathcal{F}_2^l\right) = \widetilde{\mathbb{E}}_{(p,l)}$. Consider the blow up diagram of $\widetilde{\mathbb{X}}$ along $\mathbb{M}$

$$\begin{array}{ccc} \widetilde{\mathbb{X}}' & \supset & \widetilde{\mathbb{M}} \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{X}} & \supset & \mathbb{M} \end{array} \tag{34}$$

By construction $\widetilde{\mathbb{X}}'$ embeds in $\widetilde{\mathbb{X}} \times \mathbb{G}(12, \mathcal{F}_5)$ as the closure of the graph of the rational map (33). Since $\mathbb{M}$ is disjoint from the blowup center $\widetilde{\mathbb{Y}}$ (cf. diagram 29), it follows that $\widetilde{\mathbb{Y}}$ lifts isomorphically to $\widetilde{\mathbb{Y}}' \subset \widetilde{\mathbb{X}}'$ so that the blowup of $\widetilde{\mathbb{X}}$ along $\widetilde{\mathbb{Y}}$ is naturally isomorphic to the blowup of $\widetilde{\mathbb{X}}'$ along $\widetilde{\mathbb{Y}}'$ over a neighborhood of $\widetilde{\mathbb{Y}}$. In special, only the fixed points of $\widehat{\mathbb{X}}$ over $\mathbb{M}$ are replaced by those in $\widetilde{\mathbb{M}}$. It turns out that a point like (32) is replaced by 9 fixed points in $\widetilde{\mathbb{M}}$ corresponding to ideals of the form

$$\langle x_0^2, x_0x_1, C\rangle^2 + \langle m\rangle$$

$$m \in \{x_0x_2C, x_0x_3C, \frac{C^2}{x_1}, x_0^2x_1x_2^2, x_0^2x_1x_2x_3, x_0^2x_1x_3^2, x_0^2x_0x_2^2, x_0^2x_0x_2x_3, x_0^2x_0x_3^2\}.$$

The technicalities of the final computation can be found in [28, Appendix I.1, p.177]. The polynomial that gives us the degree of $\Sigma(\mathbb{W}_{eqc}, d)$ is displayed below. Note once again that the degree is equal to $(1+1) \times \dim(\mathbb{W}_{eqc}) = 2 \times 16$, cp. (2).

$$
\begin{aligned}
\deg \Sigma_{\mathbb{W}_{eqc},d} = {} & \tfrac{77991978249}{47023181004800}d^{32} - \tfrac{142130943}{922746880}d^{31} + \tfrac{8109239447979}{1175579525120}d^{30} \\
& - \tfrac{4150267051797}{20992491520}d^{29} + \tfrac{47676232841150619}{11755795251200}d^{28} - \tfrac{6615027446596551}{104962457600}d^{27} \\
& + \tfrac{128385059997089001}{167939932160}d^{26} - \tfrac{103459871906659801}{14129561600}d^{25} + \tfrac{8937969600419178632 71}{16277254963200}d^{24} \\
& - \tfrac{3128459731517024143 13}{1017328435200}d^{23} + \tfrac{43125876092002536956 39}{4069313740800}d^{22} \\
& + \tfrac{6155781582234103357}{7266631680}d^{21} - \tfrac{1105621403101024328482787}{24415882444800}d^{20} \\
& + \tfrac{2134617904050477326290337}{5410337587200}d^{19} - \tfrac{102770429075204895153733777 1}{476109707673600}d^{18} \\
& + \tfrac{1568309607110425883232529237}{223176425472000}d^{17} + \tfrac{399314335681097660200615893191}{57133164920832000}d^{16} \\
& - \tfrac{1279119743116127875650943577 69}{396758089728000}d^{15} + \tfrac{7297607552669425891347140320 19}{238054853836800}d^{14} \\
& - \tfrac{1828532248668326451456639996724 9}{892705701888000}d^{13} + \tfrac{1505077790650358035091498239027 7}{137339338752000}d^{12} \\
& - \tfrac{836272120499064344796075142171 9}{17167417344000}d^{11} + \tfrac{1785652834399799300784848728 09}{98099527680}d^{10} \\
& - \tfrac{27317871287377170497361801712 43}{476872704000}d^{9} + \tfrac{1125598445944774654288515801691861}{74392141824000}d^{8} \\
& - \tfrac{58025484355390407710374488759691}{1743565824000}d^{7} + \tfrac{1679603946104074748281436517442 9}{278970531840}d^{6} \\
& - \tfrac{8521350244073783951990040324653}{96864768000}d^{5} + \tfrac{5994222085454702603815927073 47}{5930496000}d^{4} \\
& - \tfrac{79632703268071528722557737021 9}{9081072000}d^{3} + \tfrac{43427222707902930597970733 3}{8072064}d^{2} \\
& - \tfrac{1490642041280752415948983 9}{720720}d + 37131247788800303 20.
\end{aligned}
\tag{35}
$$

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