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Structural Properties and Applications to Sandwich Problem of the two forbidden four-vertex graph classes

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Abstract

For a set $\mathcal{F}$ of graphs, an instance of the $\mathcal{F}$-FREE GRAPH SANDWICH PROBLEM is a pair (G_1, G_2) consisting of two graphs G_1 and G_2 with the same vertex set such that G_1 is a subgraph of G_2, and the task is to decide whether there exists an $\mathcal{F}$-free graph G containing G_1 and contained in G_2. Dantas et al. (2011, 2015) completely classify the complexity of the $\{F\}$-FREE GRAPH SANDWICH PROBLEM when F is a four-vertex subgraph. In this paper we study the complexity status of several two forbidden four-vertex subgraphs sandwich problems, including the trivially perfect, $\{claw, \overline{claw}\}$-free and $\{diamond, \overline{diamond}\}$-free graph classes.

1 Introduction

For a graph property Π, that is, Π is a set of graphs, the corresponding graph sandwich problem is the following.

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Π Graph Sandwich Problem

Instance: A pair (G_1, G_2) of two graphs such that G_1 and G_2 have the same vertex set, and G_1 is a subgraph of G_2.

Task: Decide whether there exists a graph G with $G_1 \subseteq G \subseteq G_2$ and $G \in \Pi$.

Let $\mathcal{F}$ be a set of graphs. A graph G is $\mathcal{F}$-free if no induced subgraph of G is in $\mathcal{F}$. Let $\overline{\mathcal{F}}$ be $\left\{\overline{F} : F \in \mathcal{F}\right\}$, where $\overline{F}$ is the complement of a graph F.

The Π-Graph Sandwich Problem was introduced by Golumbic and Shamir in 1993, attracting much attention because of many applications, and so several graph sandwich problems were considered for different graph classes, for instance: interval graph, unit interval graph, permutation graph and comparability graph sandwich problems are all NP-complete, while the split graph, threshold graph and cograph sandwich problems are in P (see [2]).

Dantas et al. [7, 8] completely classify the complexity of the $\{F\}$-free Graph Sandwich Problem when F is a four-vertex subgraph. Motivated by a question proposed by M. C. Golumbic about the complexity status of the graph sandwich problem of the well known *trivially perfect* graph class, we study the graph sandwich problems for $\mathcal{F}$-free graphs, where $\mathcal{F}$ is a set of two non-isomorphic graphs of order four.

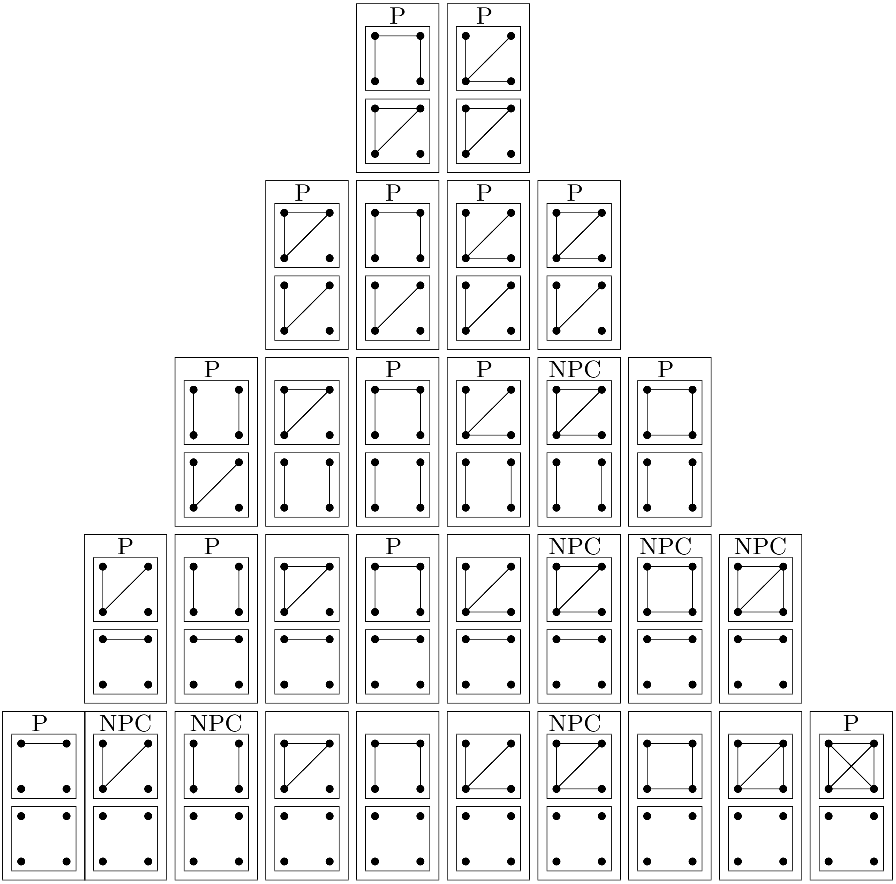


Figure 1: All 30 pairs of non-isomorphic graphs of order four up to complementation, together with the status of the corresponding sandwich decision problem, where "P" means "*polynomial time solvable*" and "NPC" means "*NP-complete*", corresponding result within this paper.

Our goal it to study the complexity of the $\mathcal{F}$-FREE GRAPH SANDWICH PROBLEM for sets $\mathcal{F}$ containing two non-isomorphic graphs of order four. It is well-known that it suffices to consider the sets $\mathcal{F}$ up to complementation. Note that P_4 is the only self-complementary graph of order four. Hence, up to complementation, there are five sets $\mathcal{F}$ that contain P_4.

There are 10 sets $\mathcal{F}$ containing two non-isomorphic graphs with less than four edges and, up to complementation, there are 15 sets $\mathcal{F}$ containing one graph with less than four edges and one graph with more than four edges.

Figure 1 represents the 30 cases of two non-isomorphic graphs of order four up to complementation, and summarizes our contributions.

This work is divided in four sections. Section 2 contains an interesting structural characterization of a two forbidden four-vertex subgraph class, and a (sketch) proof. It also contains a key lemma used in the proof of our complexity result, which we present in Section 4. In the final section, we conclude with a comment on the open cases.

2 Preliminaries

In this section, we define some concepts to the benefice of the reader, including comments of well-known properties of them. For the first definition see e.g. [3, 6].

A graph G is called *chain graph* if it is $\{2K_2, C_3, C_5\}$-free.

The following three definitions belongs to the Modular Decomposition Theory (see e.g. [4, 9]).

Let $G = (V(G), E(G))$ be a graph. A subset $H \subseteq V(G)$ is a *module set* in G if for every $v \notin H$ either v is adjacent to each vertex of H or v is non-adjacent to any vertex of H. In addition, if the module set H has at least two elements and is not the vertex set $V(G)$, then it is called a *homogeneous set* in G. A graph G is called *prime* if it contains no homogeneous sets.

Given a connected and co-connected graph G, it is well-known that the maximal homogeneous sets are pairwise disjoint. In this case, we associate the quotient graph G^*, called the *characteristic graph* of G, that arises by contracting every maximal homogeneous set to a vertex. *The Modular*

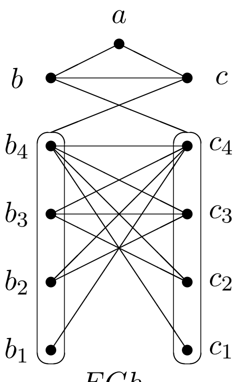
Decomposition Theorem state that G^* is prime.

Finally, we introduce the following classes of graphs which appears in [1].

A graph G is a *thin spider* if it is partitionable into a clique C and an independent set S with $|S| \leq |C| \leq |S| + 1$ such that the edges between C and S are a matching and at most one vertex in C is uncovered by the matching.

A graph G is an *enhanced bipartite chain graph* if it is partitionable into a chain graph with independent sets, say B and C, and three additional vertices a, b and c (a and c optional) such that $N_G(a) = \{b, c\}$, $N_G(b) = B \cup \{a, c\}$ and $N_G(c) = C \cup \{a, b\}$.

Figure 2 shows an example of an enhanced bipartite chain graph, which is a key ingredient in the proof of our main result.



ECh_4

Figure 2: The enhanced bipartite chain graph ECh_4 with triangle T : a, b, c. The vertex b is adjacent to all vertices in the independent set $C = \{c_1, c_2, c_3, c_4\}$, and the vertex c is adjacent to all vertices in the independent set $B = \{b_1, b_2, b_3, b_4\}$.

3 Structural Results

In order to prove the complexity results of the two forbidden four-vertex subgraph sandwich problem, we study structural properties of several such classes. Here, we prove a structural result of the $\{2K_2, paw\}$-free class and the $\{diamond, 2K_2\}$-free class, useful in the main result of Section 4. Let $G = (V(G), E(G))$ be a simple graph.

Lemma 1. *If $\delta \geq 1$, then G is $\{2K_2, paw\}$-free if and only if G is connected and satisfies one of the following properties: (i) G is a complete multipartite graph; or (ii) the homogeneous sets of G are independent and the characteristic graph G^* of G is isomorphic to C_5; or (iii) G is a chain graph.*

Before presenting the proof of Lemma 1, we characterize the $\{2K_2, C_3\}$-free prime graphs.

Proposition 1. *A prime graph is $\{2K_2, C_3\}$-free if and only if it is either isomorphic to C_5 or a (prime) chain graph.*

Now, we characterize the $\{2K_2, paw\}$-free graphs by using the Modular Decomposition Theorem.

Proof of Lemma 1: Clearly G is connected, because otherwise, since $\delta \geq 1$, we induce $2K_2$. Suppose that G is a prime graph. By Olariu's Lemma [5], it is clear that G is C_3-free. Thus, G is a $\{2K_2, C_3\}$-free prime graph. By Proposition 1, we are done.

Now, we can assume that G is not prime. If $\overline{G}$ is disconnected and remarking that $\overline{G}$ is $\overline{paw}$-free, then each one of its components is a complete graph, i.e. P_3-free. In other words, G is a complete multipartite graph. So, we assume that G is connected and $\overline{G}$ is connected. Similar to the prime case, Olariu's Lemma [5] implies that G is $\{2K_2, C_3\}$-free. In particular, the characteristic graph G^* is a $\{2K_2, C_3\}$-free prime graph. Hence, by Proposition 1, G^* is C_5 or a (prime) chain graph. We claim that every homogeneous set H of G is an independent set. Set A the set of vertices adjacent to each vertex of H. We proceed by contradiction. Let x an y two adjacent vertices of H. For any element $v \in A$, the induced subgraph $G[x, y, v] \cong C_3$, which is a contradiction.

Applying Proposition 1 to G^*, if G^* is a prime chain graph, then G is a chain graph, since every homogeneous set is independent. Therefore, in any case, we are done.

Finally, to prove the converse, we analyze each possibility: if (i) then G is $\overline{P_3}$-free, in particular G is $\{2K_2, paw\}$-free; if (ii) then G is $\{2K_2, C_3\}$-free, since C_5 is also. In particular, G is $\{2K_2, paw\}$-free; if (iii), by definition, G is $\{2K_2, C_3, C_5\}$-free, in particular G is $\{2K_2, paw\}$-free. □

In order to prove the main result of Section 4, we also state the following result.

Lemma 2. *If G is a $\{diamond, 2K_2\}$-free graph with $\delta \geq 1$, then G is connected and satisfies one of the following conditions: (i) G is complete bipartite, (ii) or G arises adding all possible edges between the center of some stars, (iii) or the homogeneous sets of G are independent sets, and the characteristic graph G^* of G either has at most 9 vertices, or is a thin spider, or is an enhanced bipartite chain graph.*

4 A Complexity Result

In this section, we present one of the complexity results obtained in this work. In the context of Π Graph Sandwich Problem, we say that an edge is forced (resp. forbidden) if it belongs to E^1 (resp. $E^3 := \overline{E^2}$).

Theorem 1. *The $\{diamond, 2K_2\}$-free Graph Sandwich Problem is NP-complete.*

In order to prove this theorem, we refer to the auxiliary graph ECh_4 given in Figure 2. Note that the graph ECh_4 is prime.

(Sketch) Proof of Theorem 1: Let Π be the class of chain graphs. We describe a polynomial reduction from Π Graph Sandwich Problem to

$\{diamond, 2K_2\}$-FREE GRAPH SANDWICH PROBLEM. Let $I = (V, E^1, E^3)$ be an instance of the Π GRAPH SANDWICH PROBLEM, where E^1 is a matching. We associate the instance $J = (U, F^1, F^3)$ as follows:
-Let U be the union of the set V with the set of new vertices of ECh_4.
-Let F^1 be the union of the new forced edges of ECh_4 with E^1.
-Let F^3 be the union of the new forbidden edges of ECh_4, with the edges av for any $v \in V$, and the previous edges of E^3.
First we prove that if there exists a chain sandwich graph solution G for the instance I, then there exists a $\{diamond, 2K_2\}$-free sandwich graph solution H for the instance J.

In fact, suppose that $G = (L, R, E)$ is a chain sandwich graph solution for the instance I. Graph H arises by the disjoint union of ECh_4 with the graph G, adding all the possible edges between C with R, the edges br with $r \in R$, and the edges $c\ell$ with $\ell \in L$. We see that H is an enhanced bipartite chain graph, in particular, H is as desired.
Now, if there exists a $\{diamond, 2K_2\}$-free sandwich graph solution H for the instance J, then there exists a sandwich graph solution G for the instance I.

In fact, suppose that H is a $\{diamond, 2K_2\}$-free sandwich graph solution for the instance J. We show that H is necessarily an enhanced bipartite chain graph with triangle T. Thus, the induced graph $H[U \setminus V(T)]$ is a bipartite chain graph, in particular $G = H[U \setminus V(ECh_4)]$ satisfies our requirement. Note that $\delta(H) \geq 1$. Since H is not complete bipartite, by Lemma 2, H satisfies the condition (ii) or (iii). Since H contains ECh_4, we show that H does not satisfy the condition (ii). Suppose that H satisfies the condition (ii) and let C be the set of centers and I the set of leaves of the stars. Clearly $V(T) \subseteq C$ but, since $N_H(a) = \{b, c\}$, then $C = V(T)$. Hence, $I = N_H(b) \cup N_H(c)$, but this contradicts the fact that I is independent. So, H satisfies the condition (iii). Let $D = ECh_4 - T$, so D is a chain prime graph. Suppose that H^* is a thin spider. Since D is bipartite, then there only exist at most two vertices of H^* in its clique, intersecting the set $V(D)$. By definition of thin spider graph, the graph

$H^*[X]$ (where X are vertices of H^* that intersect $V(D)$) is K_2 or P_3 or P_4. In any case, by the unique representation of the Mahadev and Peled Theorem (see [6]), we have a contradiction. Note that, by Lemma 2 (*iii*), $V(D)$ is not a vertex of H^*. Now, we prove that $|V(H^*)| \geq 11$. Note that, if each vertex of ECh_4 is a maximal modular set of H, then we are done. By contradiction, suppose that a maximal modular set M of H contains two different vertices of ECh_4. We have that $V(ECh_4)$ is contained in M. However, by the Lemma 2 (*iii*), M is independent, a contradiction. Hence, H^* is an enhanced bipartite chain graph with triangle $T^* : a^*b^*c^*$. Finally, we show that $a^* = \{a\}, b^* = \{b\}$ and $c^* = \{c\}$. Note that this fact completes the proof. Clearly, $a \in a^*$, $b \in b^*$ and $c \in c^*$. But $N(a) = \{b, c\}$, implying that $b^* = \{b\}$ and $c^* = \{c\}$. By contradiction, suppose that there exists a vertex $x \neq a$ with $x \in a^*$. Note that $x \notin V(ECh_4)$. So, $x \in V$ and $N_H(x) = \{b, c\}$, which contradicts the assumption that E^1 is a matching.

Therefore, since the Π GRAPH SANDWICH PROBLEM restricted to forced matching is NP-complete (see [6]), we are done. □

5 Conclusion

Analysing the complexity of problems in Figure 1, there is evidence that most of the corresponding problems are hard.

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