Matemática Contemporânea, Vol. 46, 38–46

t-Pebbling in k-connected graphs with a universal vertex

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Abstract

The t-pebbling number is the smallest integer m so that any initially distributed supply of m pebbles can place t pebbles on any target vertex via pebbling moves. The 1-pebbling number of diameter 2 graphs is well-studied. Here we investigate the t-pebbling number of diameter 2 graphs under the lens of connectivity.

1 Introduction

Graph pebbling models the transportation of consumable resources. It has an interesting history, with many challenging open problems, and with applications to zero-sum theory in abelian groups. Calculating pebbling numbers of graphs is a well known computationally difficult problem. See [4, 5] for more background.

A *configuration* C of pebbles on the vertices of a connected graph G is a function $C : V(G) \to \mathbb{N}$ (the nonnegative integers), so that $C(v)$ counts the number of pebbles placed on the vertex v. We write $|C|$ for the *size* $\sum_v C(v)$ of C; i.e. the number of pebbles in the configuration. A *pebbling step* from a vertex u to one of its neighbors v reduces $C(u)$ by two and increases $C(v)$ by one. Given a specified *root* vertex r we say

2000 AMS Subject Classification: 05C40, 05C75, 05C87 and 05C99.
Key Words and Phrases: graph pebbling, pebbling number, connectivity.

that C is *t-fold r-solvable* if some sequence of pebbling steps starting from C places t pebbles on r. We are concerned with determining $\pi_t(G, r)$, the minimum positive integer m such that every configuration of size m on the vertices of G is t-fold r-solvable. The *t-pebbling number* of G is defined to be $\pi_t(G) = \max_{r \in V(G)} \pi_t(G, r)$. We omit t when $t = 1$. Clearly, $\pi_t(G) \leq t\pi(G)$.

Pebbling number of diameter 2 graphs was solved and characterized by the following theorem. For the purpose of the present work, it is enough to know that a pyramidal graph has no *universal* vertex (a vertex adjacent to every other vertex) and has connectivity 2.

Theorem 1. *[2, 6] For a diameter 2 graph G with connectivity k and n vertices, $\pi(G) = n + 1$ if and only if $k = 1$ or G is pyramidal. Otherwise (i.e. $k = 2$ and G is not pyramidal, or $k \geq 3$), $\pi(G) = n$.*

In contrast, other than the following bound, little is known about the t-pebbling number of diameter 2 graphs.

Theorem 2. *[3] If G is a diameter 2 graph on n vertices then $\pi_t(G) \leq \pi(G) + 4t - 4$. Moreover, $\liminf_{t\to\infty} \pi_t(G)/t = 4$.*

The goal of the present paper is to determine the exact t-pebbling number of a large subfamily of diameter 2 graphs by considering their connectivity. Define $\mathcal{G}(n, k)$ to be the set of all k-connected graphs on n vertices having a universal vertex. Set $f_t(n, k) = n + 4t - k - 2$ and $h_t(n) = n + 2t - 2$. Notice that $h_t(n) \geq f_t(n, k)$ if and only if $k \geq 2t$. Define $p_t(n, k) = \max\{f_t(n, k), h_t(n)\}$. The main result is the following theorem which is proved in Section 3.

Theorem 3. *If $G \in \mathcal{G}(n, k)$ then $\pi_t(G) = p_t(n, k)$.*

We observe from our result that, for any fixed t, in the family of graphs with universal vertex, there are graphs whose t-pebbling number is much lower than the bound given by Theorem 2, and also that there are graphs reaching that bound: when $k \geq 2t$ we have $\pi_t(n, k) = (n+4t-4)-2(t-1)$; when $k < 2t$ $\pi_t(n, k) = (n + 4t - 4) - (k - 2)$.

It will be useful to take advantage of the following version of Menger's Theorem ([7], exercise 4.2.28).

Theorem 4. (Menger's Theorem) *[7] Let G be a k-connected graph and $S = \{v_1, \ldots, v_k\}$ be a multiset of vertices of G. For any $r \notin S$ there are k pairwise-internally-disjoint paths, one from each v_i to r.*

2 Technical Lemmas

We begin with a lemma that is used to prove lower bounds on the pebbling number of a graph by helping to show that certain configurations are unsolvable.

For a vertex v, define its *open neighborhood* $N(v)$ to be the set of vertices adjacent to v, and its *closed neighborhood* $N[v] = N(v) \cup \{v\}$. We say that a vertex y is a *junior sibling of* a vertex x (or, more simply, *junior to* x) if $N(y) \subseteq N[x]$, and that y is a *junior* if it is junior to some vertex x.

Lemma 5. (Junior Removal Lemma) *[1] Given the graph G with root r and t-fold r-solvable configuration C, suppose that $y \neq r$ is a junior with $C(y) = 0$. Then C (restricted to $G - y$) is t-fold r-solvable in $G - y$.*

Given a configuration C of pebbles, we say that a path $Q = (r, q_1, \ldots, q_j)$ with $j \geq 1$ is a *slide* from q_j to r if no q_i is empty and q_j has at least two pebbles.

A *potential move* is a pair of pebbles sitting on the same vertex. To say that C has j potential moves means that the j pairs are pairwise disjoint. For example, any configuration on 5 vertices with values $0, 1, 1, 2$, and 7 has 4 potential moves. The *potential* of C, $\mathsf{pot}(C)$, is the maximum j for which C has j potential moves; i.e., $pot(C) = \sum_{v \in V} \lfloor (C(v)/2) \rfloor$. Because every solution that requires a pebbling move uses a potential move, the following fact is evident.

Fact 6. *If C is a configuration with $C(r) + \mathsf{pot}(C) < t$ then C is not t-fold r-solvable.*

Basic counting yields the following lemma.

Lemma 7. (Potential Lemma) *Let G be a graph on n vertices. If C is a configuration on G of size $n+y$ ($y \geq 0$) having z zeros, then* $\mathsf{pot}(C) \geq \lceil \frac{y+z}{2} \rceil$.

A nice application of the Potential Lemma is the following result, which we will use repeatedly in the arguments that follow.

Lemma 8. (Slide Lemma) *Let r be a vertex of a k-connected graph G. Let C be a configuration on G of size $n+y$ ($y \geq 0$) with z zeros. If $\lceil \frac{y+3z}{2} \rceil \leq k$ then C is $\lceil \frac{y+z}{2} \rceil$-fold r-solvable.*

Proof. Set $p = \lceil \frac{y+z}{2} \rceil$. By Lemma 7 we can choose a set P of p potential moves. Note that the hypothesis implies that $p \leq k - z$. Delete all non-root zeros to obtain G'. Since G is k-connected, G' is p-connected. Thus Menger's Theorem 4 implies that there are p pair-wise disjoint slides in G' from P to r, which implies that C is p-fold r-solvable. □

3 Proof of Theorem 3

The proof will follow from Lemmas 9 and 10, below. Let u be a universal vertex of a graph $G \in \mathcal{G}(n,k)$. If C is a configuration of size $n+2t-3$ with $C(u)=0$ and every other vertex odd then $\mathsf{pot}(C) = t-1$, and so C is not t-fold u-solvable. Hence $\pi_t(G,u) \geq n+2t-2$. On the other hand, if $|C| \geq n+2t-2$ then $\mathsf{pot}(C) \geq t$ when u is empty, and $\mathsf{pot}(C) \geq t-1$ when u is not; either way C is t-fold u-solvable because u is universal. Thus $\pi_t(G,u) = n+2t-2$, which is at most $p_t(n,k)$ always.

3.1 Lower bound

Clearly, $\pi_t(G) \geq \pi_t(G,u) = h_t(n)$. Now let r be any non-universal vertex of G, and let s be a vertex at distance 2 from r. Let X be any (r,s)-cutset of size k (in particular, $u \in X$) and define the configuration $F_t(n,k)$

$k \backslash t$	1	2	3	4	5	6	7	8
2	0	4	8	12	16	20	24	28
3	0	3	7	11	15	19	23	27
4	0	2	6	10	14	18	22	26
5	0	2	5	9	13	17	21	26
6	0	2	4	8	12	16	20	24
7	0	2	4	7	11	15	19	23
8	0	2	4	6	10	14	18	22
9	0	2	4	6	9	13	17	21
10	0	2	4	6	8	12	16	20
11	0	2	4	6	8	11	15	19

Figure 1: The values m for which $\pi_t(G) = |V(G)| + m$.

by placing 0 on r and on every vertex in X, $4t-1$ on s, and 1 on each vertex of $V(G)-(X\cup\{r,s\})$; then $|F_t(n,k)| = (4t-1)+(n-k-2) = f_t(n,k)-1$.

Since the vertices of $X - \{u\}$ have 0 pebbles and all them are juniors to u, Lemma 5 states that if t pebbles can reach r then $2t$ pebbles can reach u. But, with exactly $2t-1$ potential moves in F, by Fact 6, we can place at most $2t-1$ pebbles on u. Therefore $\pi_t(G,r) \geq f_t(n,k)$, implying $\pi_t(G) \geq f_t(n,k)$.

We record these results as

Lemma 9. *For $G \in \mathcal{G}(n,k)$ we have $\pi_t(G) \geq p_t(n,k)$.*

3.2 Upper bound

We will prove that any configuration of size $f_t(n,k)$ when $k \leq 2t$, and of size $h_t(n)$ when $k \geq 2t$, is t-fold r-solvable for any $r \in V(G)$.

Lemma 10. *For $k \geq 2$, let $G \in \mathcal{G}(n,k)$ be a graph with a universal vertex u, and let r be any root vertex. Then $\pi_t(G,r) \leq p_t(n,k)$.*

Proof. First note that the lemma is true when $t = 1$. Indeed, in this case we have $k \geq 2t$, and so $p_t(n,k) = h_t(n) = n + 2t - 2 = n$. On the other hand, because no pyramidal graph has a universal vertex, we have from Theorem 1 that $\pi(G) = n$, hence $\pi(G, r) \leq n$.

In addition, the lemma holds for $k = 2$. Indeed, in this case we have $k \leq 2t$, and so $p_t(n, k) = f_t(n, k) = n + 4t - k - 2 = n - 4t - 4$. Also, we have by Theorem 2 that $\pi_t(G, r) \leq n + 4t - 4$.

Hence, we may assume that $t \geq 2$ and $k \geq 3$. Figure 1 shows the structure of this proof. As was noted above, the grey section has been proven before. We continue by proving the dashed-bordered, lower left section and diagonal circled entries together, and then the solid-bordered, upper right section by induction.

Base case.

We will simultaneously address the case $k = 2t-1$ (the circled entries), for which $|C| = f_t(n, k) = n + 2t - 1$, and the case $k \geq 2t$ (the dashed-bordered section), for which $|C| = h_t(n) = n + 2t - 2$, by writing $k \geq 2t - 1$ and considering a configuration of size $|C| = n + 2t - 2 + \phi$, where $\phi = 1$ if $2t - 1 = k$ and 0 otherwise. The natural idea we leverage here is repeating the argument that increased zeros force increased potential, which, combined with connectivity, yields either more solutions or more zeros.

Let $x \geq 0$ such that $k = 2t - 1 + x$. By Lemma 7, since we may assume that $C(r) = 0$ (otherwise apply induction on t), we have at least $\lceil (2t-2+1)/2 \rceil = t$ potential moves. Therefore, we have at least t solutions if there are at least t different slides from them to r.

Thus we consider the case in which there are at most $t-1$ slides; that is, from some of the vertices in which a potential move is sitting, say v, there is no path to r without an internal zero after considering the remaining $t - 1$ slides. Since G is k-connected, that implies that C has at least $k - (t - 1)$ zeros between v and r and so, because of r, C has at least $k - (t - 1) + 1 = t + 1 + x$ zeros.

Assume that there are exactly $z = t+1+j$ zeros, for some $j \geq x$. Then, by Lemma 7, C has at least

$$\left\lceil \frac{(2t-2)+(t+1+j)}{2} \right\rceil = t + \left\lceil \frac{t-1+j}{2} \right\rceil$$

potential moves. If there are at least $t - \left\lceil \frac{t-1+j}{2} \right\rceil$ slides from them to r, then we can use those slides for that many solutions. Then, the other $\left\lceil \frac{t-1+j}{2} \right\rceil$ solutions can be obtained from the remaining $2\left\lceil \frac{t-1+j}{2} \right\rceil$ potential moves, putting $2\left\lceil \frac{t-1+j}{2} \right\rceil$ pebbles on the universal vertex u and then $\left\lceil \frac{t-1+j}{2} \right\rceil$ on r.

Otherwise, there are at most $t - \left\lceil \frac{t-1+j}{2} \right\rceil - 1$ slides, from which we find, using $k = 2t - 1 + x$, at least

$$k - \left(t - \left\lceil \frac{t-1+j}{2} \right\rceil - 1\right) + 1 = t + x + \left\lceil \frac{t-1+j}{2} \right\rceil + 1$$

zeros. Clearly, this number cannot exceed the total number of zeros $z = t + 1 + j$; therefore $j \geq x + \left\lceil \frac{t-1+j}{2} \right\rceil \geq x + \frac{t-1+j}{2}$, and so $j \geq t - 1 + 2x$.

Let $j = t - 1 + 2x + i$ for some $i \geq 0$; then $z = t + 1 + j = t + 1 + t - 1 + 2x + i = 2t + 2x + i$. Applying Lemma 7 again, there are at least

$$\left\lceil \frac{(2t-2)+(2t+2x+i)}{2} \right\rceil = 2t + x - 1 + \lceil i/2 \rceil$$

potential moves.

If either $x \geq 1$ or $i \geq 1$, then we can move $2t$ pebbles to the universal vertex u, and then t to r.

Hence, we consider the case for which $x = i = 0$; i.e. $k = 2t - 1$, $z = 2t$, and $|C| = n + 2t - 1$ (because $\phi = 1$ in such a case). We let T be the star centered on u, having leaves r and the nonzero vertices of G. Clearly, T is a subgraph of G with $n + 2t - 1$ pebbles on it and with either $2 + (n - z)$ or $1 + (n - z)$ vertices, depending on whether u is empty or not. In either case $n(T) \leq 2 + n - z = 2 + n - 2t$. Therefore, since

$$\pi_t(T, r) = n(T) + 4t - 3 \leq (2 + n - 2t) + 4t - 3 = n + 2t - 1 = |C(T)|,$$

we see that C is r-solvable.

Induction step.

Finally, we consider the case $k < 2t - 1$ (the solid-bordered section); so $|C| = f_t(n, k) = n + 4t - k - 2$. Since $2(t - 1) = 2t - 1 - 1 \geq k$, we have $\pi_{t-1}(G, r) = f_{t-1}(n, k) = n + 4(t - 1) - k - 2 = n + 4t - k - 2 - 4 = |C| - 4$. Hence, if C has a solution of cost at most 4, we are done. Otherwise, there is at most one vertex v having two or more pebbles, and on such a vertex there are at most 3 pebbles. This implies the contradiction $|C| \leq 3 + (n - 2)$, which completes the proof. $\square$

In future work we intend to study k-connected diameter 2 graphs without a universal vertex, and use that work as a base step toward studying graphs of larger diameter.

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