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Bounds and Complexity for the Tessellation Problem

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Abstract

Given a graph $\Gamma = (V, E)$, a tessellation $\mathcal{T}$ is a partition of V into cliques so that the union of the cliques covers the vertex set V but not necessarily the edge set E. A graph Γ is t-tessellable if we can cover the edge set E with t tessellations. The staggered model yields a discrete-time quantum walk on a graph without using a coin operator. From the tessellations of the graph, the staggered model defines orthogonal reflexive operators which are then composed in order to obtain the evolution operator. We relate the tessellation problem to the clique operator, by showing that the chromatic number of the clique graph is a tight upper bound for the tessellation number. We show that the tessellation problem is also related to the edge-coloring problem, a hard complexity problem.

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1 Introduction

Random walks are used as an important tool in many areas, in particular, in computer science. Random walks are used in the development of algorithms to solve important problems, for example the k-SAT problem. Due to this, it is expected that random walk's counterpart, known as quantum walks, will have at least a similar importance in quantum computing [9]. Recently, it was proposed a new quantum walk model known as staggered quantum walk [3, 4] that includes the Szegedy's model [6] as a particular case, in addition to most interesting cases of the coined model [1]. Each step of the quantum walk is described by a unitary operator which is applied to a vector representing a quantum state. The tessellation process on a graph generates unitary matrices, where the product of these matrices corresponds to an evolution operator that defines the quantum walk in staggered model. In the present extended abstract, we relate the graph tessellation problem to the clique operator [7] and to the edge-coloring problem [2].

Preliminaries. In the rest of this section, we discuss basic definitions and notation. Given an undirected graph $\Gamma = (V, E)$, a **clique** of Γ is a subset $C \subseteq V$ of its vertices such that every two distinct vertices in C are adjacent. A clique of size d is called a d-clique. Given a graph Γ, its **clique graph** $K(\Gamma)$ is the intersection graph of the maximal cliques of Γ. Given a class $\mathcal{C}$ of graphs, we say that the **clique-inverse** graphs of $\mathcal{C}$ are the graphs whose clique graphs belong to class $\mathcal{C}$.

Definition 1. *A* ***tessellation*** $\mathcal{T}$ *is a partition of a graph into cliques, where no two cliques have a vertex in common. Each clique in the tessellation is called a* ***polygon*** *(or a* ***cell****). We say that an edge* ***belongs to*** *(or* ***is covered by****) the tessellation if both endpoints of the edge belong to the same polygon. The set of edges belonging to (or covered by)* $\mathcal{T}$ *is denoted by* $\mathcal{E}(\mathcal{T})$*.*

Notice that although a single tessellation always covers the vertex set, we may need more than one tessellation in order to cover the edge set. This remark is important in our underlying application, since the staggered quantum walk model describes an evolution operator for each tessellation, allowing the walker to move only across the covered edges.

Definition 2. *A* ***tessellation cover*** *of size k of a graph Γ is a set of k tessellations $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_k$ such that the union $\cup_{i=1}^{k} \mathcal{E}(\mathcal{T}_i)$ is the edge set of Γ.*

It is important to determine the minimum number of tessellations required in a tessellation cover. In the staggered quantum walk model, we are interested in obtaining the minimal tessellation cover, because it leads to more efficient algorithms.

Definition 3. *The* ***tessellation number*** *$T(\Gamma)$ is the cardinality of a smallest tessellation cover of Γ. We say graph Γ is* ***t-tessellable*** *for an integer t when $T(\Gamma) \leq t$. Given a graph Γ, the* ***t-tessellation problem*** *consists in finding if a graph Γ is t-tessellable.*

As an example, we illustrate the above definitions using the Hajós graph (Fig. 1, left), which is the graph Γ with vertex set $\{0, 1, \cdots, 5\}$ and edge set $\mathcal{E}(\Gamma) = \{\{0, 1\}, \{0, 2\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{4, 5\}\}$. Notice that, in this case, we have $T(\Gamma) = 3$ because tessellations

$$
\begin{aligned}
\mathcal{T}_1 =& \{\{0\}, \{1\}, \{3\}, \{2, 4, 5\}\},\\
\mathcal{T}_2 =& \{\{0\}, \{2\}, \{5\}, \{1, 3, 4\}\},\\
\mathcal{T}_3 =& \{\{3\}, \{4\}, \{5\}, \{0, 1, 2\}\}
\end{aligned}
$$

form a minimum tessellation cover. In fact, each of $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ is a partition of Γ into cliques which covers all vertices. The set $\mathcal{E}(\mathcal{T}_1) \cup \mathcal{E}(\mathcal{T}_2) \cup \mathcal{E}(\mathcal{T}_3)$ is equal to the edge set of Γ. In the next section, we will show that the Hajós graph cannot be covered with less than three tessellations.

Definition 4. *A* ***coloring*** *(resp. an* ***edge-coloring****) of a graph is a labeling of vertices (edges) with colors such that no two adjacent vertices*

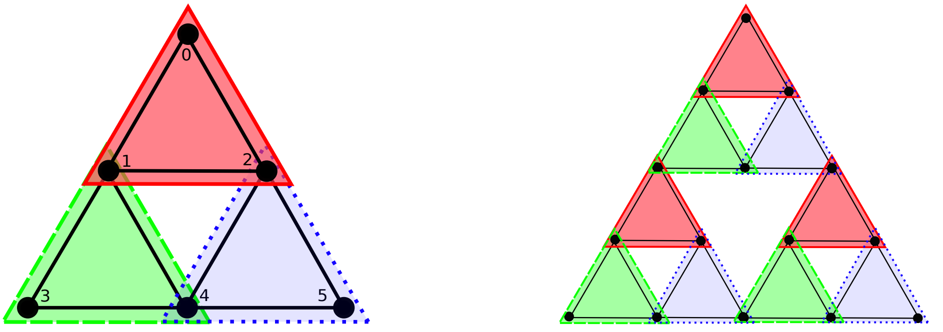

Figure 1: (Left) The Hajós graph S_2. (Right) The Sierpińsky Sieve graphs can always be covered by three tessellations. For better visualization, only maximal cliques are depicted here for each tessellation. These graphs cannot be covered by less than three tessellations.

*(incident edges) have the same color. A k-**colorable** (k**-edge-colorable**) graph is one whose vertices (edges) can be colored with at most k colors so that no two adjacent vertices (incident edges) share the same color. The **chromatic number** $\chi(\Gamma)$ (**chromatic index** $\chi'(\Gamma)$) of a graph Γ is the smallest number of colors needed to color the vertices (edges) of Γ.*

2 Results

Bounds for the tessellation number. In this subsection, we prove lower and upper bounds for the tessellation number, and show that these bounds are tight.

Portugal proved that a graph is 2-tessellable if and only if its clique graph is 2-colorable [3]. The Hajós graph is not 2-tessellable, since its clique graph is a K_4. Therefore, the tessellation number for the Hajós graph is $T(\Gamma) = 3$ (see Fig. 1, left). The Hajós graph is also known as the Sierpińsky graph S_2. The Sierpińsky Sieve graph S_n, $n \geq 3$, is constructed by three copies of S_{n-1}, as depicted in Fig. 1, right. In fact, every Sierpińsky Sieve graph S_n of order $n \geq 2$ has tessellation number $T(S_n) = 3$, which can be proved by induction on n.

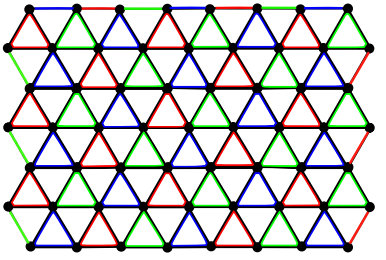

Figure 2: The triangular lattice is covered by three tessellations, which is the minimum number of tessellations required to cover the graph.

The problem is also well defined for infinite graphs. The triangular lattice (see Fig. 2) is a nice example of an infinite graph with tessellation number equal to three, which can be proved by an argument very similar to the one used with the Sierpińsky family (see Fig. 1). This result is particularly important for the applications on staggered quantum walks, because we are mostly interested on graphs with large diameter and small tessellation number, and the triangular lattice satisfies these conditions.

Theorem 1. *Let* Γ *be a graph whose clique graph is not 2-colorable. Then,* $3 \leq T(\Gamma) \leq \chi(K(\Gamma))$.

Proof. The lower bound follows from the fact that a graph is 2-tessellable if and only if its clique graph is bipartite [3]. Now we give a proof for the upper bound, by defining a family of $\chi(K(\Gamma))$ tessellations whose union covers the edge set of Γ. Consider an optimal coloring of $K(\Gamma)$, and let M_g be the set of maximal cliques corresponding to vertices of $K(\Gamma)$ colored by color g. Any pair of such maximal cliques must be disjoint, so we can define a tessellation $\mathcal{T}_g$ whose polygons are the cliques of M_g together with the missing vertices of Γ. Since every edge of Γ belogs to at least one maximal clique, the union of the $\chi(K(\Gamma))$ defined tessellations covers the edges of Γ. ■

Note that the Sierpińsky Sieve graphs and the triangular lattice provide examples where the lower bound $3 \leq T(\Gamma)$ is tight.

In order to prove that the upper bound $T(\Gamma) \leq \chi(K(\Gamma))$ is tight, consider the (k,n)-windmill graph $\mathrm{Wd}_{k,n}$, for $k \geq 2, n \geq 2$, constructed by joining n copies of k-cliques at a universal vertex. The clique graph of the windmill $\mathrm{Wd}_{k,n}$ is an n-clique, and therefore by Theorem 1, we have that $T(\mathrm{Wd}_{k,n}) \leq n$. On the other hand, it is not possible to cover the edge set of $\mathrm{Wd}_{k,n}$ with less than n tessellations, since no two edges that are incident on the universal vertex and that are not in the same clique can be covered by a single tessellation. Therefore, $T(\mathrm{Wd}_{k,n}) = \chi(K(\mathrm{Wd}_{k,n})) = n$ (see Fig. 3). The argument still holds if we generalize the standard windmill by allowing different-sized cliques.

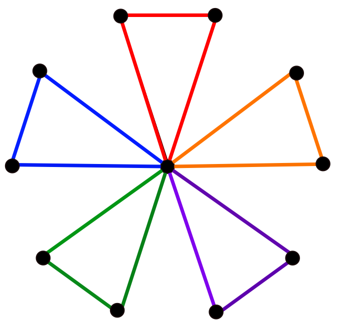

Figure 3: The class of windmill graphs is tight with respect to the upper bound of Theorem 1. The windmill $\mathrm{Wd}_{3,5}$ has 5 triangles, its clique graph $K(\mathrm{Wd}_{3,5})$ is the complete graph with 5 vertices, and the tesselation number $T(\mathrm{Wd}_{3,5})$ is 5.

Complexity of the tessellation problem. In this subsection, we review that the k-tessellation problem is polynomial for $k = 2$ and prove that it is NP-complete for $k = 3$.

Let $\Gamma = (V, E)$ be a connected graph, $n = |V(\Gamma)|$, and $m = |E(\Gamma)|$. Protti and Szwarcfiter proved that if $K(\Gamma)$ is bipartite then Γ contains at most $2n$ maximal cliques, and as a consequence of this result they described a polynomial-time algorithm to recognize if a given graph is the clique-inverse of a bipartite graph [5]. The procedure goes as follows.

Check whether Γ contains at most $2n$ maximal cliques by applying the algorithm from [8] to the complement of Γ. The task takes $O(n^2m)$ time. If Γ has more than $2n$ maximal cliques, then Γ is not clique-inverse of a bipartite graph. Otherwise, we construct $K(\Gamma)$ by taking the maximal cliques generated by the algorithm. This task takes $O(mn)$ time. Finally, check whether $K(\Gamma)$ is bipartite, which takes $O(m)$ time. Therefore, we can check whether Γ is clique-inverse of a bipartite graph in polynomial time, which proves that deciding whether a connected graph is 2-tessellable is polynomial.

On the other hand, deciding whether a graph is 3-tessellable is NP-complete. To prove this statement, we use that the 3-edge-coloring problem is NP-complete even if the input graphs are restricted to the class of triangle-free graphs [2].

Theorem 2. *Deciding whether a graph is 3-tessellable is NP-complete.*

Proof. The 3-tessellation problem is in NP: given a graph Γ and three families of subsets of $V(\Gamma)$, we can verify in deterministic polynomial time whether they are a valid tessellation cover for Γ. Moreover, in a triangle-free graph Γ, a 3-tessellation corresponds to 3 matchings of Γ covering its edge set, and so define a 3-edge-coloring of Γ. ■

3 Final remarks

We discussed the tessellation number, the tessellation problem and applications to staggered quantum walks. Given a connected graph Γ such that its clique graph is not bipartite, we have that $3 \leq T(\Gamma) \leq \chi(K(\Gamma))$, and both bounds are tight. Deciding whether a graph is 2-tessellable is polynomial. Deciding whether a graph is 3-tessellable is NP-complete.

We discussed the examples of Sierpińsky Sieve graphs and the triangular lattice, which have arbitrarily large diameter and tessellation number three. This is interesting for quantum walks. Which classes of graphs have large diameter and small tessellation number?

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