Matemática Contemporânea, Vol. 45, 77–85

On equitable total colouring of Loupekine Snarks and their products

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Abstract

In this work, we study total colourings and equitable total colourings of snarks. We obtain an equitable 4-total colouring for an infinite family of Loupekine Snarks. Also, we extend this colouring to equitable 4-total colourings for infinite families of dot products of Loupekine Snarks with Flower and Blanuša Snarks.

1 Introduction

Let G be a simple graph. A *k-total colouring* of G is an assignment of k colours to its vertices and edges such that two adjacent or incident elements have different colours. The *total chromatic number* of G - $\chi''(G)$ - is the smallest k for which G admits a k-total colouring. The Total Colouring Conjecture [1, 12] states that every simple graph admits a total colouring using at least $\Delta+1$ and at most $\Delta+2$ colours. It was proved for *cubic graphs* in 1971 independently by Rosenfeld [8] and Vijayaditya [11]. Cubic graphs with $\chi'' = 4$ are said to be *Type 1* while cubic graphs with $\chi'' = 5$ are said to be *Type 2*.

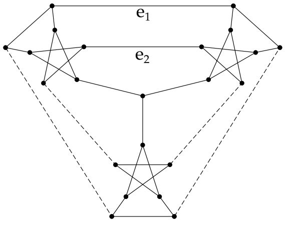
2000 AMS Subject Classification: 68R10, 68R15.
Key Words and Phrases: total colouring, equitable total colouring, snarks.
Partially supported by CNPq, CAPES, and FAPERJ.

An *equitable k-total colouring* of a graph is a k-total colouring of its elements such that the cardinalities of any two colour classes differ by at most 1. The *equitable total chromatic number* of G - $\chi''_e(G)$ - is the smallest k for which G admits an equitable k-total colouring. Wang [13] proposed the Equitable Total Colouring Conjecture, which states that every simple graph admits an equitable total colouring with at least $\Delta + 1$ and most $\Delta + 2$ colours and proved that it holds for cubic graphs. Dantas et al. [5] proved that the problem of deciding whether the equitable total chromatic number of a cubic graph is 4 or 5 is NP-complete.

Here we focus on equitable total colouring of *snarks*, that are cyclically 4-edge-connected cubic graphs with chromatic index 4. Their importance arise from the fact that many conjectures on Graph Theory would have snarks as minimal counterexamples, as shown recently by Brinkmann et al. [2]. In 2003, Cavicchioli et al. [4] proposed the question of finding the smallest Type 2 snark with girth at least 5. In the same paper, they verified that all snarks with girth at least 5 and fewer than 30 vertices are Type 1. In 2011, Brinkmann et al. [2] extended this search and verified that all snarks with such girth and fewer than 38 vertices are Type 1. In 2011, Campos, Dantas and de Mello [3] proved that all members of the infinite families of Flower and Goldberg Snarks are Type 1, and all these colourings are equitable. Sasaki et al. [10] proved that both Blanuša and Loupekine families are Type 1, but they could not determine equitable total colourings using 4 colours. A related question was proposed by Sasaki [9], who questioned if there exists a Type 1 snark with girth at least 5 that does not admit an equitable 4-total colouring. In this work, we investigate equitable total colourings of snarks by determining an equitable 4-total colouring for an infinite family of Loupekine Snarks and extending this colouring to equitable 4-total colourings for infinite families of dot products of these Loupekine Snarks with Flower and Blanuša Snarks.

2 Loupekine Snarks

Loupekine families of snarks were defined by Isaacs [7]. The first Loupekine Snark L_0 is presented on the left of Figure 1. The next members of the subfamily that will be considered here are obtained from L_0 by deleting the dashed edges and by adding a copy of the block B depicted on the right of Figure 1. The k-th member of the family is depicted in Figure 2.



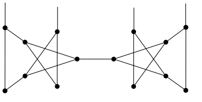


Figure 1: The snark L_0 and the block B.

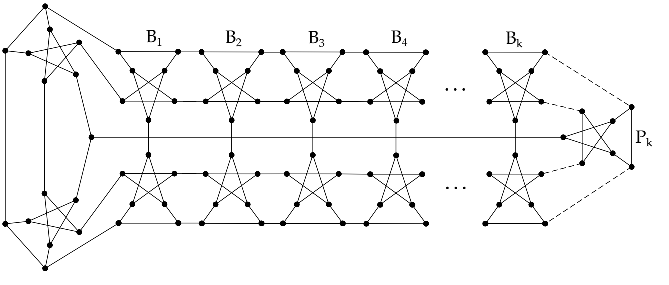


Figure 2: The snark L_k, obtained from L_0 by adding k blocks B.

In 2014, Sasaki et al. [10] proved that all members of the Loupekine families are Type 1, but these colourings are not equitable.

Theorem 1. *All members of Loupekine subfamily depicted in Figure 3 have equitable total chromatic number equal to 4.*

Sketch of the proof. The proof is by construction, based on the construction of the family itself, and four different equitable 4-total colourings of the block. Figure 3 shows the equitable 4-total colourings obtained by Theorem 1 for the second and third members of the family. For the benefit of the reader, we represent only colours for edges; vertices colours may be deducted from these. ■

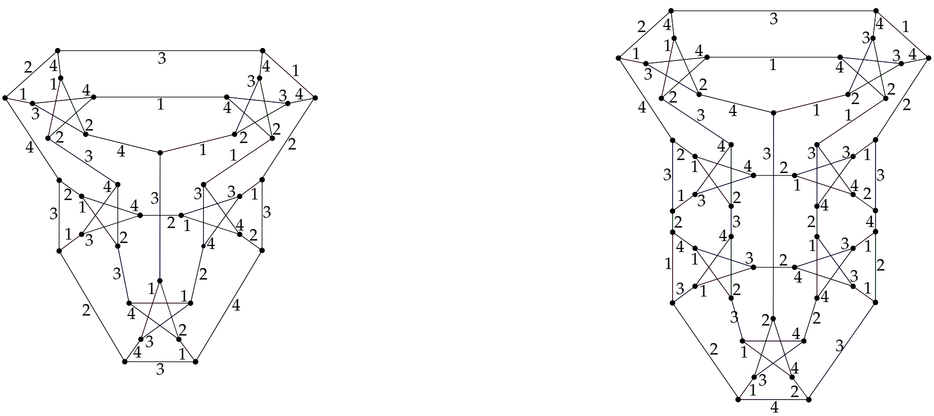


Figure 3: Equitable 4-total colourings for L_1 and L_2.

3 Dot Products of Families of Snarks

A *dot product* of two cubic graphs is a cubic graph obtained by deleting two nonadjacent edges of one graph and two adjacent vertices of the other graph, and then connecting the degree 2 vertices. A dot product of two snarks is a snark [6]. In this section, we present equitable total colourings for dot products of Loupekine Snarks considered in the previous section with snarks from other families. In all products, we remove edges e_1 and e_2 of each Loupekine Snark.

Flower Snarks The first member of the Flower family of snarks is presented on the left of Figure 4. The next elements in the family are obtained by deleting the dashed edges and adding the block depicted on the right of Figure 4, connecting the degree 2 vertices.

An equitable 4-total colouring for all members of this family was determined by Campos et al. [3] in 2011.

Figure 4: The graph F_3 and the block FL.

Theorem 2. *All graphs obtained by the dot product of Loupekine Snark L_i, $i \geq 0$, using edges e_1 and e_2 and all Flower Snarks have equitable total chromatic number 4.*

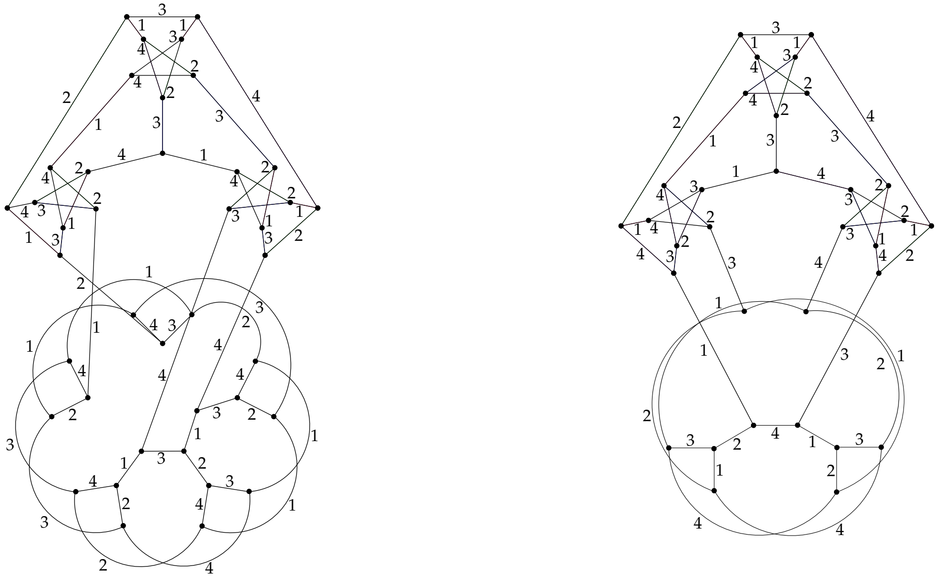


Figure 5: Two examples of equitable 4-total colourings for the dot products of Loupekine Snarks L_i, $i \geq 0$, using edges e_1 and e_2, with Flower Snarks.

Sketch of the proof. We construct equitable 4-total colourings for dot products of the first Flower Snark with the Loupekine Snarks that are compatible with the total colourings obtained by Campos et al. [3] (that is, their configurations are equivalent around the dashed edges) so they can be extended to the dot products of Loupekine Snarks with larger members of the Flower Snark family. ∎

Blanuša Snarks The two families of Blanuša Snarks were defined by Watkins [14] and its elements are obtained by dot products of Petersen graphs. The first two members of the Blanuša families are depicted in Figure 6, from which infinite families derive.

Figure 6: First and second Blanuša Snarks.

Sasaki et al. [10] determined equitable 4-total colourings for all members of Blanuša families. Since edges e and f are used in the construction of the equitable 4-total colourings obtained by Sasaki et al. [10], we considered only products which preserve these edges.

Theorem 3. *All graphs obtained by the dot product of Loupekine Snark L_i, $i \geq 0$, using edges e_1 and e_2, and all Blanuša Snarks, preserving edges e and f, have total chromatic number 4.*

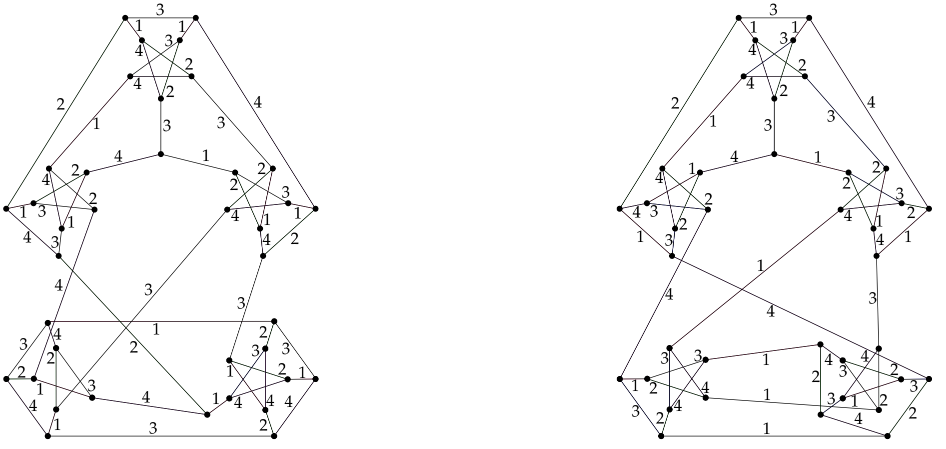


Figure 7: Two examples of equitable 4-total colourings for products of Loupekine Snarks L_i, $i \geq 0$, using edges e_1 and e_2, with Blanuša Snarks, preserving edges e and f.

In Theorem 3 we could not obtain equitable 4-total colourings for all cases. The exceptions might give us an affirmative answer to Sasaki's [9] question regarding the existence of a Type 1 snark with girth greater than 4 that does not admit an equitable 4-total colouring.

4 Concluding remarks

In this work, we contributed to the research about total colourings of snarks with girth greater than 4 by determining the total chromatic number and the equitable total chromatic number of members of infinite families of snarks. As immediate future work, we will investigate the equitable total colouring of all Loupekine Snarks and their dot products.

References

[1] Behzad, M., Chartrand, G., Cooper Jr., J. K. The colour numbers of complete graphs. J. London Math. Soc., 42, 226-228, 1967.

[2] Brinkmann, G., Goedgebeur, J., Hgglund, J., Markstrm, K. Generation and Properties of Snarks, J. Comb. Theory Ser. B 103, 468-488, 2013.

[3] Campos, C. N., Dantas, S., De Mello, C. P. The Total-chromatic Number of Some Families of Snarks, Discrete Math. 311, 984-988, 2011.

[4] Cavicchioli, A., Murgolo, T.E., Ruini, B. et al. Special Classes of Snarks. Acta Appl. Math., 76: 57, 2003.

[5] Dantas, S., De Figueiredo, C. M. H., Mazzuoccolo, G., Preissmann, M., Dos Santos, V. F., Sasaki, D. On the equitable total chromatic number of cubic graphs. Discrete Appl. Math., 209, 84-91, 2016.

[6] Isaacs, R. Infinite families of nontrivial trivalent graphs which are not Tait colourable. Amer. Math. Monthly, v. 82, n. 3, 221239, 1975.

[7] Isaacs, R. Loupekhine's Snarks: A Bi-Family of Non-Tait-Colorable Graphs (Technical Report 263). Dpt. of Math. Sci., The Johns Hopkins University, 1976.

[8] Rosenfeld, M. On the total coloring of certain graphs. Israel J. Math. 9, 396-402, 1971.

[9] Sasaki, D. Sobre Colorao Total de Grafos Cbicos. PhD thesis, Programa de Engenharia de Sistemas e Computao COPPE/UFRJ, 2013.

[10] Sasaki, D., Dantas, S., De Figueiredo, C. M. H., Preissmann, M. The hunting of a snark with total chromatic number 5. Discrete Appl. Math., 164, 470-481, 2014.

[11] Vijayaditya, N. On total chromatic number of a graph. J. London Math. Soc. 3, 405-408, 1971.

[12] Vizing, V. On an estimate of the chromatic class of a p-graph. Diskret. Analiz No. 3, 25-30, 1964.

[13] Wang, W. F. Equitable total colouring of graphs with maximum degree 3, Graphs Combin. 18, 677-685, 2002.

[14] Watkins, J. J. On The Construction of Snarks, Ars Combin. 16, 111-124, 1983.

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