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Critical ideals and applications

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Abstract

Critical ideals were defined as a generalization of the critical group, also known as sandpile group. Furthermore, the varieties associated to these ideals can be regarded as a generalization of the Laplacian and adjacency spectra of the graph. Recently, there have been found relations between the algebraic co-rank, the zero forcing number, and the minimum rank of a graph. We outlook how all these concepts are related.

1 Introduction

Let $\mathcal{R}$ be a commutative ring and consider an $n \times n$ matrix M whose entries are in the polynomial ring $\mathcal{R}[x_1, \ldots, x_m]$ with m indeterminates. For $i \in [n] := \{1, \ldots, n\}$, let $\mathcal{I} = \{r_j\}_{j=1}^{i}$ and $\mathcal{J} = \{c_j\}_{j=1}^{i}$ be two sequences such that

$$1 \leq r_1 < r_2 < \cdots < r_i \leq n \text{ and } 1 \leq c_1 < c_2 < \cdots < c_i \leq n.$$

Let $M[\mathcal{I};\mathcal{J}]$ denote the submatrix of a matrix M induced by the rows with indices in $\mathcal{I}$ and columns with indices in $\mathcal{J}$. The determinant of $M[\mathcal{I};\mathcal{J}]$ is called an *i-minor* of M. The *i-th* determinantal ideal of matrix

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M is the ideal generated by all i-minors of M, and denoted by $I_i(M)$. We denote by $\text{minors}_i(M)$ the set of all i-minors of M.

Given a graph G with n vertices and a set of variables $X_G = \{x_u : u \in V(G)\}$, the *generalized Laplacian matrix* $L(G, X_G)$ of G is the matrix whose uv-entry is given by

$$L(G, X_G)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where m_{uv} is the number of the edges between vertices u and v.

Definition 1. Let $\mathcal{R}[X_G]$ be the polynomial ring over a commutative ring $\mathcal{R}$ in the variables X_G. For $i \in [n]$, the *i-th critical ideal* $I_i^{\mathcal{R}}(G, X_G) \subseteq \mathcal{R}[X_G]$ of G is the determinantal ideal generated by $\text{minors}_i(L(G, X_G))$.

An ideal is said to be *trivial* if it is equal to $\langle 1 \rangle$ $(= \mathcal{R}[X])$. The *algebraic co-rank* $\gamma_{\mathcal{R}}(G)$ of G is the maximum integer i for which $I_i^{\mathcal{R}}(G, X_G)$ is trivial.

Example 1. Consider the generalized Laplacian matrix of the cycle with 4 vertices.

$$L(C_4, X_{C_4}) = \begin{bmatrix} x_0 & -1 & 0 & -1 \\ -1 & x_1 & -1 & 0 \\ 0 & -1 & x_2 & -1 \\ -1 & 0 & -1 & x_3 \end{bmatrix}$$

Below we give a Gröbner bases of the critical ideals over $\mathbb{Z}[X_{C_4}]$.

$$I_i^{\mathbb{Z}}(C_4, X_{C_4}) = \begin{cases} \langle 1 \rangle & \text{if } i \leq 2 \\ \langle x_0 + x_2, x_1 + x_3, x_2 x_3 \rangle & \text{if } i = 3 \\ \langle x_0 x_1 x_2 x_3 - x_0 x_1 - x_0 x_3 - x_1 x_2 - x_2 x_3 \rangle & \text{if } i = 4. \end{cases}$$

From which follows that $\gamma_{\mathbb{Z}}(C_4) = 2$. For this graph, these Gröbner bases coincide with the Gröbner bases of the critical ideals over $\mathbb{R}[X_{C_4}]$. In general they are different.

In the following, we are going to see two applications of the critical ideals. In Section 2, critical ideals are defined over $\mathcal{R}[X]$ with $\mathcal{R}$ a field, we will see that they can be used to bound the minimum rank (defined over $\mathcal{R}$) and the zero forcing number, this opens new applications of algebraic geometry. In Section 3, critical ideals are defined over $\mathbb{Z}[X]$. Here we will see that critical ideals generalize the sandpile group and the Smith group of a graph.

2 Minimum rank and zero forcing number

Let $\mathcal{R}$ be a field and $I \subseteq \mathcal{R}[X]$ be an ideal. The *variety* $V(I)$ of I is defined as the set $\{\mathbf{a} \in \mathcal{R}^n : f(\mathbf{a}) = 0 \text{ for all } f \in I\}$, that is, $V(I)$ is the set of common roots between polynomials in I. Note that these varieties can be regarded as a generalization of the Laplacian and adjacency spectra of a graph.

An important property of the critical ideals is that they form a chain of ideals.

Proposition 1. *For any graph, we have*

$$\langle 1 \rangle \supseteq I_1^{\mathcal{R}}(G, X_G) \supseteq \cdots \supseteq I_n^{\mathcal{R}}(G, X_G) \supseteq \langle 0 \rangle.$$

And

$$\emptyset = V(\langle 1 \rangle) \subseteq V(I_1^{\mathcal{R}}(G, X_G)) \subseteq \cdots \subseteq V(I_n^{\mathcal{R}}(G, X_G)) \subseteq V(\langle 0 \rangle) = \mathcal{R}^n.$$

Proof. Let M be an $(i+1) \times (i+1)$-submatrix of $L(G, X_G)$. We have

$$\det M = \sum_{j=1}^{i+1} M_{j,1} \det M[j;1],$$

where $M_{j,1}$ denotes the $(j,1)$ entry of the M and $M[j;1]$ denotes the submatrix of M whose j-th row and 1st column were deleted. This gives that $I_{i+1}^{\mathcal{R}}(G, X_G) \subseteq I_i^{\mathcal{R}}(G, X_G)$. From which in turn follows that $V(I_{i+1}^{\mathcal{R}}(G, X_G)) \supseteq V(I_i^{\mathcal{R}}(G, X_G))$. □

There are several problems where bounds depend on the spectra of a matrix associated to a graph, it could be that we could obtain bounds by finding roots in the varieties of the critical ideals. An example comes from minimum rank and zero forcing number.

The *zero forcing game* is a color-change game where vertices can be blue or white. At the beginning, the player can pick a set of vertices B and color them blue while others remain white. The goal is to color all vertices blue through repeated applications of the *color change rule*: If u is a blue vertex and v is the only white neighbor of u, then v turns blue. An initial set of blue vertices B is called a *zero forcing set* if starting with B one can make all vertices blue. The *zero forcing number* $Z(G)$ is the minimum cardinality of a zero forcing set.

Example 2. For any cycle graph with $n \geq 3$ vertices, any pair of two adjacent vertices form a zero forcing set of minimum cardinality. From which follows $Z(C_n) = 2$ for $n \geq 3$.

For a graph G on n vertices, the family $\mathcal{S}_{\mathcal{R}}(G)$ collects all $n \times n$ symmetric matrices with entries in the ring $\mathcal{R}$, whose i, j-entry $(i \neq j)$ is nonzero whenever i is adjacent to j and zero otherwise. Note that the diagonal entries can be any element in the ring $\mathcal{R}$. The *minimum rank* $\mathrm{mr}_{\mathcal{R}}(G)$ of G is the smallest possible rank among matrices in $\mathcal{S}_{\mathcal{R}}(G)$. Let $\mathrm{mz}(G) = |V(G)| - Z(G)$. In [1], it was proved that $\mathrm{mz}(G) \leq \mathrm{mr}_{\mathcal{R}}(G)$ for any field $\mathcal{R}$.

Theorem 1. *[3] For any commutative ring $\mathcal{R}$ with unity,* $\mathrm{mz}(G) \leq \gamma_{\mathcal{R}}(G)$.

The relation between $\mathrm{mr}_{\mathcal{R}}(G)$ and $\gamma_{\mathcal{R}}(G)$ is still not completely understood. However, we have the following.

Proposition 2. *[3] If there exist $k \in \mathbb{N}$ and $\mathbf{a} \in \mathcal{R}^n$ such that $I_k^{\mathcal{R}}(G, \mathbf{a}) = \langle 0 \rangle$ for some k, then* $\mathrm{mr}_{\mathcal{R}}(G) \leq k - 1$.

In particular, we have the following.

Corollary 1. *If $V\left(I^{\mathcal{R}}_{\gamma_{\mathcal{R}}(G)+1}(G, X_G)\right)$ is not empty, then* $\mathrm{mr}_{\mathcal{R}}(G) \leq \gamma_{\mathcal{R}}(G)$.

Therefore, mz and γ can be used to compute exact values of the minimum rank, as next example shows.

Example 3. Continuing Example 1, we observe vector $\mathbf{0}$ is a root of all polynomials in $I_3^{\mathbb{R}}(C_4, X_{C_4})$, then $\mathrm{mr}_{\mathbb{R}}(C_4) \leq \gamma_{\mathbb{R}}(C_4)$. Thus, by Theorem 1 and Example 2, it follows that $\mathrm{mz}(G) = \mathrm{mr}_{\mathbb{R}}(C_4) = \gamma_{\mathbb{R}}(C_4) = 2$.

In [3], it was noted the Weak Nullstellensatz implies that if $\mathcal{R}$ is an algebraically closed field, then $\mathrm{mr}_{\mathcal{R}}(G) \leq \gamma_{\mathcal{R}}(G)$. Also, there exist graphs for which $\mathrm{mr}_{\mathbb{Z}}(G) > \gamma_{\mathbb{Z}}(G)$. For the field of real numbers, it is conjectured [3] that $\mathrm{mr}_{\mathbb{R}}(G) \leq \gamma_{\mathbb{R}}(G)$. In [3] it was proved that if a graph G have $\mathrm{mr}_{\mathbb{R}}(G) \leq 2$, then $\mathrm{mr}_{\mathbb{R}}(G) \leq \gamma_{\mathbb{R}}(G)$. And in [2] that if a graph G have $\mathrm{mr}_{\mathbb{R}}(G) = 3$, then $\mathrm{mr}_{\mathbb{R}}(G) \leq \gamma_{\mathbb{R}}(G)$. Moreover, in [3] there were found families of graphs where mz, the minimum rank and the algebraic co-rank coincide. Among these graph families are the trees, cycles, and line graphs of trees.

3 Sandpile group and Smith group

Another interest in computing critical ideals comes when they are evaluated, since we recover information of the graph. For instance, the *adjacency matrix* $A(G)$ and *Laplacian matrix* $L(G)$ of G are the evaluation of $-L(G, X_G)$ and $L(G, X_G)$ at $X_G = \mathbf{0}$ and at $X_G = \deg(G)$, respectively, where $\deg(G)$ is the degree vector of G. A subtler relation is obtained with the sandpile group and the Smith group. We will explore these relations in more detail.

By considering an $m \times n$ matrix M with integer entries as a linear map $M : \mathbb{Z}^n \to \mathbb{Z}^m$, the *cokernel* of M is defined as the quotient module $\mathbb{Z}^m / \mathrm{Im}\, M$. By the fundamental theorem for finitely generated Abelian groups, the cokernel of M can be described as

$$\mathit{coker}(M) \cong \mathbb{Z}_{f_1(M)} \oplus \mathbb{Z}_{f_2(M)} \oplus \cdots \oplus \mathbb{Z}_{f_r(M)} \oplus \mathbb{Z}^{m-r},$$

where the integer number $f_1(M), f_2(M), ..., f_r(M)$ are the *invariant factors* of M. One way to compute the invariant factors is by means of the following formula.

Proposition 3. *[8, Theorem 3.9] Let $\Delta_i(M)$ denote the greatest common divisor of the i-minors of a $m \times n$ matrix M with integer entries. Then, the i-th invariant factor $f_i(M)$ of M is equal to $\Delta_i(M)/\Delta_{i-1}(M)$, where $\Delta_0(M) = 1$.*

This finitely generated Abelian group becomes a graph invariant when we take the matrix M to be a matrix associated with a graph, say, the adjacency or Laplacian matrices. The cokernel of $A(G)$ is known as the *Smith group* of G and is denoted $S(G)$, and the torsion part of the cokernel of $L(G)$ is known as the *sandpile group* $K(G)$ of G. It is known that $|K(G)|$ is equal to the number of spanning trees of G. For an introduction to the sandpile group and the smith group we refer the reader to [9] and [10], respectively.

The relation between invariant factors and the critical ideals is given in the following result. Here we present a general form of [7, Proposition 3.6].

Proposition 4. *Let $\mathcal{R}$ be a principal ideal domain, and M a matrix with entries in $\mathcal{R}$. If $M - \operatorname{diag}(M) = -A(G)$, then*

$$I_i^{\mathcal{R}}(G, X_G)|_{X_G=\operatorname{diag}(M)} = \left\langle \prod_{j=1}^{i} f_j(M) \right\rangle = \left\langle \Delta_i^{\mathcal{R}}(M) \right\rangle \text{ for all } 1 \leq i \leq r,$$

where r is the rank of M, $\Delta_i^{\mathcal{R}}(M)$ is the greatest common divisor of the i-minors of M over $\mathcal{R}$, and $f_1(M) \mid \cdots \mid f_r(M)$ are the invariant factors in the Smith normal form of M.

Proof. After evaluating the ideal $I_i^{\mathcal{R}}(G, X_G)$ at $X_G = \operatorname{diag}(M)$, the ideal obtained is the generated by the i-minors of M. Since $\mathcal{R}$ is p.i.d., then the evaluated ideal is principal and generated by the g.c.d. of the i-minors of M. The last equality follows since $f_i(M) = \Delta_i(M)/\Delta_{i-1}(M)$. □

Therefore, by evaluating the i-th critical ideal $I_i^{\mathbb{Z}}(G, X_G)$ at the degree vector or the zero-vector, we get that the ideal is generated by $\Delta_i(L(G))$

and $\Delta_i(A(G))$, respectively. Thus, objects like the sandpile group and the Smith group can be recovered from critical ideals.

Example 4. Consider again Example 1. By evaluating the critical ideals over $\mathbb{Z}$ of C_4 at $X_{C_4} = \deg(C_4) = (2,2,2,2)$, we obtain $\Delta_i(L(C_4)) = 1$ for $i \leq 2$, $\Delta_3(L(C_4)) = 4$ and $\Delta_4(L(C_4)) = 0$. Thus the sandpile group $K(C_4) \cong \mathbb{Z}_4$. An evaluation of the critical ideals of C_4 at $X_{C_4} = (0,0,0,0)$, we obtain $\Delta_i(A(C_4)) = 1$ for $i \leq 2$, and $\Delta_i(A(C_4)) = 0$ for $i \in \{3,4\}$. Therefore, the Smith group $S(C_4) \cong \mathbb{Z}_1 \oplus \mathbb{Z}^2$.

In general, sandpile group is not induced monotone, for instance $K(K_4) \cong \mathbb{Z}_4^2$ is not a subgroup of $K(K_5) \cong \mathbb{Z}_5^3$. One interesting feature of the critical ideals is that they are induce monotone.

Proposition 5. *Let H be an induced subgraph of G. Then $I_i^{\mathcal{R}}(H, X_H) \subseteq I_i^{\mathcal{R}}(G, X_G)$.*

Proof. Let M be an $i \times i$ submatrix of $L(H, X_H)$. Then M is a submatrix of $L(G, X_G)$. From which follows that $I_i^{\mathcal{R}}(H, X_H) \subseteq I_i^{\mathcal{R}}(G, X_G)$. □

This implies that if H is an induced subgraph of G, then $\gamma_{\mathcal{R}}(H) \leq \gamma_{\mathcal{R}}(G)$. Therefore, for each k, the class of graphs with $\gamma_{\mathcal{R}}(G) \leq k$ can be characterized by a collection (possibly infinite) of forbidden induced subgraphs. A consequence of Proposition 4 is that $\gamma_{\mathcal{R}}(G) \leq f_1(M)$, where M is such that $M - \operatorname{diag}(M) = -A(G)$. Therefore, critical ideals can also be used in finding characterizations of graphs whose associated adjacency or Laplacian matrices have few invariant factors equal to 1. These ideas have been used to characterize the graphs whose Laplacian matrix have at most 2 or 3 invariant factors equal to one, see [4, 5]. In the following we give the characterization of the digraphs with at most one trivial critical ideals, which in turn is related with mz and the minimum rank.

Let Λ_{n_1,n_2,n_3} be the digraph defined in the following way: The vertex set $V(\Lambda_{n_1,n_2,n_3})$ is partitioned in three sets T, T' and K with n_1, n_3 and n_2 vertices, respectively, such that T and T' are two trivial digraphs (which have no arcs), and K is a complete digraph (which has double arcs between each pair of vertices). Additionally, the arc sets $(T, K)_{\Lambda_{n_1,n_2,n_3}}$, $(T, T')_{\Lambda_{n_1,n_2,n_3}}$ and $(K, T')_{\Lambda_{n_1,n_2,n_3}}$ are complete.

Theorem 2. *[3, 6] Let $\mathcal{R}$ be a commutative ring with unity. The following are equivalent:*

(1) *D is isomorphic to Λ_{n_1,n_2,n_3},*

(2) $\mathrm{mr}_{\mathcal{R}}(D) \leq 1$,

(3) $\mathrm{mz}(D) \leq 1$,

(4) $\gamma_{\mathcal{R}}(D) \leq 1$.

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