Matemática Contemporânea, Vol. 46, 65–73

Equitable total chromatic number of infinite classes of complete tripartite non balanced graphs

Anderson G. da Silva **Simone Dantas**
Diana Sasaki

Abstract

An equitable total coloring is the assignment of colors to the vertices and edges of a graph such that incident and adjacent elements receive different colors and the difference between the cardinalities of any two color classes is at most 1. The equitable total chromatic number of a graph (χ''_e) is the smallest integer for which the graph has an equitable total coloring. Wang (2002) conjectured that $\Delta + 1 \leq \chi''_e \leq \Delta + 2$. In this work, we contribute to this conjecture by proving that infinite classes of tripartite complete non balanced graphs have equitable total coloring with $\Delta + 1$ colors using coloring matrices.

2000 AMS Subject Classification: 05C15 and 05C85.

Key Words and Phrases: equitable total coloring, complete tripartite graphs, graph coloring.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, CNPq, FAPERJ (Brazilian research agencies) and L'oreal-UNESCO-ABC Para mulheres na Ciência 2017.

1 Introduction

Let $G = (V, E)$ be a graph. A *k-total coloring* of G is an assignment of k colors to the vertices and edges of G such that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by χ'', is the smallest k for which G has a k-total coloring. From the definition of total coloring, we have that $\chi'' \geq \Delta+1$ and the Total Coloring Conjecture (TCC) (Behzad [1], Vizing [10]) states that the total chromatic number of any graph is at most $\Delta + 2$, where Δ is the maximum degree of the graph. In 1989, Sànchez-Arroyo [5] proved that the problem of determining the total chromatic number of an arbitrary graph is NP-hard, and it remains NP-hard even for cubic bipartite graphs.

A *k-equitable total coloring* is an assignment of k colors to the edges and vertices of the graph such that adjacent and incident elements receive different colors and the difference between the cardinalities of any two color classes is at most one. The smallest integer k for which a graph G has a k-equitable total coloring is called the *equitable total chromatic number* of G and it is denoted by $\chi''_e(G)$. In 2002, Wang [11] conjectured that the equitable total chromatic number of any graph is at most $\Delta+2$ (Equitable Total Coloring Conjecture (ETCC)). In 2016, Dantas et al. [3] proved that the problem of determining the equitable total chromatic number of a cubic bipartite graph is NP-complete.

A graph is said to be *r-partite* if its vertex set can be partitioned into r sets such that no two vertices within the same part are adjacent. An r-partite graph is *complete* if there is an edge between any two vertices of different parts of the partition. When the cardinalities of the parts of the partition are equal, we say that the graph is *balanced*.

In 1974, the total chromatic number of all complete r-partite balanced graphs was determined by Bermond [2]. In 1994, Fu [4] determined that the equitable total coloring of complete bipartite graphs is $\Delta + 2$ and proved that there exist equitable $(\Delta + 2)$-total colorings for all complete r-partite graphs of odd order.

Recently, for all complete r-partite balanced graphs the Equitable Total Coloring Conjecture was verified in 2018 [7] and the sharp value for the equitable total chromatic number for this class of graphs was determined in 2019 [6].

In order to contribute with the ETCC, we investigate equitable total colorings of complete 3-partite (tripartite) non balanced graphs. These graphs have also been investigated in the context of different kinds of colorings such as the adjacent vertex distinguishing total coloring and the adjacent vertex distinguishing edge coloring. The results from such investigations can be found, respectively, in [9] and [8].

The complete tripartite non balanced graphs are denoted by $K_{a,b,c}$ meaning that the parts of the partition of the vertex set have, respectively, a, b and c vertices. For convenience we adopt the convention that $a \leq b \leq c$, without a, b and c being equal simultaneously. We verify the ETCC for the following classes of complete tripartite non balanced graphs:

1. $K_{a,b,c}$ with $a < b = c$ has $\chi''_e = \Delta + 1$;
2. $K_{a,b,c}$ with $a = b$ and $c \geq b^2$ if $b \neq 1$ or $c \geq 2$ if $b = 1$ has $\chi''_e = \Delta + 1$;
3. $K_{a,b,c}$ with $a < b$ and $c \geq b^2$ has $\chi''_e = \Delta + 1$.

2 Main results

Throughout this paper we analyze the equitable total chromatic number of complete tripartite non balanced graphs, which we denote by $K_{a,b,c}$, where $a \leq b \leq c$ and not all of them are equal. The partition of the vertex set is denoted by $V = \{X_1, X_2, X_3\}$ and vertices are labeled v_i, where $i = 1, 2, \cdots, a+b+c$. The vertices of the graph are denoted by v_i, where $v_i \in X_1$ (resp. $v_i \in X_2$, $v_i \in X_3$) if $1 \leq i \leq a$ (resp. $a+1 \leq i \leq a+b$, resp. $a+b+1 \leq i \leq a+b+c$).

In this section we show that some classes of $K_{a,b,c}$, where $a \leq b \leq c$ (not all of them are equal) have an equitable total coloring with $\Delta + 1$

colors. We determine the equitable total chromatic number of the cases: $a < b = c$; $a = b$ and $c \geq b^2$ if $b \neq 1$ or $a = b$ and $c \geq 2$ if $b = 1$; and $a < b$ and $c \geq b^2$.

Theorem 2.1. *The graph $K_{a,b,c}$ with $a < b = c$ has $\chi''_e = \Delta + 1$.*

Proof. We obtain the coloring of this class of graphs from the graphs of type $K_{3\times b}$. For $K_{a,b,b}$ $(a < b)$ we have that $\Delta(K_{a,b,b}) = 2b = \Delta(K_{b,b,b})$. Since $K_{b,b,b}$ is a regular graph with $\chi''_e = \Delta+1$, as shown in [7], this means that each color is represented in all vertices. When removing k vertices and their incident edges from the part X_1 of $K_{b,b,b}$, each color class has its cardinality reduced in k unities. Thus, the difference between the cardinalities of any two color classes is not changed in $K_{a,b,b}$ $(a = b - k)$. Also, because no incident or adjacent elements received the same color in $K_{b,b,b}$ and their colors were not changed in the subgraph $K_{a,b,b}$, we conclude that, indeed, $K_{a,b,b}$ has an equitable total coloring with $\Delta + 1$ colors. In Figure 1, we show, on the left, the graph $K_{3\times 2}$ totally colored according to algorithm presented in [6]; on the right, we show the graph $K_{1,2,2}$ totally colored according to Theorem 2.1. We observe that the partition into independent sets of the graphs presented in Figure 1 are, respectively: $X_1 = \{x_{11}, x_{12}\}, X_2 = \{x_{21}, x_{22}\}, X_3 = \{x_{31}, x_{32}\}$ (we followed the notation of [6]); and $X_1 = \{v_1\}, X_2 = \{v_2, v_3\}, X_3 = \{v_4, v_5\}$. $\square$

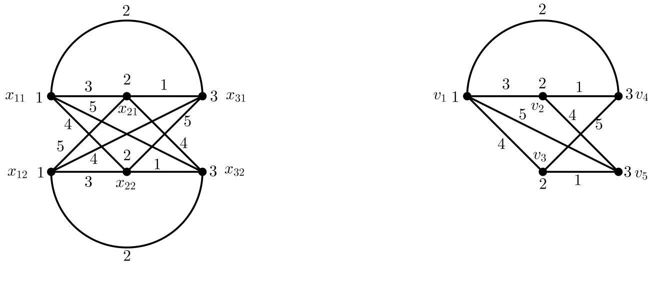


Figure 1: The graph $K_{1,2,2}$ with an equitable total coloring using 5 colors.

Theorem 2.2. *The graph $K_{a,b,c}$ with $a = b$ and $c \geq b^2$ if $b \neq 1$ or $a = b$ and $c \geq 2$ if $b = 1$ has $\chi''_e = \Delta + 1$.*

Proof. We define a *coloring matrix* as a matrix whose entries determine the colors of elements of a graph. Let M be a matrix of order $2b+c$. Our goal is to fill the entries of M with $\Delta+1 = b+c+1$ different elements. The entry m_{ij} of M represents the color that the vertex v_i receives if $i = j$. If $i \neq j$ the entry m_{ij} will either be left empty (if v_iv_j is not an edge of $K_{b,b,c}$) or represent the color of the edge v_iv_j otherwise. The first b rows refer to set X_1 since $a = b$ and it has vertices 1 to b then. The same thing can be made for the next rows ($b+1$ to $2b$ refer to X_2 and $2b+1$ to $2b+c$, to X_3). Note that M is symmetric, since if $i \neq j$, then entry m_{ij} and m_{ji} will either both be left empty or will represent the color of the edge v_iv_j $(= v_jv_i)$. Therefore, we are only interested in filling the entries m_{ij} where $j \geq i$.

The total number of elements in the graph $K_{b,b,c}$ is $b^2 + 2bc + 2b + c$ and $\Delta + 1 = b + c + 1$. We refer the reader to [6, 7], where it is explained in details how to determine the number of elements and Δ in the case where all the independent sets have the same cardinality. The reasoning to obtain such values when the cardianlities of the independent sets is not the same is similar. One can easily check that dividing the number of elements by the number of colors the quotient is $2b$ and the remainder is $c - b^2$. Since in this case, by assumption, $c \geq b^2$ we ensure that the remainder is non negative. It can be easily seen that the remainder is strictly less than the divisor. This means that $c - b^2$ colors must be used $2b + 1$ times, whereas $b^2 + b + 1$ colors are used $2b$ times.

As we said previously, we fill the entries m_{ij} where $j \geq i$ and, since the matrix is symmetric, the other entries are automatically filled. To fill the coloring matrix, we distribute the numbers from 1 to $\Delta + 1$ in the first row in ascending order in the entries that represent either the color of a vertex or the color of an edge. In the second row, we shift all elements of the first row one column to the right, except for the color $\Delta + 1$, which occupies the entry below the number 2 of the first row. The process is analogous until the b-th row. We shift one unity to the right at every new row because repetition of colors in column would imply that incident or adjacent elements were assigned to the same color. If a color occupies the entry $m_{k,2b+c}$ ($1 \leq k \leq b-1$), in the next row such color occupies the entry $m_{k+1,b+1}$. Since the matrix is symmetric, after this step, the first b columns have their entries determined. Also, it can be easily seen that each color was used b times so far.

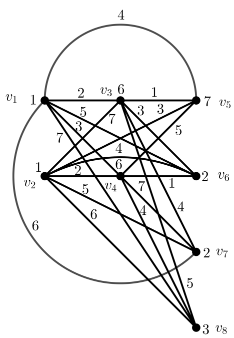
For the next b rows, the process of shifting is repeated except for two cases. First, the colors 1 and the color in the entry $m_{b,2b+c}$ are switched, because if color 1 kept being shifted 1 column to the right at every new row, it would imply that such color was assigned to the vertices of parts X_1 and X_2, which are adjacent. Also, we remark that the first b columns were already filled by the previous step. So, when filling an entry of a row from $b+1$ to $2b$, if a color has occurred in that row by the previous step, the entry is left empty at first. Such entries will be filled with the last $b-1$ colors, that is, by colors $\Delta+1, \Delta, \Delta-1, \cdots, \Delta-b+3$ as follows:

Color $\Delta-b+3$ goes on entries $m_{b+2,2b+3}, m_{b+3,2b+4}, m_{b+4,2b+5}, \cdots,$ $m_{2b,3b+1}$, whereas color $\Delta-b+4$ occupies entries $m_{b+3,2b+5}, m_{b+4,2b+6},$ $m_{b+5,2b+7}, \cdots, m_{2b+1,3b+3}$. Color $\Delta+1$ appears in the entry $m_{2b,4b-1}$. In general, color $\Delta-b+i$ is used in entries $m_{b+(i-1)+j,2b+3+2(i-3)+j}$, with $0 \leq j \leq b+1-i$.

After that step, it only lacks to determine the colors of the vertices of X_3 since that, by symmetry, the first $2b$ columns were filled. The colors that are less used so far are the ones in the entries $m_{b+1,1}, m_{b+1,2}, \cdots, m_{b+1,b},$ $m_{b+2,1}, \cdots, m_{b+2,b}, \cdots, m_{2b,1}, \cdots, m_{2b,b}$. We begin using the colors of the referred entries in the vertices of X_3 so that all the colors are used the same number of times. We observe that the colors of the submatrix of order b below the submatrix that represents the colors of the vertices of part X_1 are $2, 3, 4, \cdots, b+1, \Delta-b+3, \Delta-b+4, \Delta-b+5, \cdots, \Delta+1$ and we use such colors in the following order: $\Delta+1$ in the first vertex of part X_3, 2 in the b^2-th vertex of part X_3 and for the vertices in between the first and the b^2-th, we use the colors $2, 3, 4, \cdots, b+1, \Delta-b+3, \Delta-b+4, \Delta-b+5, \cdots, \Delta+1$ as many times as they are used in the above described submatrix, except for 2 and $\Delta+1$, since they have already been used once each. However, if c is strictly greater than b^2 we have that $c - b^2 > 0$ and those vertices would remain without a color, but since this is an equitable total coloring, the difference between the cardinalities of two color classes might be 1, which allows us to use the color of the entry $m_{2b,i}$ in the $(i+1)$-th vertex of part X_3 for $i \geq b^2$. By construction, we get that no incident or adjacent elements receive the same color and by the last step we ensure that the coloring is equitable, as desired. For instance, observe the coloring matrix presented below, which is for the graph $K_{2,2,4}$. □

1		2	3	4	5	6	7
	1	7	2	3	4	5	6
2	7	6		1	3	4	5
3	2		6	5	1	7	4
4	3	1	5	7			
5	4	3	1		2		
6	5	4	7			3	
7	6	5	4				2

(a) Coloring matrix of $K_{2,2,4}$



(b) Coloring of $K_{2,2,4}$

Theorem 2.3. *The graph $K_{a,b,c}$ with $a < b$ and $c \geq b^2$ has $\chi''_e = \Delta + 1$.*

Proof. We obtain the coloring of this class of graphs from the class described in Theorem 2.2. Consider now the graph $K_{b,b,c}$ $(c \geq b^2)$. By removing k vertices from X_1 $(1 \leq k < b)$ and their incident edges, we get a subgraph $K_{a,b,c}$ $(a = b - k)$ such that $\Delta(K_{b,b,c}) = \Delta(K_{a,b,c})$. Analogously to the first class of graphs to have its equitable total chromatic number determined, we get that $K_{a,b,c}$ $(a < b$ and $c \geq b^2)$ has an equitable total coloring with $\Delta + 1$ colors. The graph $K_{1,2,4}$ with an equitable $(\Delta + 1)$-total coloring is presented in Figure 3. It can be easily seen that, by Theorem 2.3, the coloring of $K_{1,2,4}$ can be obtained from the coloring matrix of $K_{2,2,4}$ by excluding the first column and row of that matrix. □

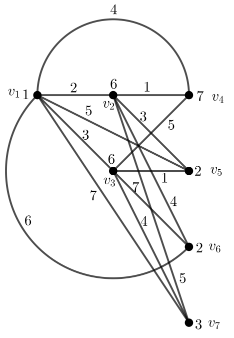


Figure 3: The graph $K_{1,2,4}$ with an equitable total coloring using 7 colors.

3 Final considerations

In this paper we proved that three infinite classes of tripartite complete non balanced graphs have equitable total coloring with $\Delta + 1$ colors using coloring matrices. Future works include determining the equitable total chromatic number of the graphs of the remaining cases $a = b < c < b^2$ and $a < b < c < b^2$, as well as other r-partite non balanced graphs.

References

[1] M. Behzad. Graphs and their chromatic numbers. PhD thesis, Michigan State University, 1965.

[2] J. C. Bermond. Nombre chromatique total du graphe r-parti complet (Chromatic number of the complete r-partite graph). *J. London Math. Soc.*, 9:279–285, 1974.

[3] S. Dantas, C. M. H. de Figueiredo, G. Mazzuoccolo, M. Preissmann, V. F. dos Santos, and D. Sasaki. On the equitable total chromatic number of cubic graphs. *Discrete Appl. Math.*, 209:84–91, 2016.

[4] H. L. Fu. Some results on equalized total coloring. *Congr. Numer.*, 102:111–119, 1994.

[5] A. Sànchez-Arroyo. Determining the total colouring number is NP-hard. *Discrete Math.*, 78:315–319, 1989.

[6] A. G. da Silva, S. Dantas and D. Sasaki. Equitable total chromatic number of $K_{r\times p}$ for p even. *Electron. Notes Theor. Comput. Sci.* (accepted), 2019.

[7] A. G. da Silva, S. Dantas and D. Sasaki. Equitable total coloring of complete r-partite p-balanced graphs. *Discrete Appl. Math.*, 2018 (`https://doi.org/10.1016/j.dam.2018.03.009`).

[8] D. F. Silva. Coloração de arestas distinta na vizinhança (Adjacent vertex distinguishing edge coloring). (2017) Universidade Tecnológica Federal do Paraná, Ponta Grossa, Brazil.

[9] I. R. Tiburcio. Coloração total semiforte de grafos tripartidos completos (Adjacent vertex distinguishing total coloring of complete tripartite graphs). (2016) Universidade Tecnológica Federal do Paraná, Ponta Grossa, Brazil.

[10] V. G. Vizing. Some unsolved problems in graph theory. *Russian Math. Surveys*, 23(6):125–141, 1968.

[11] W. F. Wang. Equitable total coloring of graphs with maximum degree 3. *Graphs Combin.*, 18:677–685, 2002.

Anderson G. da Silva
Department of Mathematics
Pontifical Catholic University of Rio de Janeiro
Rio de Janeiro, Brazil
andersongs@mat.puc-rio.br

Simone Dantas
IME
Fluminense Federal University
Niteroi, Brazil
sdantas@im.uff.br

Diana Sasaki
IME
Rio de Janeiro State University
Rio de Janeiro, Brazil
diana.sasaki@ime.uerj.br