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Critical ideals of digraphs

Carlos A. Alfaro **Carlos E. Valencia**
Adrián Vázquez-Ávila

Abstract

Critical ideals generalize the critical group and the characteristic polynomials of the adjacency and Laplacian matrices of a graph. We aim at discussing some numerical experiments providing information about the critical ideals of small digraphs. We also show the complete characterization of the digraphs with at most one trivial critical ideal. Which implies a characterization of the digraphs with exactly one invariant factor equal to one.

1 Introduction

Given a digraph $D = (V, A)$ and a set of indeterminates $X_D = \{x_u : u \in V(D)\}$, the *generalized Laplacian matrix* $L(D, X_D)$ of D is the matrix with rows and columns indexed by the vertices of D given by

$$L(D, X_D)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where m_{uv} is the number of arcs going from u to v.

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The *Laplacian matrix* $L(D)$ of D is the evaluation of $L(D, X_D)$ at $X = deg^+(D)$, where $deg^+(D)$ is the out-degree vector of D. By considering $L(D)$ as a linear map $L(D) : \mathbb{Z}^V \to \mathbb{Z}^V$, the *cokernel* of $L(D)$ is the quotient module $\mathbb{Z}^V / \mathrm{Im}\, L(D)$. The torsion part of this module is the *critical group* $K(D)$ of D. It is known [7, Theorem 1.4] that the critical group of a digraph D can be described as: $K(D) \cong \mathbb{Z}_{f_1} \oplus \mathbb{Z}_{f_2} \oplus \cdots \oplus \mathbb{Z}_{f_r}$, where $f_1, f_2, ..., f_r$ are positive integers such that $f_i \mid f_j$ for all $i \leq j$. These integers are called *invariant factors* of the Laplacian matrix of D. The characterization of the family $\mathcal{G}_i$ of simple connected graphs with i invariant factors equal to 1 has been of great interest, see [1, 4, 9, 10].

Now we will focus on the critical ideals, which were defined in [5], and studied in [1, 3, 5] as a generalization of the characteristic polynomial and the critical group.

Definition 1.1. *For all $1 \leq i \leq |V|$, the i-th critical ideal of D is the determinantal ideal given by*

$$I_i(D, X_D) = \langle \{\det(m) : m \text{ is an } i \times i \text{ submatrix of } L(D, X_D)\} \rangle \subseteq \mathbb{Z}[X_D].$$

We say that a critical ideal is *trivial* when it is equal to $\langle 1 \rangle$. The following result serves as a bridge between the critical groups and critical ideals.

Theorem 1.2. *[5] If $\deg^+(D) = (\deg^+_D(v_1), ..., \deg^+_D(v_n))$ is the out–degree vector of D, and $f_1 \mid \cdots \mid f_{n-1}$ are the invariant factors of $K(D)$, then*

$$I_i(D, \deg^+(D)) = \left\langle \prod_{j=1}^{i} f_j \right\rangle \text{ for all } 1 \leq i \leq n-1.$$

Thus if the critical ideal $I_i(D, X_D)$ is trivial, then f_i is equal to 1. Equivalently, if f_i is not equal to 1, then the critical ideal $I_i(D, X_D)$ is not trivial.

Definition 1.3. *The algebraic co-rank $\gamma(D)$ of a digraph D is the number of trivial critical ideals of D.*

The counterpart to the algebraic co-rank, in critical ideals, is the number of invariant factors equal to 1, denoted by $f_1(D)$.

Most of the basic properties of the critical ideals were obtained in [5]. For instance, it was proven that if H is an induced subdigraph of G, then $I_i(H, X_H) \subseteq I_i(G, X_G)$ for all $i \leq |V(H)|$. Thus $\gamma(H) \leq \gamma(G)$. The algebraic co-rank allows to define the following graph families:

Definition 1.4.

$$\overrightarrow{\Gamma_{\leq i}} = \{D : D \textit{ is a connected digraph with } \gamma(D) \leq i\}$$

$$\mathcal{D}_i = \{D : D \textit{ is a connected digraph with } f_1(D) = i\}$$

We have that $\overrightarrow{\Gamma_{\leq i}}$ is closed under induced subdigraphs. Moreover, $\mathcal{D}_i \subseteq \overrightarrow{\Gamma_{\leq i}}$ for all $i \geq 0$. Therefore, after an analysis of the *i-th* invariant factor of the Laplacian matrix of the graphs in $\overrightarrow{\Gamma_{\leq i}}$ the characterization of $\mathcal{D}_i$ can be obtained.

In what follows we will give the characterization of the digraphs with at most 1 trivial critical ideals. Firstly, we will study in Section 2 the minimal forbidden digraphs for $\overrightarrow{\Gamma_{\leq k}}$. These digraphs have been playing a crucial role in the understating the critical ideals and their classification. And this allow us to give the characterization of the digraphs with 1 trivial critical ideal. Finally, in Section 3, we will give the complete characterization of $\overrightarrow{\Gamma_{\leq 1}}$ and $\mathcal{D}_1$. The full details and the missing proofs of some results appear in the complete version [2] of this extended abstract.

2 γ-critical digraphs

The major advantage of the critical ideals over the critical group is that critical ideals behave well under induced subdigraph property. This property allow us to define the following concepts.

Definition 2.1. *A digraph D is forbidden for $\overrightarrow{\Gamma_{\leq k}}$ if and only if $\gamma(D) \geq k+1$.*

Let $\mathbf{Forb}(\overrightarrow{\Gamma_{\leq k}})$ denote the set of minimal (under induced subdigraphs property) forbidden digraphs for $\overrightarrow{\Gamma_{\leq k}}$. Given a family $\mathcal{F}$ of digraphs, a digraph D is called $\mathcal{F}$-free if no induced subdigraph of D is isomorphic to a member of $\mathcal{F}$. Thus $D \in \overrightarrow{\Gamma_{\leq k}}$ if and only if D is $\mathbf{Forb}(\overrightarrow{\Gamma_{\leq k}})$-free. And equivalently, D belongs to $\overrightarrow{\Gamma_{\geq k+1}}$ if and only if D contains a digraph of $\mathbf{Forb}(\overrightarrow{\Gamma_{\leq k}})$ as an induced subgraph. Hence characterizing $\mathbf{Forb}(\overrightarrow{\Gamma_{\leq k}})$ leads to a characterization of $\overrightarrow{\Gamma_{\leq k}}$.

As k grows the combinatorial properties make difficult to completely describe $\mathbf{Forb}(\Gamma_{\leq k})$. An alternative technique of computing the elements of $\mathbf{Forb}(\overrightarrow{\Gamma_{\leq k}})$ is by means of the following definition. A digraph D is called γ-*critical* if $\gamma(D \setminus v) < \gamma(D)$ for all $v \in V(D)$. That is, $D \in \mathbf{Forb}(\overrightarrow{\Gamma_{\leq k}})$ if and only if $\gamma(D) \geq k+1$ and $\gamma(D-v) \leq k$, for all $v \in V(D)$. We implemented this criterion in the software Macaulay2 [6] and Nauty [8], Table 1 shows the number of γ-critical digraphs with at most 6 vertices.

$k\backslash n$	2	3	4	5	6
0	2				
1		7	10		
2			61	1308	414
3				1183	542437
4					38229

Table 1: The number of γ-critical digraphs with n vertices and algebraic co-rank $k+1$.

We have that the directed path $\overrightarrow{P_2}$ with 2 vertices and the directed cycle $\overrightarrow{C_2}$ with 2 vertices are the minimal forbidden digraphs for $\overrightarrow{\Gamma_{\leq 0}}$. Since any other connected digraph with more than 2 vertices contains $\overrightarrow{P_2}$ or $\overrightarrow{C_2}$ as induced digraphs, we have that $\mathbf{Forb}(\overrightarrow{\Gamma_{\leq 0}}) = \{\overrightarrow{P_2}, \overrightarrow{C_2}\}$. Therefore, the digraph P_1 consisting of an isolated vertex is the only connected digraph such that is $(\overrightarrow{P_2}, \overrightarrow{C_2})$-free, that is, $\overrightarrow{\Gamma_{\leq 0}} = \{T_1\}$.

Figure 1: The γ-critical digraphs with 2 vertices and algebraic co-rank equal to 1.

Using the data computation, we found 7 digraphs with 3 vertices and 10 digraphs with 4 vertices with algebraic co-rank equal to 2 that are γ-critical, which are shown in Figure 2. This will be used to completely characterize $\overrightarrow{\Gamma_{\leq 1}}$.

3 Digraphs with one trivial critical ideal

The main goal of this section is to give the characterization of the digraphs with at most 1 trivial critical ideal. After, using the fact that $\mathcal{D}_1 \subseteq \overrightarrow{\Gamma_{\leq 1}}$, we will give the classification of the digraphs whose critical group has one invariant factors equal to 1. As in the previous case, the characterization of $\overrightarrow{\Gamma_{\leq 1}}$ relies heavely in the fact that $\overrightarrow{\Gamma_{\leq 1}}$ is closed under induced subdigraphs and that we have previously computed the γ-critical digraphs with algebraic co-rank equal to 2.

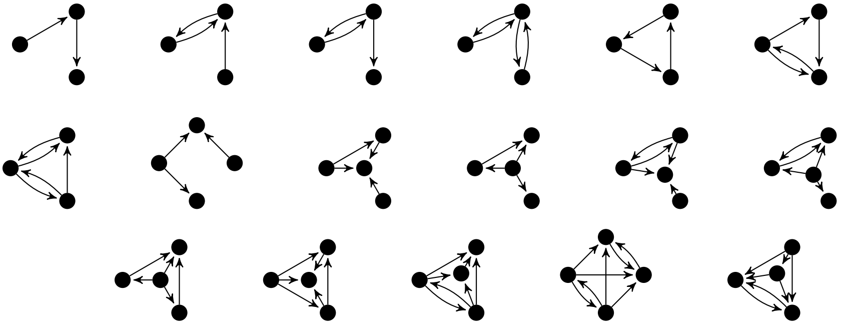

Figure 2: The γ-critical digraphs with 3 and 4 vertices that have algebraic co-rank equal to 2.

We have the following lemma.

Lemma 3.1. *Let* $\mathfrak{F}$ *be the family of digraphs shown in Figure 2. Then* $\mathfrak{F} \subseteq \mathbf{Forb}(\overrightarrow{\Gamma_{\leq 1}})$.

A digraph D is *complete* if, for every pair u, v of distinct vertices of D, both arcs uv and vu are in D. The complete digraph with n vertices is denoted by $\overrightarrow{K_n}$. The trivial digraph with n vertices and no arcs is denoted by T_n. Let D_1 and D_2 be vertex-disjoint subdigraphs of D. The set of arcs with tails in $V(D_1)$ and heads in $V(D_2)$ is denoted by $(D_1, D_2)_D$. We say that $(D_1, D_2)_D$ is *complete* when is equal to $\{uv : u \in V(D_1) \text{ and } v \in V(D_2)\}$.

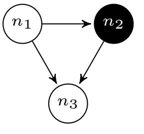

Figure 3: The digraph Λ_{n_1,n_2,n_3}.

Let $\Lambda_{n_1,n_2 n_3}$ be the digraph defined as follows. We start with two trivial digraphs T_{n_1} and T_{n_3}, and one complete digraph $\overrightarrow{K_{n_2}}$. Additionally, the arc sets $(T_{n_1}, \overrightarrow{K_{n_2}})_\Lambda$, $(T_{n_1}, T_{n_3})_\Lambda$ and $(\overrightarrow{K_{n_2}}, T_{n_3})_\Lambda$ are complete. See Fig. 3.

Lemma 3.2. *If* Λ_{n_1,n_2,n_3} *is connected such that* $n_1 + n_2 + n_3 \geq 2$*, then* $I_1(\Lambda_{n_1,n_2,n_3}, \{X_{n_1}, Y_{n_2}, Z_{n_3}\})$ *is trivial, and* $I_2(\Lambda_{n_1,n_2,n_3}, \{X_{n_1}, Y_{n_2}, Z_{n_3}\})$ *is equal to*

$$\begin{cases} \langle \cup_{i=1}^{n_1} x_i, \cup_{i=1}^{n_2} (y_i+1), \cup_{i=1}^{n_3} z_i \rangle\,, & \textit{if } n_1, n_2, n_3 \geq 1, \\ \langle x_1 y_1 \rangle\,, & \textit{if } n_1 = n_2 = 1, n_3 = 0, \\ \langle x_1 z_1 \rangle\,, & \textit{if } n_1 = n_3 = 1, n_2 = 0, \\ \langle y_1 z_1 \rangle\,, & \textit{if } n_2 = n_3 = 1, n_1 = 0, \\ \langle \cup_{i=1}^{n_1} x_i \rangle\,, & \textit{if } n_1 \geq 2, n_2 = 1, n_3 = 0, \\ \langle \cup_{i=1}^{n_1} x_i, \cup_{i=1}^{n_2} (y_i+1) \rangle\,, & \textit{if } n_1 \geq 1, n_2 \geq 2, n_3 = 0, \\ \langle \cup_{i<j} x_i x_j \rangle\,, & \textit{if } n_1 = 0, n_2 \geq 2, n_3 = 0, \\ \langle \cup_{i=1}^{n_1} z_i \rangle\,, & \textit{if } n_1 = 0, n_2 = 1, n_3 \geq 2, \\ \langle \cup_{i=1}^{n_2} (y_i+1), \cup_{i=1}^{n_3} z_i \rangle\,, & \textit{if } n_1 = 0, n_2 \geq 2, n_3 \geq 1, \\ \langle \cup_{i=1}^{n_3} z_i \rangle\,, & \textit{if } n_1 = 1, n_2 = 0, n_3 \geq 2, \\ \langle \cup_{i=1}^{n_1} x_i \rangle\,, & \textit{if } n_1 \geq 2, n_2 = 0, n_3 = 1, \\ \langle \cup_{i=1}^{n_1} x_i, \cup_{i=1}^{n_3} z_i \rangle\,, & \textit{if } n_1 \geq 2, n_2 = 0, n_3 \geq 2. \end{cases}$$

Now we present the characterization of the digraphs with at most one trivial critical ideal.

Theorem 3.3. *Let D be a connected digraph. Then the following statements are equivalent:*

(i) $D \in \overrightarrow{\Gamma_{\leq 1}}$,

(ii) *D is $\mathfrak{F}$-free.*

(iii) *D is isomorphic to Λ_{n_1,n_2,n_3}.*

By Theorem 1.2, to obtain the classification of the digraphs whose critical group has exactly one invariant factor equal to one. For this, we only need to evaluate the out-degree of the vertices in the second critical ideal corresponding to each digraph in Lemma 3.2.

Theorem 3.4. *The critical group of a connected digraph has exactly 1 invariant factor equal to 1 if and only if is isomorphic to the digraph Λ_{n_1,n_2,n_3} where n_1, n_2, n_3 satisfy one of the following conditions.*

- $n_1, n_2, n_3 \geq 1$,
- $n_1 = n_2 = 1$, $n_3 = 0$,
- $n_1 = n_3 = 1$, $n_2 = 0$,
- $n_2 = n_3 = 1$, $n_1 = 0$,
- $n_1 \geq 1$, $n_2 \geq 2$, $n_3 = 0$,
- $n_1 = 0$, $n_2 \geq 2$, $n_3 = 0$,
- $n_1 = 0$, $n_2 = 1$, $n_3 \geq 2$,
- $n_1 = 0$, $n_2 \geq 2$, $n_3 \geq 1$,
- $n_1 = 1$, $n_2 = 0$, $n_3 \geq 2$,
- $n_1 \geq 2$, $n_2 = 0$, $n_3 \geq 2$.

References

[1] C.A. Alfaro and C.E. Valencia, Graphs with two trivial critical ideals, Discrete Applied Mathematics 167 (2014) 33–44.

[2] C.A. Alfaro, C.E. Valencia and A. Vázquez-Ávila, Digraphs with at most one trivial critical ideal. Linear and Multilinear Algebra, 2017. DOI: http://dx.doi.org/10.1080/03081087.2017.1383349.

[3] C.A. Alfaro, H.H. Corrales and C.E. Valencia, Critical ideals of signed graphs with twin vertices, Advances in Applied Mathematics 86 (2017) 99–131.

[4] W.H. Chan, Y. Hou, W.C. Shiu, Graphs whose critical groups have larger rank, Acta Math. Sinica 27 (2011) 1663–1670.

[5] H.H. Corrales and C.E. Valencia, On the critical ideals, Linear Algebra and its Applications 439 (2013) 3870–3892.

[6] D.R. Grayson and M.E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at www.math.uiuc.edu/Macaulay2/.

[7] D.J. Lorenzini, Arithmetical Graphs, Math. Ann. 285 (1989) 481–501.

[8] B.D. McKay, *nauty User's Guide (Version 2.4)*, available at http://cs.anu.edu.au/∼bdm/nauty/.

[9] C. Merino, The chip-firing game, Discrete Math. 302 (2005) 188–210.

[10] Y. Pan and J. Wang, A note on the third invariant factor of the Laplacian matrix of a graph, preprint arXiv:0912.3608 [math.CO].

Banco de México,
Mexico City, Mexico
carlos.alfaro@banxico.org.mx, alfaromontufar@gmail.com

Departamento de Matemáticas,
Centro de Investigación y de Estudios Avanzados del IPN,
Apartado Postal 14-740, 07000 Ciudad de México, Mexico
cvalencia@math.cinvestav.edu.mx

Subdirección de Ingeniería y Posgrado
Universidad Aeronáutica en Querétaro
adrian.vazquez@unaq.edu.mx