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TESSELLABILITY and TESSELLABILITY COMPLETION of graphs with few P_4

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Abstract

A tessellation of a graph $G = (V, E)$ is a set of disjoint cliques that covers $V(G)$. A tessellation cover of G is a set of tessellations that covers $E(G)$. The tessellation cover number of G, denoted by $T(G)$, is the minimum size of a smallest tessellation cover of G. The t-TESSELLABILITY of G aims to decide whether $T(G) \leq t$. In this work, we present a polynomial time algorithm for t-TESSELLABILITY for quasi-threshold graphs. Next, we introduce the t-TESSELLABILITY COMPLETION of G, which aims to decide whether there is a tessellation cover $\mathcal{T}$ of G with t tessellations given a partial tessellation cover $\mathcal{T}'$ of G that must be part of $\mathcal{T}$. Finally, we compare the behavior of the computational complexity of t-TESSELLABILITY COMPLETION and k-EDGE PRECOLORING in some subclasses of graphs with few P_4, such as complete bipartite graphs, triangulated of complete graphs, and complete graphs.

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1 Introduction

Nowadays quantum computation receives a lot of attention from the scientific community. An important concept in this computational paradigm is the quantum walk. This concept is defined as a mathematical model of a particle's walk thought the edges of a graph. Recently, Portugal *et al.* [11] proposed the *Staggered Quantum Walk Model*, that includes Szegedy Model and an important part of Coined Model. The Staggered Model uses the concept of tessellations on graphs to generate the evolution operators that rules the corresponding quantum walk. Given a graph $G = (V, E)$, a *tessellation* is a set of disjoint cliques of G that covers $V(G)$. A set of tessellations $\mathcal{T} = \{C_1, \ldots, C_j\}$ is a *tessellation cover* when $\mathcal{T}$ covers $E(G)$. The size of a smallest tessellation cover in a graph G is denoted by $T(G)$. The t-TESSELLABILITY problem aims to decide whether a graph G has $T(G) \leq t$ [1].

Let $K(G)$ be the clique graph of G, i.e., the vertices of $K(G)$ are related to maximal cliques of G and two vertices are adjacent if the related maximal cliques are non-disjoint in G. Abreu *et al.* [1] proved that $T(G) \leq \min\{\chi(K(G)), \chi'(G)\}$, where $\chi(K(G))$ and $\chi'(G)$ denote the chromatic number and chromatic index of graphs $K(G)$ and G, respectively. They also showed NP-completeness proofs of the t-TESSELLABILITY problem for several graph classes. Moreover, they showed that this problem is polynomial-time solvable for *threshold graphs* $G = (C \cup S, E)$. A threshold graph G has $K(G)$ that is a complete graph, and $T(G) = \chi(K(G)) = |S| + 1$ (C is a largest maximum clique of G, $S = V \setminus C$ is a stable set of G).

Note that the computational complexity of t-TESSELLABILITY is still open for cographs, whereas it is polynomial time solvable for threshold graphs [1]. In this work, we present the tessellation cover number for *quasi-threshold* graphs and the polynomial-time algorithm for t-TESSELLABILITY for this graph class, in Sec. 2. We also present the definition of t-TESSELLABILITY COMPLETION relating it to k-EDGE PRECOL-

ORING, in Sec. 3. Finally, in Sec. 4, we present the concluding remarks.

2 Tessellability for quasi-threshold graphs

Note that the tessellation cover number of a disconnected graph is given by the maximum of the parameters of their connected components, i.e., if $G = G_1 \cup G_2$, then $T(G) = \max\{T(G_1), T(G_2)\}$.

A graph G is a *cograph*, *quasi-threshold*, *threshold* if G is $\{P_4\}$-free, $\{P_4, C_4\}$-free, and $\{P_4, C_4, 2K_2\}$-free, respectively [3]. Let G be a graph with a vertex u. The addition of a twin vertex v of u in G includes v in G with the same neighborhood of u, and there is an edge uv in $E(G)$ if v is a *true twin*. Otherwise, v is a *false twin*. Let G' be obtained from a graph G by adding a true twin v of $u \in V(G)$. The cliques containing u in G will become cliques in G' that also contain v. So, we can use the same cliques of tessellations that cover the edges incident to u in G to cover the edges incident to v in G'.

Lemma 1. *If G is a graph with a vertex u and G' is obtained from G by the addition of a true twin vertex v of u, then $T(G) = T(G')$.*

Quasi-threshold graphs can be recursively obtained by the following operations from a K_1: adding universal vertices, and; the union operation of two quasi-threshold graphs [12].

Theorem 1. *Let G be a quasi-threshold graph and G' be a quasi-threshold graph constructed by adding a universal vertex v to G. Hence, $T(G') = \sum_i T(C_i)$, where C_i is a connected component of G.*

Proof. The vertex v is universal, and we have two cases:

(I) Consider G connected. Therefore, G has a universal vertex u such that v and u are true twin vertices in G'. So, by Lemma 1, $T(G') = \sum_i T(C_i) = T(G)$.

(II) Consider G disconnected. Therefore each connected component C_i of G is a subgraph that is a quasi-threshold graph with a universal vertex u_i. So we can consider that vertices v and u_i are true twin vertices in each connected component C_i' (that is related to each C_i before adding

vertex v), thus the tessellation cover number in each connected component C_i' remains equal to $T(C_i)$. Since each connected component shares the vertex v in G', the cliques in each connected component share the vertex v. Then, to cover the incident edges of v we cannot use the same tessellations for each connected component, so $T(G') = \sum_i T(C_i)$. □

Every quasi-threshold graph is also a cograph, which have a *cotree*, that is a tree where the internal nodes represent operations of union or join, and the nodes that are leaves represent the vertices of the cograph [3]. We can construct the cotree of quasi-threshold graphs in such a way that every join operation occurs between a vertex and a quasi-threshold graph, and the cotree be binary where, w.l.o.g., the left side is a cotree and the right side is a leaf. Thereby, we are able to calculate the tessellation cover number of graphs of this class using its cotree by climbing this tree until the root. When the internal node of this cotree is a union operation, we know the value is the maximum among the parameters of the connected components. Otherwise, the internal node represents the join operation, so we use the result provided in Theorem 1. Note that the number of connected components in this situation is exactly the number of union operations in sequence until the next join operation in this cotree plus one. Therefore, we can calculate the tessellation cover number for quasi-threshold graphs in polynomial time.

3 Tessellability completion

We now introduce the t-TESSELLABILITY COMPLETION problem, which has a graph G and a partial tessellation cover $\mathcal{T}'$ of G as instance and aims to decide whether G has a tessellation cover $\mathcal{T}$ with t tessellations such that the tessellations of $\mathcal{T}'$ are part of $\mathcal{T}$. Note that in this work we consider that the cliques of tessellations of $\mathcal{T}'$ in $\mathcal{T}$ may expand, including new vertices. The k-EDGE PRECOLORING problem has a graph G and a partial edge coloring of G as instance and aims to decide whether G has an edge coloring with k colors such that the colors used in the partial edge coloring given by the instance are maintained.

The *Latin Square* problem has a $n \times n$ matrix M as instance and a set of elements of M with values in $\{1, ..., n\}$ and aims to decide whether it is possible to fill the remaining elements of M with values in $\{1, \ldots, n\}$ in such way that there is no repeated value in any line or column of M. Colbourn [4] proved that k-EDGE PRECOLORING is $\mathcal{NP}$-complete for complete bipartite graphs $K_{x,y}$ showing a polynomial transformation from the LATIN SQUARE. The idea of the proof is that the lines of M will be vertices of one stable set of the complete bipartite, the columns will be vertices of the other stable set. Moreover, the set of given values of M is related to the colors of the partial edge coloring of the complete bipartite graph such as if $M_{i,j} = \alpha$, then the edge ij of the complete bipartite graph receives the color α. It is not hard to verify that this complete bipartite graph has a n-edge coloring using that partial n-edge coloring if and only if the matrix can be filled with values in $\{1, \ldots, n\}$ given a set of elements of M already labeled. Figure 1(a) illustrates this construction.

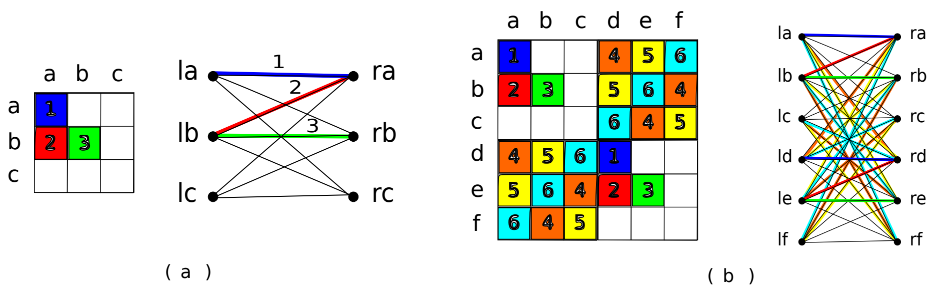

Figure 1: LATIN SQUARE and K-EDGE PRECOLORING on complete bipartite.

Bonomo et al. [2] proved that $(n-1)$-EDGE PRECOLORING of complete split (resp. complete) graphs is $\mathcal{NP}$-complete. The key idea of that proof is that given an instance I of LATIN SQUARE with a $n \times n$ matrix M, it is possible to create another instance I' with a $2n \times 2n$ matrix M' in such way I has a **YES** answer if and only if I' also has a **YES** answer. The matrix M' is obtained by adding two $n \times n$ elements in the top right and bottom left of M' with permutations of the values in $\{n+1, \ldots, 2n\}$ and by copying the values of M in the bottom right positions of M' (see Figure 1(b)). Moreover, given an instance I' of LATIN SQUARE with even n' we can construct a complete bipartite graph as described before. Next, we include all the missing edges of one clique (resp. two disjoint cliques) of size n' such that all these edges of the clique (resp. cliques) of even size n also appears in the partial edge coloring using colors in $\{n'+1,\dots,2n'-1\}$. Now, I' has a **YES** answer if and only if the $(2n'-1)$-EDGE PRECOLORING of this complete split graph of $2n'$ vertices (resp. complete graph of $2n'$ vertices) also is **YES**. Therefore, $(n-1)$-EDGE PRECOLORING is $\mathcal{NP}$-complete for complete bipartite graphs (a superclass of cographs), complete graphs, and complete split graphs.

In triangle-free graphs a tessellation cover behaves just like an edge coloring [1], the same holds for t-TESSELLABILITY COMPLETION and PARTIAL k-EDGE COLORING. Therefore, the computational complexity of k-EDGE PRECOLORING and k-TESSELLABILITY COMPLETION for triangle-free graphs are the same. Moreover, since k-EDGE PRECOLORING of Star graphs S_n is always **YES** for $k \geq \Delta(S_n) = n$ and **NO** otherwise, both k-EDGE PRECOLORING and k-TESSELLABILITY COMPLETION are in $\mathcal{P}$ for star graphs S_n. Marx [7] proved that k-EDGE PRECOLORING is $\mathcal{NP}$-complete for planar 3-regular bipartite graphs; bipartite outerplanar graphs; and bipartite series-parallel graphs. Thus, t-TESSELLABILITY COMPLETION is also hard for these graph classes.

Consider t-TESSELLABILITY COMPLETION for a complete graph G. If there is an edge without any available tessellation, then we know that the answer is **NO**. Otherwise, each edge has at least one available color and we obtain a tessellation cover of G by selecting one color for each unlabeled edge, and then covering all the endpoints of these unlabeled edges with a same color as a single clique in the tessellation related to this color, repeating this process for all colors.

The triangulated $TR(G)$ of a graph $G=(E,V)$ is obtained by adding to G, for each $e = uv \in E$, a vertex e_{uv} adjacent only to u and to v. Note that the $TR(K_n)$ of complete graphs K_n are split graphs. Let I

be an instance of $(n-1)$-EDGE PRECOLORING of a complete graph K_n with even n, which is $\mathcal{NP}$-complete [2]. Now, consider an instance I' of t-TESSELLABILITY COMPLETION of the graph $TR(K_n)$. Moreover, for each edge uv in the partial edge coloring of I we relate the triangle e_{uv}, u, v to a tessellation of the same label of the color of uv in the partial t-tessellation cover of $TR(K_n)$. Since $TR(K_n)$ has an induced star of size $n-1$, all the triangles with e-vertices incident to any vertex of the original clique K_n in $TR(K_n)$ need to be entirely covered by some tessellation. Note that each of these triangles of $TR(K_n)$ are related to an edge of K_n. Therefore, I has a **YES** answer if and only if I' also is **YES**.

Theorem 2. t-TESSELLABILITY COMPLETION *for Stars, Completes are in* $\mathcal{P}$ *whereas it is* $\mathcal{NP}$*-complete for Complete Bipartite and Triangulated Complete.*

Table 1: Computational Complexities Behaviors

	S_n	K_n	$K_{x,y}$	$TR(K_n)$	threshold	cograph
t-TESSELLATION COMPLETION	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{NP}-c$	$\mathcal{NP}-c$	Open	$\mathcal{NP}-c$
k-PARTIAL EDGE COLORABILITY	$\mathcal{P}$ [4]	$\mathcal{NP}$-c [4]	$\mathcal{NP}$-c [4]	$\mathcal{NP}$-c [4]	$\mathcal{NP}$-c [4]	$\mathcal{NP}$-c [4]
t-TESSELLABILITY	$\mathcal{P}$ [1]	$\mathcal{P}$ [1]	$\mathcal{P}$ [1]	$\mathcal{P}$ [1]	$\mathcal{P}$ [1]	Open
k- EDGE COLORABILITY	$\mathcal{P}$ [5]	$\mathcal{P}$ [5]	$\mathcal{P}$ [6]	$\mathcal{P}$ [5]	$\mathcal{P}$ [9]	Open

4 Final Remarks

In this work, we show that the tessellation cover number of quasi-threshold graphs is $T(G) = \sum_i T(C_i)$, where C_i is a connected component of G. Using these results we also prove that the t-TESSELLABILITY is polynomial-time solvable for quasi-threshold graphs.

There exist polynomial algorithms for k-EDGE COLORING restricted to complete graphs [5], complete bipartite graphs [6], complete split graphs [10], split indifference [8], and threshold graphs [9]. Similarly, in this work we have established polynomial time solutions for t-TESSELLABILITY COMPLETION restricted to star graphs and complete graphs. Moreover, we showed the hardness of t-TESSELLABILITY COMPLETION for complete bipartite graphs and triangulated complete graphs, a subclass of split graphs. Table 1 summarizes these results.

All the proofs for t-TESSELLABILITY $\mathcal{NP}$-complete also hold in the case of t-TESSELLABILITY COMPLETION. Therefore, it is only interesting to investigate graph classes for which t-TESSELLABILITY is in $\mathcal{P}$ or its computational complexity is open. We are close to establish a polynomial time algorithm for t-TESSELLABILITY COMPLETION restricted to line graphs of bipartite graphs, complete split graphs, and split indifference graphs, all graph classes for which we know t-TESSELLABILITY has linear time solution [1].

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